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NOETHER ALMOST DISTRIBUTIVE LATTICES

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Abstract. In this paper, we introduce the concept of Noether Almost Distributive Lattice (Noether ADL). We prove important results in a Noether ADL and also prove the fundamental Theorem of Noether ADL.

1. Introduction

Swamy, U.M. and Rao, G.C. [**5**] introduced the concept of an Almost Distributive Lattice as a common abstraction of almost all the existing ring theoretic generalizations of a Boolean algebra (like regular rings, $p-$rings, biregular rings, associate rings, P_1-rings etc.) on one hand and distributive lattices on the other.

In [**1**], Dilworth, R.P., has introduced the concept of a residuation in lattices and in [**6, 7**] Ward, M. and Dilworth, R.P., have studied residuated lattices. We introduced the concepts of a residuation and a multiplication in an ADL and the concept of a residuated ADL in our earlier paper [**3**]. We have proved some important properties of residuation ' : ' and multiplication ' . ' in a residuated ADL L in [**4**]. In this paper, we intrtoduce the concept of Noether ADL. We prove important results in a Noether ADL.

In section 2, we recall the definition of an Almost Distributive Lattice (ADL) and certain elementary properties of an ADL from Swamy, U.M. and Rao, G.C. [**5**], Rao, G.C. [**2**] and the concepts of residuation and multiplication in an ADL L and the definition of a residuated almost distributive lattice from our earlier paper [**4**]. In section 3, we introduce the concept of Noether Almost Distributive Lattice (Noether ADL). We prove important results in a Noether ADL and also prove the fundamental Theorem of Noether ADL.

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2. Preliminaries

In this section we collect a few important definitions and results which are already known and which will be used more frequently in the paper. We begin with the definition of an ADL :

DEFINITION 2.1. ([**2**]) An Almost Distributive Lattice(ADL) is an algebra $(L, \vee, \wedge)$ of type $(2, 2)$ satisfying

(1) $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$

(2) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

(3) $(a \vee b) \wedge b = b$

(4) $(a \vee b) \wedge a = a$ (5) $a \vee (a \wedge b) = a$, for all $a, b, c \in L$.

It can be seen directly that every distributive lattice is an ADL. If there is an element $0 \in L$ such that $0 \wedge a = 0$ for all $a \in L$, then $(L, \vee, \wedge, 0)$ is called an ADL with 0.

EXAMPLE 2.1. ([**2**]) Let X be a non-empty set. Fix $x_0 \in X$.
For any $x, y \in L$, define

$$x \wedge y = \begin{cases} x_0, & \text{if } x = x_0 \\ y, & \text{if } x \neq x_0 \end{cases} \quad x \vee y = \begin{cases} y, & \text{if } x = x_0 \\ x, & \text{if } x \neq x_0. \end{cases}$$

Then $(X, \vee, \wedge, x_0)$ is an ADL, with x_0 as its zero element. This ADL is called a discrete ADL.

For any $a, b \in L$, we say that a is less than or equals to b and write $a \leqslant b$, if $a \wedge b = a$. Then "$\leqslant$" is a partial ordering on L.

THEOREM 2.1. ([**2**]) *Let $(L, \vee, \wedge, 0)$ be an ADL with ' 0 '. Then, for any $a, b \in L$, we have*

(1) $a \wedge 0 = 0$ *and* $0 \vee a = a$

(2) $a \wedge a = a = a \vee a$

(3) $(a \wedge b) \vee b = b$, $a \vee (b \wedge a) = a$ *and* $a \wedge (a \vee b) = a$

(4) $a \wedge b = a \Longleftrightarrow a \vee b = b$ *and* $a \wedge b = b \Longleftrightarrow a \vee b = a$

(5) $a \wedge b = b \wedge a$ *and* $a \vee b = b \vee a$ *whenever* $a \leqslant b$

(6) $a \wedge b \leqslant b$ *and* $a \leqslant a \vee b$

(7) $\wedge$ *is associative in* L

(8) $a \wedge b \wedge c = b \wedge a \wedge c$

(9) $(a \vee b) \wedge c = (b \vee a) \wedge c$

(10) $a \wedge b = 0 \Longleftrightarrow b \wedge a = 0$

(11) $a \vee (b \vee a) = a \vee b$.

