NOETHER ALMOST DISTRIBUTIVE LATTICES

G. C. Rao and S. S. Raju

Abstract. In this paper, we introduce the concept of Noether Almost Distributive Lattice (Noether ADL). We prove important results in a Noether ADL and also prove the fundamental Theorem of Noether ADL.

1. Introduction

Swamy, U.M. and Rao, G.C. [5] introduced the concept of an Almost Distributive Lattice as a common abstraction of almost all the existing ring theoretic generalizations of a Boolean algebra (like regular rings, \( p \)-rings, biregular rings, associate rings, \( P_1 \)-rings etc.) on one hand and distributive lattices on the other.

In [1], Dilworth, R.P., has introduced the concept of a residuation in lattices and in [6, 7] Ward, M. and Dilworth, R.P., have studied residuated lattices. We introduced the concepts of a residuation and a multiplication in an ADL and the concept of a residuated ADL in our earlier paper [3]. We have proved some important properties of residuation ‘’ and multiplication ‘’ in a residuated ADL L in [4]. In this paper, we introduce the concept of Noether ADL. We prove important results in a Noether ADL.

In section 2, we recall the definition of an Almost Distributive Lattice (ADL) and certain elementary properties of an ADL from Swamy, U.M. and Rao, G.C. [5], Rao, G.C. [2] and the concepts of residuation and multiplication in an ADL L and the definition of a residuated almost distributive lattice from our earlier paper [4]. In section 3, we introduce the concept of Noether Almost Distributive Lattice (Noether ADL). We prove important results in a Noether ADL and also prove the fundamental Theorem of Noether ADL.

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2. Preliminaries

In this section we collect a few important definitions and results which are already known and which will be used more frequently in the paper. We begin with the definition of an ADL:

**Definition 2.1.** ([2]) An Almost Distributive Lattice (ADL) is an algebra \((L, \lor, \land)\) of type \((2, 2)\) satisfying

(1) \((a \lor b) \land c = (a \land c) \lor (b \land c)\)
(2) \(a \land (b \lor c) = (a \land b) \lor (a \land c)\)
(3) \((a \lor b) \land b = b\)
(4) \((a \lor b) \land a = a \lor (a \land b) = a\), for all \(a, b, c \in L\).

It can be seen directly that every distributive lattice is an ADL. If there is an element \(0 \in L\) such that \(0 \land a = 0\) for all \(a \in L\), then \((L, \lor, \land, 0)\) is called an ADL with \(0\).

**Example 2.1.** ([2]) Let \(X\) be a non-empty set. Fix \(x_0 \in X\).

For any \(x, y \in L\), define

\[ x \land y = \begin{cases} x_0, & \text{if } x = x_0 \\ y, & \text{if } x \neq x_0 \end{cases} \quad x \lor y = \begin{cases} y, & \text{if } x = x_0 \\ x, & \text{if } x \neq x_0 \end{cases} \]

Then \((X, \lor, \land, x_0)\) is an ADL, with \(x_0\) as its zero element. This ADL is called a discrete ADL.

For any \(a, b \in L\), we say that \(a\) is less than or equals to \(b\) and write \(a \leq b\), if \(a \land b = a\). Then "\(\leq\)" is a partial ordering on \(L\).

**Theorem 2.1.** ([2]) Let \((L, \lor, \land, 0)\) be an ADL with ‘\(0\)’. Then, for any \(a, b \in L\), we have

(1) \(a \land 0 = 0\) and \(0 \lor a = a\)
(2) \(a \land a = a \lor a\)
(3) \((a \land b) \lor b = b\), \(a \lor (b \land a) = a\) and \(a \land (a \lor b) = a\)
(4) \(a \lor b = a \iff a \land b = b\) and \(a \land b = b \iff a \lor b = a\)
(5) \(a \lor b = b \land a\) and \(a \lor b = b \lor a\) whenever \(a \leq b\)
(6) \(a \land b \leq b\) and \(a \leq a \lor b\)
(7) \(\land\) is associative in \(L\)
(8) \(a \land b \lor c = b \lor a \land c\)
(9) \((a \lor b) \land c = (b \lor a) \land c\)
(10) \(a \land b = 0 \iff b \land a = 0\)
(11) \(a \lor (b \lor a) = a \lor b\).

