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# SECURE DOMINATION IN MIDDLE GRAPHS

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**Abstract.** A set $S \subseteq V(G)$ is a dominating set of G if every $u \in V \setminus S$, there exists a $v \in S$ such that $uv \in E(G)$. The domination number of G, denoted by $\gamma(G)$ is the minimum cardinality of a dominating set of G. A dominating set $S \subseteq V(G)$ of a graph $G = (V, E)$ is a secure dominating set if for each $u \in V \setminus S$ there exists a $v \in S \cap N(u)$ such that $(S \setminus \{v\}) \cup \{u\}$ is a dominating set. The minimum cardinality of a secure dominating set is called secure domination number and is denoted by $\gamma_s(G)$ (or shortly $\gamma_s$). In this paper we evaluate the exact values of $\gamma_s$ for middle graphs of certain graph families and further we determine a sharp lower bound for middle graph of trees.

**Keywords:** Secure domination, secure domination number, middle graphs.

**2010 Mathematics Subject Classification:** 05C69.

## 1. Introduction

Domination in graph theory has many applications in both mathematical and real-world problems, in particular, in monitoring communication or electrical networks, facility location problems, in defense to safe guard an area or a region etc. In view of many varied applications in the field of communication networks, algorithm designs, computational complexity etc., the study of several domination parameter is the fastest growing area in graph theory. For further study on domination one can refer to [12].

Various papers have considered the problems associated with defending the vertices of a graph. In secure domination problem at most one guard per vertex is placed such that each unguarded vertex is adjacent to a guarded vertex with a guard. When an unguarded vertex is attacked, a guard moves along an edge from a vertex with a guard to the attacked vertex. After the move, each unguarded vertex must be adjacent to a guarded vertex. Hence this defending model consists of placing a minimum number of guards on the vertices of a graph G in order to defend it against a single attack, such that the resulting placement of guards before and after an attack induces a dominating set. The concept of secure domination was introduced by Cockayne et al. [9] and explored in the papers [5, 6, 7,8, 13]. For further study on various defending models one can refer to [14, 15, 16].

All graphs considered in this paper are simple, finite, connected and undirected graphs G = (V, E) with the vertex set V(G) and the edge set E(G) of orders n and m respectively. A set $S \subseteq V(G)$ is a dominating set of G if every $u \in V \setminus S$, there exists a $v \in S$ such that $uv \in E(G)$. The domination number of G, denoted by $\gamma(G)$ (or shortly $\gamma$) is the minimum cardinality of a dominating set of G. A dominating set S of G with $|S| = \gamma(G)$ is called a $\gamma$-set of G (or simply $\gamma(G)$-set). A secure dominating set (SDS) $S \subseteq V(G)$ is a dominating set with the property that for each $u \in V \setminus S$, there exists a $v \in S \cap N(u)$ such that $(S \setminus \{v\}) \cup \{u\}$ is dominating. The minimum cardinality of a secure dominating set is called secure domination number and it is denoted by $\gamma_s(G)$ (or shortly $\gamma_s$) and the set is called $\gamma_s$-set of G (or simply $\gamma_s(G)$-set). In this case we say that v-S defends u or v is an S-defender.

The middle graph M(G) of a graph G is the graph obtained by subdividing each edge of G exactly once and joining all these newly introduced vertices of adjacent edges of G. The basic definition of M(G) is as follows.

The vertex set of M(G) is $V(G) \cup E(G)$. The two vertices x and y in the vertex set of M(G) are adjacent in M(G) if either (i) x, y are in E(G) and x, y are adjacent in G or (ii) x is in V(G), y is in E(G) and x, y are incident in G.

In 1976, it was Hamada and Yoshimura [10] defined the middle graph M(G) of a graph G. In [10], they give some other properties of middle graphs. Further characterization of the middle graph of a graph is given by Akiyama et al. [1]. For further study on middle graphs one can refer to [2,3,4] and elsewhere.

In this paper we evaluate secure domination number for middle graphs of certain graph families such as paths, cycles, wheels and complete bipartite graphs. Further we obtain a sharp lower bound of $\gamma_s$ for middle graph of trees.

The following are the basic definitions and few preliminary results required for our study.