It can be observed that an ADL L satisfies almost all the properties of a distributive lattice except, possible the right distributivity of $\vee$ over $\wedge$, the commutativity of $\vee$, the commutativity of $\wedge$ and the absorption law $(a \wedge b) \vee a = a$. Any one of these properties convert L into a distributive lattice.

THEOREM 2.2. ([**2**]) *Let* $(L, \vee, \wedge, 0)$ *be an ADL with* 0. *Then the following are equivalent:*

(1) $(L, \vee, \wedge, 0)$ *is a distributive lattice*

(2) $a \vee b = b \vee a$, *for all* $a, b \in L$

(3) $a \wedge b = b \wedge a$, *for all* $a, b \in L$

(4) $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$, *for all* $a, b, c \in L$.

PROPOSITION 2.1. ([**2**]) *Let* $(L, \vee, \wedge)$ *be an ADL. Then for any* $a, b, c \in L$ *with* $a \leqslant b$, *we have*

(1) $a \wedge c \leqslant b \wedge c$

(2) $c \wedge a \leqslant c \wedge b$

(3) $c \vee a \leqslant c \vee b$.

DEFINITION 2.2. ([**2**]) An element $m \in L$ is called maximal if it is maximal as in the partially ordered set $(L, \leqslant)$. That is, for any $a \in L$, $m \leqslant a$ implies $m = a$.

THEOREM 2.3. ([**2**]) *Let* L *be an ADL and* $m \in L$. *Then the following are equivalent:*

(1) m *is maximal with respect to* $\leqslant$

(2) $m \vee a = m$, *for all* $a \in L$

(3) $m \wedge a = a$, *for all* $a \in L$.

LEMMA 2.1. ([**2**]) *Let* L *be an ADL with a maximal element* m *and* $x, y \in L$. *If* $x \wedge y = y$ *and* $y \wedge x = x$ *then* x *is maximal if and only if* y *is maximal.Also the following conditions are equivalent:*

(i) $x \wedge y = y$ *and* $y \wedge x = x$ (ii) $x \wedge m = y \wedge m$.

DEFINITION 2.3. ([**2**]) If $(L, \vee, \wedge, 0, m)$ is an ADL with 0 and with a maximal element m, then the set $I(L)$ of all ideals of L is a complete lattice under set inclusion. In this lattice, for any $I, J \in I(L)$, the l.u.b. and g.l.b. of I, J are given by $I \vee J = \{(x \vee y) \wedge m \mid x \in I, y \in J\}$ and $I \wedge J = I \cap J$. The set $PI(L) = \{(a] \mid a \in L\}$ of all principal ideals of L forms a sublattice of $I(L)$. (Since $(a] \vee (b] = (a \vee b]$ *and* $(a] \cap (b] = (a \wedge b]$)

In the following, we give the concepts of residuation and multiplication in an almost distributive lattice (ADL) L and the definition of a residuated almost distributive lattice taken from our earlier paper [**3**].

DEFINITION 2.4. ([**3**]) Let L be an ADL with a maximal element m. A binary operation : on an ADL L is called a residuation over L if, for $a, b, c \in L$ the following conditions are satisfied:

$(R1)$ $a \wedge b = b$ if and only if $a : b$ is maximal

$(R2)$ $a \wedge b = b \implies$ (i) $(a : c) \wedge (b : c) = b : c$ and (ii) $(c : b) \wedge (c : a) = c : a$

$(R3)$ $[(a : b) : c] \wedge m = [(a : c) : b] \wedge m$

$(R4)$ $[(a \wedge b) : c] \wedge m = (a : c) \wedge (b : c) \wedge m$

$(R5)$ $[c : (a \vee b)] \wedge m = (c : a) \wedge (c : b) \wedge m$

Definition 2.5. ([**3**]) Let L be an ADL with a maximal element m. A binary operation . on an ADL L is called a multiplication over L if, for $a, b, c \in L$ the following conditions are satisfied:

$(M1)$ $(a.b) \wedge m = (b.a) \wedge m$

$(M2)$ $[(a.b).c] \wedge m = [a.(b.c)] \wedge m$

$(M3)$ $(a.m) \wedge m = a \wedge m$

$(M4)$ $[a.(b \vee c)] \wedge m = [(a.b) \vee (a.c)] \wedge m$

Definition 2.6. ([**3**]) An ADL L with a maximal element m is said to be a residuated almost distributive lattice (residuated ADL), if there exists two binary operations ' : ' and ' . ' on L satisfying conditions R1 to R5, M1 to M4 and the following condition (A).