It can be observed that an ADL \(L\) satisfies almost all the properties of a distributive lattice except, possible the right distributivity of \(\lor\) over \(\land\), the commutativity of \(\lor\), the commutativity of \(\land\) and the absorption law \((a \land b) \lor a = a\). Any one of these properties convert \(L\) into a distributive lattice.
THEOREM 2.2. ([2]) Let \((L, \lor, \land, 0)\) be an ADL with 0. Then the following are equivalent:

1. \((L, \lor, \land, 0)\) is a distributive lattice
2. \(a \lor b = b \lor a\), for all \(a, b \in L\)
3. \(a \land b = b \land a\), for all \(a, b \in L\)
4. \((a \land b) \lor c = (a \lor c) \land (b \lor c)\), for all \(a, b, c \in L\).

PROPOSITION 2.1. ([2]) Let \((L, \lor, \land)\) be an ADL. Then for any \(a, b, c \in L\) with \(a \leq b\), we have

1. \(a \land c \leq b \land c\)
2. \(c \land a \leq c \land b\)
3. \(c \lor a \leq c \lor b\).

DEFINITION 2.2. ([2]) An element \(m \in L\) is called maximal if it is maximal as in the partially ordered set \((L, \leq)\). That is, for any \(a \in L\), \(m \leq a\) implies \(m = a\).

THEOREM 2.3. ([2]) Let \(L\) be an ADL and \(m \in L\). Then the following are equivalent:

1. \(m\) is maximal with respect to \(\leq\)
2. \(m \lor a = m\), for all \(a \in L\)
3. \(m \land a = a\), for all \(a \in L\).

LEMMA 2.1. ([2]) Let \(L\) be an ADL with a maximal element \(m\) and \(x, y \in L\). If \(x \land y = y\) and \(y \land x = x\) then \(x\) is maximal if and only if \(y\) is maximal. Also the following conditions are equivalent:

1. \(x \land y = y\) and \(y \land x = x\)
2. \(x \land m = y \land m\).

DEFINITION 2.3. ([2]) If \((L, \lor, \land, 0, m)\) is an ADL with 0 and with a maximal element \(m\), then the set \(I(L)\) of all ideals of \(L\) is a complete lattice under set inclusion. In this lattice, for any \(I, J \in I(L)\), the l.u.b. and g.l.b. of \(I, J\) are given by \(I \lor J = \{(x \lor y) \land m \mid x \in I, y \in J\}\) and \(I \land J = I \cap J\). The set \(PI(L) = \{a \mid a \in L\}\) of all principal ideals of \(L\) forms a sublattice of \(I(L)\). (Since \(\{a\} \lor \{b\} = \{a \lor b\}\) and \(\{a\} \cap \{b\} = \{a \land b\}\))

In the following, we give the concepts of residuation and multiplication in an almost distributive lattice (ADL) \(L\) and the definition of a residuated almost distributive lattice taken from our earlier paper [3].

DEFINITION 2.4. ([3]) Let \(L\) be an ADL with a maximal element \(m\). A binary operation \(\cdot\) on an ADL \(L\) is called a residuation over \(L\) if, for \(a, b, c \in L\) the following conditions are satisfied:

1. \(a \land b = b\) if and only if \(a : b\) is maximal
2. \(a \land b = b \implies (i) (a : c) \land (b : c) = b : c\) and (ii) \((c : b) \land (c : a) = c : a\)
3. \([a : b : c] \land m = [a : c] : [b : c] \land m\)
4. \([a \land b : c] \land m = [a : c] \land (b : c) \land m\)
operations ' : ' and ' . ' on a residuated almost distributive lattice (residuated ADL), if there exists two binary
operations on L. These are taken from our earlier paper
satisfying conditions R1 to R5, M1 to M4 and the following condition (A).
\[
(A) \quad (x : a) \land b = b \text{ if and only if } x \land (a : b) = a b, \text{ for any } x, a, b \in L.
\]

We use the following properties frequently later in the results.