For notation and graph theory terminology in general, we follow [11]. We denote the *degree of* v in G by *deg*(v), if the graph G is clear from context. A vertex of degree 0 is called an *isolated* vertex. A *leaf* u of G is a vertex of degree one and the *support* vertex of the leaf u is the unique vertex v such that $uv \in E$. A vertex of degree greater than one, which is not a support is a non leaf vertex. For a set $S \subseteq V$, the subgraph induced by S is denoted by G[S]. $P_n$ is a path on n vertices and $C_n$ is a cycle on n vertices. A wheel graph $W_n$ on n+1vertices is defined to be the graph $K_1 + C_n$. A complete bipartite graph is a graph whose vertices can be divided into two disjoint sets U and V such that every vertex of U is adjacent to every vertex of V and is denoted by $K_{p,q}$, where $|U| = p$, $|V| = q$. A *star graph* $K_{1,n-1}$ has one vertex of degree n–1 and n–1 vertices of degree one. A graph G is a *complete graph* if every pair of its vertices are adjacent and is denoted by $K_n$. A clique of a graph is a maximal complete subgraph.

For $v \in S \subseteq V(G)$, $u \in V \setminus S$ is an S-external private neighbor of v, if $N(u) \cap S = \{v\}$. Let $P_n(v, S)$ be the set of all S-external private neighbors of v.

**Proposition 1.1.** [9] Let S be a dominating set. A vertex v S-defends u if and only if $G[P_n(v, S) \cup \{u, v\}]$ is complete.

**Corollary 1.1.** [9] S is a secure dominating set if and only if for each $u \in V \setminus S$ there exists $v \in S$ such that $G[P_n(v, S) \cup \{u, v\}]$ is complete.

## 2. Middle Graphs of Paths and Cycles

In this section we determine $\gamma_s$ values for middle graphs of paths and cycles.

**Lemma 2.1.** For the graph H given in Figure 1, $\gamma_s(H) = 3$.

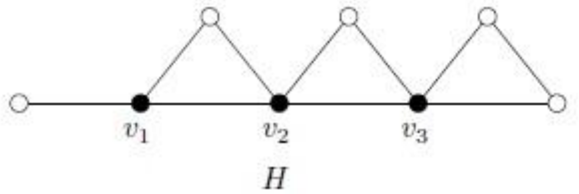


**Figure 1**. Shaded vertices indicate a $\gamma_s(H)$-set.

*Proof.* Clearly $\gamma(H) = 2$ and it has a unique $\gamma$-set $D = \{v_1, v_3\}$. Hence $\gamma_s(H) \geq 2$. Suppose $\gamma_s(H) = 2$. Neither of the vertices $v_1$ and $v_3$ satisfy the hypothesis of Proposition 1.1. Hence D is not a secure dominating set. Therefore $\gamma_s(H) \geq 3$. Clearly $\{v_1, v_2, v_3\}$ is a secure dominating set. Hence $\gamma_s(H) = 3$. □

**Theorem 2.1.** For paths $P_n$, $n \geq 3$

$$\gamma_s(M(P_n)) = \begin{cases} \left\lceil \frac{3(2n-1)}{8} \right\rceil + 1 & \text{if } n+1 \equiv 0 \pmod 4 \\ \left\lceil \frac{3(2n-1)}{8} \right\rceil & \text{otherwise} \end{cases}$$

*Proof.* Let $G = M(P_n)$ and $V(P_n) = \{v_1, v_2, ..., v_n\}$ and $E(P_n) = \{e_1, e_2, ..., e_{n-1}\}$, where $e_i = v_i v_{i+1}$, $1 \le i \le n-1$. By definition of $M(G)$, $V(G) = V(P_n) \cup E(P_n) = \{v_i : 1 \le i \le n\} \cup \{e_i : 1 \le i \le n-1\}$, in which each $e_i$ is adjacent to $v_i$ and $v_{i+1}$, $1 \le i \le n-1$ and also adjacent to $e_{i+1}$, $1 \le i \le n-2$.

We now partition the vertex set $V(G)$ into sets $V_1, V_2, ..., V_k$ such that the subgraphs induced by each $V_i$, $1 \le i \le k-1$ is isomorphic to the graph H as given in Figure 1 and let $G[V_k] = H'$, where $H'$ is isomorphic to one of the graphs $G_1$ or $G_2$ as given in Figure 2 or $K_1$ or $P_3$.