(A) $(x : a) \wedge b = b$ if and only if $x \wedge (a.b) = a.b$, for any $x, a, b \in L$.

We use the following properties frequently later in the results.

Lemma 2.2. ([**3**]) *Let L be an ADL with a maximal element m and . a binary operation on L satisfying the conditions $M1 - M4$. Then for any $a, b, c, d \in L$,*

(i) $a \wedge (a.b) = a.b$ *and* $b \wedge (a.b) = a.b$

(ii) $a \wedge b = b \implies (c.a) \wedge (c.b) = c.b$ *and* $(a.c) \wedge (b.c) = b.c$

(iii) $d \wedge [(a.b).c] = (a.b).c$ *if and only if* $d \wedge [a.(b.c)] = a(b.c)$

(iv) $(a.c) \wedge (b.c) \wedge [(a \wedge b).c] = (a \wedge b).c$

(v) $d \wedge (a.c) \wedge (b.c) = (a.c) \wedge (b.c) \implies d \wedge [(a \wedge b).c] = (a \wedge b).c$

(vi) $d \wedge [(a.c) \vee (b.c)] = (a.c) \vee (b.c) \Leftrightarrow d \wedge [(a \vee b).c] = (a \vee b).c$

The following result is a direct consequence of M1 of definition 2.15.

Lemma 2.3. ([**3**]) *Let L be an ADL with a maximal element m and . a binary operation on L satisfying the condition $M1$. For $a, b, x \in L$, $a \wedge (x.b) = x.b$ if and only if $a \wedge (b.x) = b.x$*

In the following, we give some important properties of residuation ' : ' and multiplication ' . ' in a residuated ADL L. These are taken from our earlier paper [**4**].

Lemma 2.4. ([**4**]) *Let L be a residuated ADL with a maximal element m. For $a, b, c, d \in L$, the following hold in L.*

(1) $(a : b) \wedge a = a$

(2) $[a : (a : b)] \wedge (a \vee b) = a \vee b$

(3) $[(a : b) : c] \wedge [a : (b.c)] = a : (b.c)$

(4) $[a : (b.c)] \wedge [(a : b) : c] = (a : b) : c$

(5) $[(a \wedge b) : b] \wedge (a : b) = a : b$

(6) $(a : b) \wedge [(a \wedge b) : b] = (a \wedge b) : b$

(7) $[a:(a\vee b)]\wedge m=(a:b)\wedge m$

(8) $[c:(a\wedge b)]\wedge[(c:a)\vee(c:b)]=(c:a)\vee(c:b)$

(9) *If* $a:b=a$ *then* $a\wedge(b.d)=b.d\Longrightarrow a\wedge d=d$

(10) $\{a:[a:(a:b)]\}\wedge(a:b)=a:b$

(11) $[(a\vee b):c]\wedge[(a:c)\vee(b:c)]=(a:c)\vee(b:c)$

(12) $a\wedge m\geqslant b\wedge m\Longrightarrow(a:c)\wedge m\geqslant(b:c)\wedge m$

(13) $(a:b)\wedge\{a:[a:(a:b)]\}=a:[a:(a:b)]$

(14) $a\wedge b=b\Longrightarrow(a.c)\wedge(b.c)=b.c$

(15) $a\wedge b\wedge(a.b)=a.b$

(16) $[(a.b):a]\wedge b=b$

(17) $(a.b)\wedge[(a\wedge b).(a\vee b)]=(a\wedge b).(a\vee b)$

(18) $a\vee b$ *is maximal* $\Longrightarrow(a.b)\wedge a\wedge b=a\wedge b$

3. Noether Almost Distributive Lattices

In this section, we introduce the concept of Noether Almost Distributive Lattice (Noether ADL). We prove important results in a Noether ADL and also prove the fundamental Theorem of Noether ADL.