**Lemma 2.2.** ([3]) Let L be an ADL with a maximal element m and a binary operation on L satisfying the conditions M1 – M4. Then for any a, b, c, d ∈ L,

(i) a \land (a b) = a b and b \land (a b) = a b

(ii) a \land b = b \implies (c a) \land (c b) = c b and (a c) \land (b c) = b c

(iii) d \land [(a b) c] = (a b) c if and only if d \land [a (b c)] = a (b c)

(iv) (a c) \land (b c) \land [(a \land b) c] = (a \land b) c

(v) d \land (a c) \land (b c) = (a c) \land (b c) \implies d \land [(a \land b) c] = (a \land b) c

(vi) d \land [(a c) \lor (b c)] = (a c) \lor (b c) \iff d \land [(a \lor b) c] = (a \lor b) c

The following result is a direct consequence of M1 of definition 2.15.

**Lemma 2.3.** ([3]) Let L be an ADL with a maximal element m and a binary operation on L satisfying the condition M1. For a, b, x ∈ L, a \land (x b) = x b if and only if a \land (b x) = b x.

In the following, we give some important properties of residuation ' : ' and multiplication ' . ' in a residuated ADL L. These are taken from our earlier paper [4].

**Lemma 2.4.** ([4]) Let L be a residuated ADL with a maximal element m. For a, b, c, d ∈ L, the following hold in L.

(1) (a : b) \land a = a

(2) [a : (a b)] \land (a \lor b) = a \lor b

(3) [(a b) : c] \land [a : (b c)] = a : (b c)

(4) [a : (b c)] \land [(a b) : c] = (a : b) : c

(5) [(a \land b) : b] \land (a : b) = a : b

(6) (a : b) \land [(a \land b) : b] = (a \land b) : b
(7) \[a : (a \lor b)] \land m = (a : b) \land m

(8) \[c : (a \land b)] \land [(c : a) \lor (c : b)] = (c : a) \lor (c : b)

(9) If \(a : b = a\) then \(a \land (b : d) = b : d \iff a \land d = d\)

(10) \{(a : (a : b))\} \land (a : b) = a : b

(11) \[(a : b) : c\] \land \[(a : c) \lor (b : c)\] = \((a : c) \lor (b : c)\)

(12) \(a \land m \geq b \land m \implies (a : c) \land m \geq (b : c) \land m\)

(13) \((a : b) \land \{a : (a : b)\} = a : \{a : (a : b)\}\)

(14) \(a \land b = b \implies (a : c) \land (b : c) = b : c\)

(15) \(a \land b \land (a : b) = a : b\)

(16) \[[a : b] : a\land b = b\]

(17) \((a : b) \land ((a \land b) \lor (a \lor b)) = (a \land b) \lor (a \lor b)\)

(18) \(a \lor b\) is maximal \(\iff (a : b) \land a \land b = a \land b\)

3. Noether Almost Distributive Lattices

In this section, we introduce the concept of Noether Almost Distributive Lattice (Noether ADL). We prove important results in a Noether ADL and also prove the fundamental Theorem of Noether ADL.

We first give the following concepts on a residuated ADL L:

**Definition 3.1.** An element \(c\) of L is called irreducible, if \(f \land g = c,\) for \(f, g \in L \implies f = c\) or \(g = c\).

**Definition 3.2.** An element \(p\) of L is called prime, if \(p \land (a : b) = a : b \implies p \land a = a\) or \(p \land b = b\), for any \(a, b \in L\)

**Definition 3.3.** An element \(p\) of L is called primary, if \(p \land (a : b) = a : b\) and \(p \land a \neq a\) \(\implies p \land b^s = b^s\), for some \(s \in Z^+\)

**Definition 3.4.** An ADL L is said to satisfy the ascending chain condition (a.c.c.), if for every increasing sequence \(x_1 \leq x_2 \leq x_3 \leq \ldots \ldots\), in L, there exists a positive integer n such that \(x_n = x_{n+1} = x_{n+2} = \ldots \ldots\)

**Definition 3.5.** A residuated ADL L is said to be a Noether ADL, if

(N1) the ascending chain condition (a.c.c.) holds in L and

(N2) every irreducible element of L is primary.

**Definition 3.6.** An element \(a\) of a residuated ADL L is said to have a primary decomposition, if there exists primary elements \(p_1, p_2, \ldots, p_m\) in L such that \(a = p_1 \land p_2 \land \ldots \ldots \land p_m\).