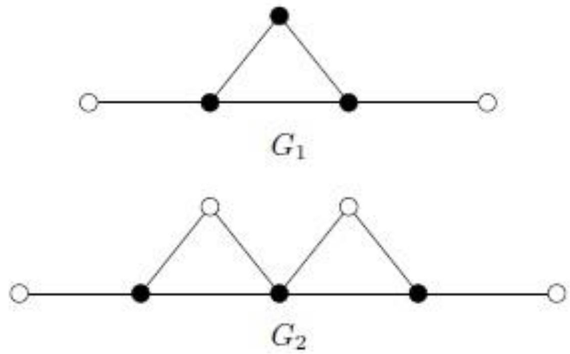


**Figure 2**. Shaded vertices indicate $\gamma_s$-sets for the respective graphs.

Hence by Lemma 2.1, we obtain $\gamma_s(G) \ge 3(k-1) + \gamma_s(H')$. Now we have the following cases.

**Case (i):** $H' \cong G_1$. In this case, $n+1 \equiv 0 \pmod 4$. Let u be the vertex in $G_1$ of degree two. Now $\gamma(G_1) = 2$ and for the unique $\gamma$-set D of $G_1$ neither of the vertices in D can defend u. Therefore $\gamma_s(G_1) \ge 3$ and the set of all non leaf vertices in $G_1$ is a $\gamma_s(G_1)$-set. Hence $\gamma_s(G_1) = 3$. Therefore $\gamma_s(G) \ge 3(k-1) + 3$. As there are $2n-1$ vertices in G and for every eight vertices in H, at least three vertices belong to $\gamma_s(G)$-set, we have $2n-1 = 8(k-1) + |V_k|$. Let $|V_k| = x$. We now obtain, $\frac{3(2n-1)}{8} = 3(k-1) + \frac{3x}{8}$ and since $|V_k| = 5$, we have $\left\lceil \frac{3(2n-1)}{8} \right\rceil = 3(k-1) + \left\lceil \frac{15}{8} \right\rceil$ which implies that $\left\lceil \frac{3(2n-1)}{8} \right\rceil + 1 = 3(k-1) + 3$. Therefore we get $\gamma_s(G) \ge \left\lceil \frac{3(2n-1)}{8} \right\rceil + 1$.

Now, $S = \{e_{4i-3}, e_{4j-2}, e_{4k-1} : 1 \le i, j, k \le \frac{n-3}{4}\} \cup \{e_{n-2}, e_{n-1}, v_n\}$, is an SDS of cardinality $\left\lceil \frac{3(2n-1)}{8} \right\rceil + 1$. Hence $\gamma_s(G) = \left\lceil \frac{3(2n-1)}{8} \right\rceil + 1$.

**Case (ii)**: $H' \cong G_2$. In this case, $n+1 \equiv 1 \pmod 4$. Now $\gamma(G_2) = 2$ and the support vertices form a unique $\gamma$-set of $G_2$, say D. Clearly neither of the vertices in D can defend the vertices of degree two in $G_2$. Hence $\gamma_s(G_2) \geq 3$ and $D \cup \{z\}$, where z is the vertex in $G_2$ of degree four, is a $\gamma_s$-set of $G_2$. Hence $\gamma_s(G_2) = 3$ and therefore $\gamma_s(G) \geq 3(k-1) + 3$. As discussed in Case (i), we obtain $\frac{3(2n-1)}{8} = 3(k-1) + \frac{21}{8}$, as $|V_k| = 7$. Hence $\left\lceil \frac{3(2n-1)}{8} \right\rceil = 3(k-1) + 3$. Therefore $\gamma_s(G) \geq \left\lceil \frac{3(2n-1)}{8} \right\rceil$. Now $S = \{e_{4i-3}, e_{4j-2}, e_{4k-1} : 1 \leq i, j, k \leq \frac{n}{4}\}$ is an SDS of cardinality $\left\lceil \frac{3(2n-1)}{8} \right\rceil$. Hence $\gamma_s(G) = \left\lceil \frac{3(2n-1)}{8} \right\rceil$.