We first give the following concepts on a residuated ADL L:

Definition 3.1. An element c of L is called irreducible, if

$$f\wedge g=c,\text{ for }f,g\in L\Longrightarrow\text{either }f=c\text{ or }g=c.$$

Definition 3.2. An element p of L is called prime, if

$$p\wedge(a.b)=a.b\Longrightarrow\text{either }p\wedge a=a\text{ or }p\wedge b=b,\text{ for any }a,b\in L$$

Definition 3.3. An element p of L is called primary, if

$$p\wedge(a.b)=a.b\text{ and }p\wedge a\neq a\Longrightarrow p\wedge b^s=b^s,\text{ forsome }s\in Z^+$$

Definition 3.4. An ADL L is said to satisfy the ascending chain condition(a.c.c.), if for every increasing sequence $x_1\leqslant x_2\leqslant x_3\leqslant$ ...., in L, there exists a positive integer n such that $x_n=x_{n+1}=x_{n+2}=$ ....

Definition 3.5. A residuated ADL L is said to be a Noether ADL, if

(N1) the ascending chain condition(a.c.c.) holds in L and

(N2) every irreducible element of L is primary.

Definition 3.6. An element a of a residuated ADL L is said to have a primary decomposition, if there exists primary elements $p_1,p_2,......,p_m$ in L such that $a=p_1\wedge p_2\wedge......\wedge p_m$.

Theorem 3.1. *If a and b are any two elements of a Noether ADL L and a.b has a primary decomposition, then there exists an exponent s such that*

$$(a.b)\wedge a\wedge b^s=a\wedge b^s$$

.

Proof. Let L be a Noether ADL and $a, b \in L$. Suppose $a.b$ has a primary decomposition, say $a.b = p_1 \wedge p_2 \wedge ... \wedge p_k$ Then for $1 \leqslant i \leqslant k$, $p_i \wedge (a.b) = a.b$. So that, either $p_i \wedge a = a$ or $p_i \wedge a \neq a$, $p_i \wedge b^{s_i} = b^{s_i}$, for some $s_i \in Z^+$.

Case (i): Suppose $p_k \wedge a = a$, then we rearrange the primary elements p_1, p_2, ... ,p_{k-1} such that $p_i \wedge b^{s_i} = b^{s_i}$, for $1 \leqslant i \leqslant l$ and $p_i \wedge a = a$, for $l+1 \leqslant i \leqslant k$. Now, take $s = Max\{s_1, s_2, ..., s_l\}$. Then $p_i \wedge b^s = b^s$, for $1 \leqslant i \leqslant l$. Now,

$(a.b) \wedge a \wedge b^s = p_1 \wedge p_2 \wedge ... \wedge p_k \wedge a \wedge b^s =$
$p_{l+1} \wedge p_{l+2} \wedge ... \wedge p_k \wedge a \wedge p_1 \wedge p_2 \wedge ... \wedge p_l \wedge b^s = a \wedge b^s$.

Case (ii): Suppose $p_k \wedge a \neq a$ and $p_k \wedge b^{s_k} = b^{s_k}$, then we rearrange the primary elements $p_1, p_2, ..., p_{k-1}$ such that $p_i \wedge a = a$, for $1 \leqslant i \leqslant j$ and $p_i \wedge b^{s_i} = b^{s_i}$, for $j+1 \leqslant i \leqslant k-1$. Now, take $s = Max\{s_{j+1}, s_{j+2}, ..., s_k\}$. Then $p_i \wedge b^s = b^s$, for $j+1 \leqslant i \leqslant k$. Now,

$(a.b) \wedge a \wedge b^s = p_1 \wedge p_2 \wedge ... \wedge p_k \wedge a \wedge b^s =$
$p_1 \wedge p_2 \wedge ... \wedge p_j \wedge a \wedge p_{j+1} \wedge p_{j+2} \wedge ... \wedge p_k \wedge b^s = a \wedge b^s$. □

Theorem 3.2. *Let L be a Noether ADL with a maximal element m and $a, b, c \in L$. If $b.b = b$ then*
(i) $a \wedge b \wedge m = (a.b) \wedge m$ *and*
(ii) $[(a \wedge c).b] \wedge m = (a.b) \wedge (c.b) \wedge m$.