**Theorem 3.1.** If \(a \land b\) are any two elements of a Noether ADL L and \(a : b\) has a primary decomposition, then there exists an exponent \(s\) such that

\((a : b) \land a \land b^s = a \land b^s\)
Proof. Let \( L \) be a Noether ADL and \( a, b \in L \). Suppose \( a, b \) has a primary decomposition, say \( a, b = p_1 \wedge p_2 \wedge \ldots \wedge p_k \) Then for \( 1 \leq i \leq k \), \( p_i \wedge (a, b) \neq a, b \). So that, either \( p_i \wedge a = a \) or \( p_i \wedge a \neq a \), \( p_i \wedge b^* = b^* \), for some \( s_i \in \mathbb{Z}^+ \).

Case (i): Suppose \( p_k \wedge a = a \), then we rearrange the primary elements \( p_1, p_2, \ldots, p_{k-1} \) such that \( p_i \wedge b^* = b^* \), for \( 1 \leq i \leq l \) and \( p_i \wedge a = a \), for \( l + 1 \leq i \leq k \).

Now, take \( s = \text{Max}\{s_1, s_2, \ldots, s_l\} \). Then \( p_i \wedge b^* = b^* \), for \( 1 \leq i \leq l \). Now, \[
(a, b) \wedge a \wedge b^* = p_1 \wedge p_2 \wedge \ldots \wedge p_k \wedge a \wedge b^* = p_{l+1} \wedge p_{l+2} \wedge \ldots \wedge p_k \wedge a \wedge p_1 \wedge p_2 \wedge \ldots \wedge p_l \wedge b^* = a \wedge b^*.
\]

Case (ii): Suppose \( p_k \wedge a \neq a \) and \( p_k \wedge b^* = b^* \), then we rearrange the primary elements \( p_1, p_2, \ldots, p_{k-1} \) such that \( p_i \wedge a = a \), for \( 1 \leq i \leq j \) and \( p_i \wedge b^* = b^* \), for \( j + 1 \leq i \leq k - 1 \). Now, take \( s = \text{Max}\{s_{j+1}, s_{j+2}, \ldots, s_k\} \). Then \( p_i \wedge b^* = b^* \), for \( j + 1 \leq i \leq k \).

\[
(a, b) \wedge a \wedge b^* = p_1 \wedge p_2 \wedge \ldots \wedge p_{j-1} \wedge a \wedge p_{j+1} \wedge p_{j+2} \wedge \ldots \wedge p_k \wedge b^* = a \wedge b^*.
\]

Theorem 3.2. Let \( L \) be a Noether ADL with a maximal element \( m \) and \( a, b, c \in L \). If \( b, b = b \) then

(i) \( a \wedge b \wedge m = (a, b) \wedge m \) and

(ii) \( [(a \wedge c), b] \wedge m = (a, b) \wedge (c, b) \wedge m \).

Proof. Let \( a, b, c \in L \) and suppose \( b, b = b \)

(i) By property (15) of Lemma 2.4, we have \( a \wedge b \wedge (a, b) = a, b \) and by Theorem 3.1, we have \( (a, b) \wedge a \wedge b^* = a \wedge b^* \). So that, \( (a, b) \wedge a \wedge b = a \wedge b \) (Since \( b, b = b \))

Hence \( a \wedge b \wedge m = (a, b) \wedge m \)

(ii) By (i) above, we have

\[
[(a \wedge c), b] \wedge m = a \wedge c \wedge b \wedge m
\]

\[
= a \wedge b \wedge m \wedge c \wedge b \wedge m
\]

\[
= (a, b) \wedge m \wedge (c, b) \wedge m \text{ (By (i), above )}
\]

\[
= (a, b) \wedge (c, b) \wedge m.
\]

Corollary 3.1. Let \( L \) be a Noether ADL with a maximal element \( m \). If \( a \) and \( b \) are any two idempotent elements of \( L \), then \( (a, b) \wedge m = a \wedge b \wedge m \).