**Case (iii):** $H' \cong K_1$. In this case, $n+1 \equiv 2 \pmod 4$. It is clear that $\gamma_s(K_1) = 1$. Hence $\gamma_s(G) \geq 3(k-1) + 1$. As discussed in case (i), we have $\frac{3(2n-1)}{8} = 3(k-1) + \frac{3}{8}$, as $|V_k| = 1$. Further we have $\left\lceil \frac{3(2n-1)}{8} \right\rceil = 3(k-1) + 1$. Therefore $\gamma_s(G) \geq \left\lceil \frac{3(2n-1)}{8} \right\rceil$. Now $S = \{e_{4i-3}, e_{4j-2}, e_{4k-1} : 1 \leq i, j, k \leq \left\lfloor \frac{n}{4} \right\rfloor \cup \{e_{n-1}\}$ is an SDS of cardinality $\left\lceil \frac{3(2n-1)}{8} \right\rceil$. Hence $\gamma_s(G) = \left\lceil \frac{3(2n-1)}{8} \right\rceil$.

**Case (iv):** $H' \cong P_3$. In this case, $n+1 \equiv 3 \pmod 4$. It is clear that $\gamma_s(P_3) = 2$. Hence $\gamma_s(G) \geq 3(k-1) + 2$. As discussed in case (i), we have $\frac{3(2n-1)}{8} = 3(k-1) + \left\lceil \frac{9}{8} \right\rceil$, as $|V_k| = 3$. Further we have $\left\lceil \frac{3(2n-1)}{8} \right\rceil = 3(k-1) + 2$. Therefore $\gamma_s(G) \geq 3(k-1) + 2$. Now $S = \{e_{4i-3}, e_{4j-2}, e_{4k-1} : i, j, k = 1, 2, ..., \left\lfloor \frac{n}{4} \right\rfloor\} \cup \{e_{n-1}, e_{n-2}\}$, is an SDS of cardinality $\left\lceil \frac{3(2n-1)}{8} \right\rceil$. Hence $\gamma_s(G) = \left\lceil \frac{3(2n-1)}{8} \right\rceil$. Hence the proof. □

**Theorem 2.2.** For cycles $C_n$, $n \geq 3$,

$$\gamma_s(M(C_n)) = \left\lceil \frac{3n}{4} \right\rceil$$

*Proof.* Let $G = M(C_n)$. For the cycle $C_n$, $n \geq 3$, let $V(C_n) = \{v_1, v_2, ..., v_n\}$ and $E(C_n) = \{e_1, e_2, ..., e_n\}$ where $e_i = v_i v_{i+1}$, $1 \leq i \leq n-1$ and $e_n = v_n v_1$. By definition, we have $V(G) = V(C_n) \cup E(C_n)$.

We now partition $V(G)$ into sets $V_1, V_2, ..., V_k$ such that the subgraphs induced by $V_i$, $1 \leq i \leq k-1$ is isomorphic to the graph H as given in Figure 1 and let $G[V_k] \cong H'$, where $H'$ is isomorphic to one of the graphs $P_2$ or $P_4$ or $P_6$. By Lemma 2.1, we obtain $\gamma_s(G) \geq 3(k-1) + \gamma_s(H')$. We now discuss the following cases.

**Case (i):** $H' \cong P_2$ . It is clear that $\gamma_s(H') = 1$. Therefore $\gamma_s(G) \geq 3(k-1) + 1$. As there are 2n vertices in G, we get $2n = 8(k-1) + |V_k|$. This further implies that $\left\lceil \frac{3n}{4} \right\rceil = 3(k-1) + \left\lceil \frac{3}{4} \right\rceil$, as $|V_k| = 2$. Hence $\left\lceil \frac{3n}{4} \right\rceil = 3(k-1) + 1$. Therefore $\gamma_s(G) \geq \left\lceil \frac{3n}{4} \right\rceil$. Now $S = \{e_{4i-3}, e_{4j-2}, e_{4j-1} : 1 \leq i, j, k \leq \left\lfloor \frac{n}{4} \right\rfloor \cup \{e_n\}$ is an SDS of cardinality $\left\lceil \frac{3n}{4} \right\rceil$. Hence $\gamma_s(G) = \left\lceil \frac{3n}{4} \right\rceil$.