Proof. Let $a, b, c \in L$ and suppose $b.b = b$
(i) By property (15) of Lemma 2.4, we have $a \wedge b \wedge (a.b) = a.b$ and by Theorem 3.1, we have $(a.b) \wedge a \wedge b^s = a \wedge b^s$. So that, $(a.b) \wedge a \wedge b = a \wedge b$ (Since $b.b = b$) Hence $a \wedge b \wedge m = (a.b) \wedge m$
(ii) By (i) above, we have
$[(a \wedge c).b] \wedge m = a \wedge c \wedge b \wedge m$
$= a \wedge b \wedge m \wedge c \wedge b \wedge m$
$= (a.b) \wedge m \wedge (c.b) \wedge m$ (By (i), above)
$= (a.b) \wedge (c.b) \wedge m$. □

Corollary 3.1. *Let L be a Noether ADL with a maximal element m. If a and b are any two idempotent elements of L, then $(a.b) \wedge m = a \wedge b \wedge m$.*

Proof. Let $a, b \in L$ be idempotent elements of L. Then $a^2 = a, b^2 = b$. By property (15) of Lemma 2.4, we have $a \wedge b \wedge (a.b) = a.b$ By Theorem 3.1, we have $(a.b) \wedge a \wedge b^s = a \wedge b^s$. So that, $(a.b) \wedge a \wedge b = a \wedge b$. Hence $(a.b) \wedge m = a \wedge b \wedge m$. □

The following Theorem is converse of Theorem 3.1. under special conditions.

Theorem 3.3. *Let L be a residuated ADL with ascending chain condition(a.c.c.) such that for any $a, b \in L$, there exists $s \in Z^+$ such that, $(a.b) \wedge a \wedge b^s = a \wedge b^s$ then L is a Noether ADL.*

Proof. Let p be an irreducible element of L. Let $a, b \in L$ such that $p \wedge (a.b) = a.b$ and $p \wedge a \neq a$. Choose $s \in Z^+$ such that $(a.b) \wedge a \wedge b^s = a \wedge b^s$. Now,

$p \wedge (a.b) = a.b \Longrightarrow p \wedge (a.b) \wedge a \wedge b^s = (a.b) \wedge a \wedge b^s$
$\Longrightarrow p \wedge a \wedge b^s = a \wedge b^s$

$\Longrightarrow p \vee (a \wedge b^s) = p$
$\Longrightarrow (p \vee a) \wedge (p \vee b^s) = p$
$\Longrightarrow$ either $p \vee a = p$ or $p \vee b^s = p$ (Since p is irreducible)

But $p \vee a \neq p$ since $p \wedge a \neq a$. Therefore $p \vee b^s = p$. Thus $p \wedge b^s = b^s$, forsome $s \in Z^+$. Therefore p is primary. Hence every irreducible element of L is primary. Thus L is a Noether ADL. □

Definition 3.7. Let L be a residuated ADL. An element a of L is called principal, if $a \wedge b = b$, forsome $b \in L$, then $a.c = b$, forsome $c \in L$.

Lemma 3.1. *Let L be a residuated ADL with a maximal element m. If $a, b \in L$ such that a is principal and $a \wedge b = b$, then $[(b : a).a] \wedge m = b \wedge m$.*

Proof. Let $a, b \in L$ such that a is principal and $a \wedge b = b$. Then there exists an element $c \in L$ such that $a.c = b$. Now,

$b \wedge (a.c) = a.c \Longrightarrow (b : a) \wedge c = c$ (By definition 2.6)
$\Longrightarrow [(b : a).a] \wedge (c.a) = c.a$ (By Lemma 2.2 (ii))
$\Longrightarrow [(b : a).a] \wedge (a.c) = a.c$ (By Lemma 2.3)
$\Longrightarrow [(b : a).a] \wedge b = b$ (Since a.c = b)

Now,

$(b : a) \wedge (b : a) = b : a \Longrightarrow\ b \wedge [a.(b : a)] = a.(b : a)$ (By definition 2.6)
$\Longrightarrow\ b \wedge [(b : a).a] = (b : a).a$ (By Lemma 2.3)

Hence $[(b : a).a] \wedge m = b \wedge m$. □

We shall now prove the following fundamental Theorem :

Theorem 3.4. *Let L be an ADL with a maximal element m satisfying the following conditions:*

(1) *L is residuated.*

(2) *L satisfies a.c.c.*

(3) *Every element of L is principal.*

Then L is a Noether ADL.