Proof. Let \( a, b \in L \) be idempotent elements of \( L \). Then \( a^2 = a, b^2 = b \). By property (15) of Lemma 2.4, we have \( a \wedge b \wedge (a, b) = a, b \). By Theorem 3.1, we have

\( (a, b) \wedge a \wedge b^* = a \wedge b^* \). So that, \( (a, b) \wedge a \wedge b = a \wedge b \). Hence \( (a, b) \wedge m = a \wedge b \wedge m \).

The following Theorem is converse of Theorem 3.1. under special conditions.

Theorem 3.3. Let \( L \) be a residuated ADL with ascending chain condition(a.c.c.) such that for any \( a, b \in L \), there exists \( s \in \mathbb{Z}^+ \) such that \( (a, b) \wedge a \wedge b^* = a \wedge b^* \) then \( L \) is a Noether ADL.

Proof. Let \( p \) be an irreducible element of \( L \). Let \( a, b \in L \) such that \( p \wedge (a, b) = a, b \) and \( p \wedge a \neq a \). Choose \( s \in \mathbb{Z}^+ \) such that \( (a, b) \wedge a \wedge b^* = a \wedge b^* \). Now,

\[
p \wedge (a, b) = a, b \implies p \wedge (a, b) \wedge a \wedge b^* = (a, b) \wedge a \wedge b^*
\]

\[
\implies p \wedge a \wedge b^* = a \wedge b^*.
\]
\\[\Rightarrow p \lor (a \land b^\ast) = p\]
\\[\Rightarrow (p \lor a) \land (p \lor b^\ast) = p\]
\\[\Rightarrow \text{either } p \lor a = p \text{ or } p \lor b^\ast = p \quad (\text{Since } p \text{ is irreducible})\]

But \(p \lor a \neq p\) since \(p \land a \neq a\). Therefore \(p \lor b^\ast = p\). Thus \(p \land b^\ast = b^\ast\), for some \(s \in Z^+\). Therefore \(p\) is primary. Hence every irreducible element of L is primary. Thus L is a Noether ADL.

\[\text{□}\]

**Definition 3.7.** Let L be a residuated ADL. An element \(a\) of L is called principal, if \(a \land b = b\), for some \(b \in L\), then \(a.c = b\), for some \(c \in L\).

**Lemma 3.1.** Let L be a residuated ADL with a maximal element \(m\). If \(a, b \in L\) such that \(a\) is principal and \(a \land b = b\), then \([(b : a).a] \land m = b \land m\).

**Proof.** Let \(a, b \in L\) such that \(a\) is principal and \(a \land b = b\). Then there exists an element \(c \in L\) such that \(a.c = b\). Now,

\[b \land (a.c) = a.c \Rightarrow (b : a) \land c = c \quad (\text{By definition } 2.6)\]
\[\Rightarrow [(b : a).a] \land (c.a) = c.a \quad (\text{By Lemma } 2.2 \; (ii))\]
\[\Rightarrow [(b : a).a] \land (a.c) = a.c \quad (\text{By Lemma } 2.3)\]
\[\Rightarrow [(b : a).a] \land b = b \quad (\text{Since } a.c = b)\]

Now,

\[(b : a) \land (b : a) = b : a \Rightarrow b \land [a.(b : a)] = a.(b : a) \quad (\text{By definition } 2.6)\]
\[\Rightarrow b \land [(b : a).a] = (b : a).a \quad (\text{By Lemma } 2.3)\]

Hence \([(b : a).a] \land m = b \land m\).

\[\text{□}\]

We shall now prove the following fundamental Theorem:

**Theorem 3.4.** Let L be an ADL with a maximal element \(m\) satisfying the following conditions:

1. L is residuated.
2. L satisfies a.c.c.
3. Every element of L is principal.

Then L is a Noether ADL.

**Proof.** Suppose the conditions (1), (2) and (3) hold in L. Let \(p\) be a non-primary element of L. Then there exists \(a, b \in L\) such that \(p \land (a.b) = a.b, p \land a \neq a\) and \(p \land b^\ast \neq b^\ast\), for any \(s \in Z^+\). Let \(k \in Z^+\). Then by Lemma 1.23 (i), we get \(b^{k-1} \lor b^k = b^k\). So that, \((p : b^k) \land (p : b^{k-1}) = p : b^{k-1}\). Hence \((p : b^k) \land m \geq (p : b^{k-1}) \land m\). Since L satisfies a.c.c., the chain \((p : b) \land m \leq (p : b^2) \land m \leq (p : b^3) \land m, \ldots\) terminates. Then there exists \(k \in Z^+\) such that \((p : b^k) \land m = (p : b^{k+1}) \land m = \ldots\) \(\Rightarrow\) Write \(c = (p \lor a) \land (p \lor b^k)\). Then \(p \lor b^k \geq c \geq p\). Now,

\[c = c \land (p \lor b^k) = p \lor (c \land b^k) \quad (\text{By definition } 2.6)\]