**Case (ii):** $H' \cong P_4$ . It is clear that $\gamma_s(H') = 2$. Therefore $\gamma_s(G) \geq 3(k-1) + 2$. As discussed in case (i), we have $2n = 8(k-1) + 4$, as $|V_k| = 4$. This further implies that $\left\lceil \frac{3n}{4} \right\rceil = 3(k-1) + 2$. Hence $\gamma_s(G) \geq \left\lceil \frac{3n}{4} \right\rceil$. Now $S = \{e_{4i-3}, e_{4j-2}, e_{4k-1} : 1 \leq i, j, k \leq \left\lfloor \frac{n}{4} \right\rfloor \cup \{e_{n-1}, e_n\}$ is an SDS of cardinality $\left\lceil \frac{3n}{4} \right\rceil$. Hence $\gamma_s(G) = \left\lceil \frac{3n}{4} \right\rceil$.

**Case (iii):** $H' \cong P_6$ . It is clear that $\gamma_s(H') = 3$. Therefore $\gamma_s(G) \geq 3(k-1) + 3$. Since $|V_k| = 6$ in this case, $2n = 8(k-1) + 6$. This further implies that $\left\lceil \frac{3n}{4} \right\rceil = 3(k-1) + \left\lceil \frac{9}{4} \right\rceil$. Further we have $\left\lceil \frac{3n}{4} \right\rceil = 3(k-1) + 3$. Hence $\gamma_s(G) \geq \left\lceil \frac{3n}{4} \right\rceil$. Now $S = \{e_{4i-3}, e_{4j-2}, e_{4k-1} : 1 \leq i, j, k \leq \left\lceil \frac{3n}{4} \right\rceil\}$ is an SDS of cardinality $\left\lceil \frac{3n}{4} \right\rceil$. Hence $\gamma_s(G) = \left\lceil \frac{3n}{4} \right\rceil$. Hence the proof. □

### 3. Middle Graphs of Wheels

In this section we evaluate $\gamma_s(M(W_n))$.

**Lemma 3.1.** For the graph $H_1$, given in the Figure 3, $\gamma_s(H_1) = 3$.

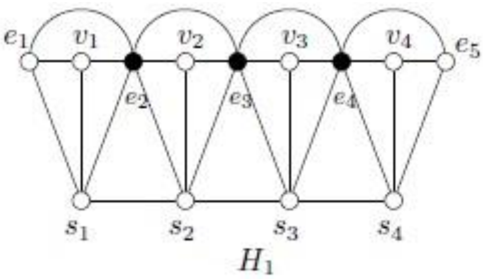


**Figure 3**. Shaded vertices indicate a $\gamma_s(H_1)$-set.

*Proof.* Let $V(H_1) = \{e_i : 1 \leq i \leq 5\} \cup \{v_j : 1 \leq j \leq 4\} \cup \{s_k : 1 \leq k \leq 4\}$. Clearly $D = \{e_2, e_4\}$ is the unique $\gamma(H_1)$-set. Neither of the vertices in D can defend $e_3$. Hence $\gamma_s(H_1) \geq 3$. Let $S = \{e_2, e_3, e_4\}$. We see that $e_2$ S-defends $v_1$, $e_1$, $s_1$, $e_4$ S-defends $v_4$, $e_5$, $s_4$ and $e_3$ S-defends $v_2$, $v_3$, $s_2$, $s_3$. Hence S is an SDS of cardinality 3. Hence $\gamma_s(H_1) = 3$ (Refer Figure 3). □

As the proof of the following Lemma 3.2 is similar to the proof of Lemma 3.1, we omit the proof.