Proof. Suppose the conditions (1), (2) and (3) hold in L. Let p be a non-primary element of L. Then there exists $a, b \in L$ such that $p \wedge (a.b) = a.b$, $p \wedge a \neq a$ and $p \wedge b^s \neq b^s$, for any $s \in Z^+$. Let $k \in Z^+$. Then by Lemma 1.23 (i), we get $b^{k-1} \wedge b^k = b^k$. So that, $(p : b^k) \wedge (p : b^{k-1}) = p : b^{k-1}$. Hence $(p : b^k) \wedge m \geqslant (p : b^{k-1}) \wedge m$. Since L satisfies a.c.c., the chain $(p : b) \wedge m \leqslant (p : b^2) \wedge m \leqslant (p : b^3) \wedge m$, ...terminates. Then there exists $k \in Z^+$ such that $(p : b^k) \wedge m = (p : b^{k+1}) \wedge m =$ .... $\rightarrow$(i) Write $c = (p \vee a) \wedge (p \vee b^k)$. Then $p \vee b^k \geqslant c \geqslant p$. Now, $c = c \wedge (p \vee b^k) = p \vee (c \wedge b^k) \longrightarrow$ (ii)

First, we prove $p \wedge c \wedge b^k = c \wedge b^k$ Since b^k is principal and $b^k \wedge c \wedge b^k = c \wedge b^k$ we get $c \wedge b^k \wedge m = ([(c \wedge b^k) : b^k].b^k) \wedge m \longrightarrow$ (iii) (By Lemma 3.1) Since $(p \vee a) \wedge c = c$, we get $[b.(p \vee a)] \wedge (b.c) = b.c$ (By Lemma 2.2 (ii)) Now,

$p \wedge (b.c) = p \wedge [b.(p \vee a)] \wedge (b.c) = p \wedge [(b.p) \vee (b.a)] \wedge (b.c)$

$= [p \wedge (b.p)] \vee [p \wedge (b.a)] \wedge (b.c) = [(b.p) \vee (b.a)] \wedge (b.c)$
$= [b.(p \vee a)] \wedge (b.c) = b.c$ (Since $(p \vee a) \wedge c = c$)
$\Longrightarrow p \wedge (b.c) \wedge [b.(c \wedge b^k)] = (b.c) \wedge [b.(c \wedge b^k)]$
$\Longrightarrow p \wedge [b.(c \wedge b^k)] = b.(c \wedge b^k)$ (Since $c \wedge c \wedge b^k = c \wedge b^k$)
$\Longrightarrow (p : b) \wedge c \wedge b^k = c \wedge b^k$ (By definition 2.6)
$\Longrightarrow (p : b) \wedge c \wedge b^k \wedge m = c \wedge b^k \wedge m$
$\Longrightarrow (p : b) \wedge ([(c \wedge b^k) : b^k].b^k) \wedge m = ([(c \wedge b^k) : b^k].b^k) \wedge m$ (By (iii))
$\Longrightarrow (p : b) \wedge ([(c \wedge b^k) : b^k].b^k) = [(c \wedge b^k) : b^k].b^k$
$\Longrightarrow (p : b) \wedge \{b^k.[(c \wedge b^k) : b^k]\} = b^k.[(c \wedge b^k) : b^k]$ (By Lemma 2.3)
$\Longrightarrow (p : b^{k+1}) \wedge [(c \wedge b^k) : b^k] = (c \wedge b^k) : b^k$ (By definition 2.6)
$\Longrightarrow (p : b^k) \wedge [(c \wedge b^k) : b^k] = (c \wedge b^k) : b^k$ (By (i))
$\Longrightarrow p \wedge \{b^k.[(c \wedge b^k) : b^k]\} = b^k.[(c \wedge b^k) : b^k]$ (By definition 2.6)
$\Longrightarrow p \wedge \{[(c \wedge b^k) : b^k].b^k\} = [(c \wedge b^k) : b^k].b^k$ (By Lemma 2.3)
$\Longrightarrow p \wedge \{[(c \wedge b^k) : b^k].b^k\} \wedge m = \{[(c \wedge b^k) : b^k].b^k\} \wedge m$
$\Longrightarrow p \wedge c \wedge b^k \wedge m = c \wedge b^k \wedge m$ (By (iii))
$\Longrightarrow p \wedge c \wedge b^k = c \wedge b^k$
$\Longrightarrow p \vee (c \wedge b^k) = p$

Now, $c = p \vee (c \wedge b^k) = p$. Then $p = (p \vee a) \wedge (p \vee b^k)$. But $p \vee a \neq p$ and $p \vee b^k \neq p$. Hence p is reducible. Thus L is a Noether ADL. □

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