First, we prove \(p \land a \land b^k = c \land b^k\). Since \(b^k\) is principal and \(b^k \land c \land b^k = c \land b^k\) we get \(c \land b^k \land m = ([c \land b^k] : b^k) \land m \quad (\text{By Lemma } 3.1)\). Since \((p \lor a) \land c = c\), we get \([b \land (p \lor a)] \land (b.c) = b.c \quad (\text{By Lemma } 2.2 \; (ii))\) Now,

\[p \land (b.c) = p \land [b \land (p \lor a)] \land (b.c) = p \land [(b.p) \lor (b.a)] \land (b.c)\]
\[ p \land (b.p) \lor [p \land (b.a) \land (b.c) = [(b.p) \lor (b.a)] \land (b.c) \]
\[ = [b.(p \lor a)] \land (b.c) = b.c \quad \text{(Since} \quad (p \lor a) \land c = c) \]
\[ \Rightarrow p \land (b.c) \land [b.(c \land b^k)] = (b.c) \land [b.(c \land b^k)] \]
\[ \Rightarrow p \land [b.(c \land b^k)] = b.(c \land b^k) \quad \text{(Since} \quad c \land c \land b^k = c \land b^k) \]
\[ \Rightarrow (p : b) \land c \land b^k = c \land b^k \quad \text{(By definition 2.6)} \]
\[ \Rightarrow (p : b) \land c \land b^k \land m = c \land b^k \land m \]
\[ \Rightarrow (p : b) \land [(c \land b^k) \land b^k] \land m = [(c \land b^k) \land b^k] \land m \quad \text{(By (iii))} \]
\[ \Rightarrow (p : b) \land [(c \land b^k) \land b^k] = [(c \land b^k) \land b^k] \land b^k \]
\[ \Rightarrow (p : b^k+1) \land [(c \land b^k) \land b^k] = (c \land b^k) \land b^k \quad \text{(By definition 2.6)} \]
\[ \Rightarrow (p : b^k) \land [(c \land b^k) \land b^k] = (c \land b^k) \land b^k \quad \text{(By (i))} \]
\[ \Rightarrow p \land [(c \land b^k) \land b^k] = b^k,[(c \land b^k) \land b^k] \quad \text{(By definition 2.6)} \]
\[ \Rightarrow p \land [(c \land b^k) \land b^k] = [(c \land b^k) \land b^k] \land b^k \quad \text{(By Lemma 2.3)} \]
\[ \Rightarrow \quad \text{(By (iii))} \]
\[ \Rightarrow \quad \text{(By (ii))} \]
\[ \Rightarrow \quad \text{(By (iii))} \]
\[ \Rightarrow \quad \text{(By (ii))} \]
\[ \Rightarrow \quad \text{(By (iii))} \]
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\[ \Rightarrow \quad \text{(By (ii))} \]
\[ \Rightarrow \quad \text{(By (iii))} \]
\[ \Rightarrow \quad \text{(By (ii))} \]
\[ \Rightarrow \quad \text{(By (iii))} \]
\[ \Rightarrow \quad \text{(By (ii))} \]
Now, \( c = p \lor (c \land b^k) = p \). Then \( p = (p \lor a) \land (p \lor b^k) \). But \( p \lor a \neq p \) and \( p \lor b^k \neq p \). Hence \( p \) is reducible. Thus \( L \) is a Noether ADL. \( \square \)

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G. C. Rao: Department of Mathematics, Andhra University, Visakhapatnam - 530003, A.P., India
E-mail address: gcraomaths@yahoo.co.in

S. S. Raju: Department of Mathematics, Andhra University, Visakhapatnam - 530003, A.P., India
E-mail address: ssrajumaths@gmail.com