**Lemma 3.2.** For the graph $H_2$ given in the Figure 4, $\gamma_s(H_2) = 3$.

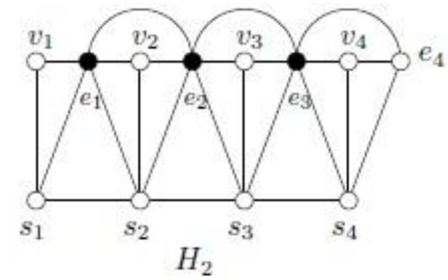


**Figure 4.** Shaded vertices indicate a $\gamma_s(H_2)$-set.

**Theorem 3.1.** For wheels $W_n$ on n+1 vertices, $n \geq 3$

$$\gamma_s(M(W_n)) = \begin{cases} \left\lceil \frac{3n}{4} \right\rceil, & \text{if } n \equiv 3 \pmod 4 \\ \left\lceil \frac{3n}{4} \right\rceil + 1, & \text{otherwise.} \end{cases}$$

*Proof.* Let $G = M(W_n)$ and v be the vertex at the center and $v_1, v_2, \ldots, v_n$ be the vertices on the rim of $W_n$. Let the vertices introduced on the edges $vv_i$, $1 \leq i \leq n$ of $W_n$ be $s_i$, $1 \leq i \leq n$ and the vertices introduced on $v_iv_{i+1}$, $1 \leq i \leq n-1$ be $e_i$, $1 \leq i \leq n-1$ and $e_n$ be the vertex introduced on the edge $v_nv_1$. Hence $V(G) = \{v_i, s_i, e_i : 1 \leq i \leq n\} \cup \{v\}$.

We now partition V(G) into sets $V_1, V_2, ..., V_k$ such that $G[V_1] \cong H_1$, $G[V_i] \cong H_2$, $2 \le i \le k-1$ and $G[V_k] \cong H'$, where $H_1$ is the graph as given in Figure 3, $H_2$ is the graph as given in Figure 4 and $H'$ is isomorphic to one of the graphs $G_1$ or $G_2$ or $G_3$ or $G_4$ as given in Figure 5. Using Lemmas 3.1 and 3.2 we obtain $\gamma_s(G) \ge 3(k-1) + \gamma_s(H_k)$.

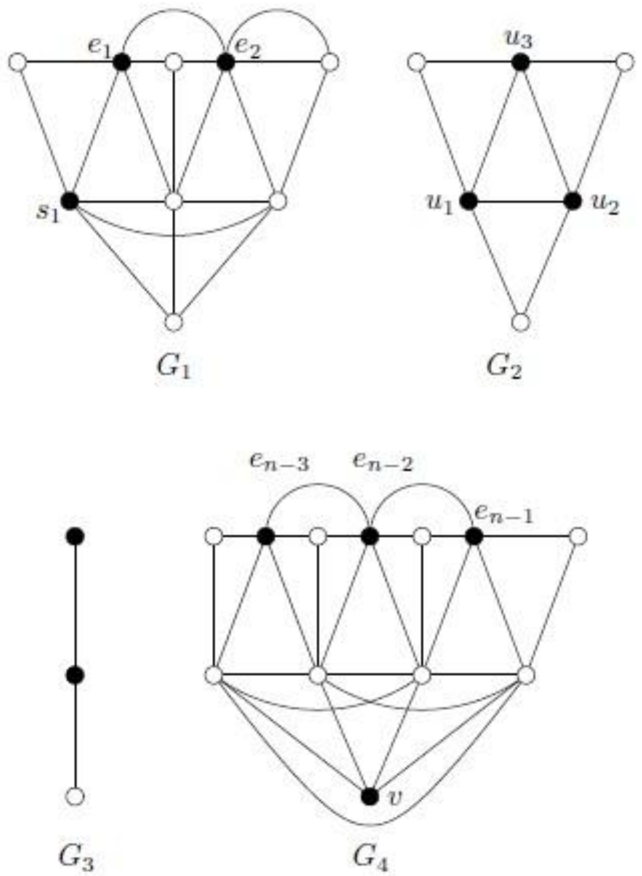


**Figure 5**. Shaded vertices indicate $\gamma_s$-sets for the respective graphs.

We now discuss the following cases.

**Case (i):** $H' \cong G_1$ . In this case, $n \equiv 3 \pmod 4$. It is clear that $\gamma(G_1) = 2$ and there are only two $\gamma$-sets in $G_1$. Further neither of the vertices in each of the $\gamma(G_1)$-sets satisfy the Proposition 1.1. Hence $\gamma_s(G_1) \ge 3$. Now $S = \{s_1, e_1, e_2\}$ is a $\gamma_s(G_1)$-set. Therefore $\gamma_s(G_1) = 3$. Hence we have $\gamma_s(G) \ge 3(k-1) + 3$. Since $|V(G)| = 3n+1$, we get $3n+1 = 13 + 12(k-2) + 9$, as $|V(G_1)| = 9$. This further implies that $\left\lceil \frac{3n}{4} \right\rceil =$

$3(k-1) + \left\lceil \frac{9}{4} \right\rceil$ or $\left\lceil \frac{3n}{4} \right\rceil = 3(k-1) + 3$. Hence $\gamma_s(G) \geq \left\lceil \frac{3n}{4} \right\rceil$. Now $S = \{e_{4i-3}, e_{4j-2}, e_{4k-1} : 1 \leq i, j, k \leq \left\lfloor \frac{n}{4} \right\rfloor\} \cup \{s_{n-2}, e_{n-2}, e_{n-1}\}$ is an SDS of cardinality $\left\lceil \frac{3n}{4} \right\rceil$. Hence $\gamma_s(G) = \left\lceil \frac{3n}{4} \right\rceil$ (Refer Figure 5)

**Case (ii):** $H' \cong G_2$ . In this case, $n \equiv 2 \pmod 4$. Clearly $\gamma(G_2) = 2$. Hence $\gamma_s(G_2) \geq 2$. For any $\gamma$-set D of $G_2$, neither of the vertices in D can defend the vertices of degree two in $G_2$. Hence $\gamma_s(G_2) \geq 3$ and $\{u_1, u_2, u_3\}$ is a $\gamma_s(G_2)$-set. Therefore $\gamma_s(G_2) = 3$. Hence $\gamma_s(G) \geq 3(k-1) + 3$. As discussed in the previous case, we obtain $3n+1 = 13 + 12(k-2) + 6$, since $|V(G_2)| = 6$. This further implies that $\left\lceil \frac{3n}{4} \right\rceil = 3(k-1) + \left\lceil \frac{3}{2} \right\rceil$ or $\left\lceil \frac{3n}{4} \right\rceil + 1 = 3(k-1) + 3$. Hence $\gamma_s(G) \geq \left\lceil \frac{3n}{4} \right\rceil + 1$. Now $S = \{e_{4i-3}, e_{4j-2}, e_{4k-1} : 1 \leq i, j, k \leq \left\lfloor \frac{n}{4} \right\rfloor\} \cup \{e_{n-1}, s_{n-1}, s_n\}$ is an SDS of cardinality $\left\lceil \frac{3n}{4} \right\rceil + 1$. Hence $\gamma_s(G) = \left\lceil \frac{3n}{4} \right\rceil + 1$ (Refer Figure 5).

**Case (iii):** $H' \cong G_3$ . In this case, $n \equiv 1 \pmod 4$. Since $G_3 \cong P_3$, it is clear that $\gamma(G_3) = 2$. Hence $\gamma_s(G) \geq 3(k-1) + 2$. Further as discussed in the earlier cases we have $\left\lceil \frac{3n}{4} \right\rceil = 3(k-1) + 1$. Hence $\left\lceil \frac{3n}{4} \right\rceil + 1 = 3(k-1) + 2$. Therefore $\gamma_s(G) \geq \left\lceil \frac{3n}{4} \right\rceil + 1$. Now $S = \{e_{4i-3}, e_{4j-2}, e_{4k-1} : 1 \leq i, j, k \leq \left\lfloor \frac{n}{4} \right\rfloor\} \cup \{e_{n-1}, s_{n-1}\}$ is an SDS of cardinality $\left\lceil \frac{3n}{4} \right\rceil + 1$. Hence $\gamma_s(G) = \left\lceil \frac{3n}{4} \right\rceil + 1$ (Refer Figure 5).

**Case (iv):** $H' \cong G_4$ . In this case, $n \equiv 0 \pmod 4$. It is clear that $\gamma(G_4) = 3$ and $G_4$ has a unique $\gamma$-set. Further neither of the vertices in the $\gamma$-set satisfy the Proposition 1.1. Hence $\gamma_s(G_4) \geq 4$. Now $S = \{e_{n-1}, e_{n-2}, e_{n-3}, v\}$ is a $\gamma_s(G_4)$-set. Therefore $\gamma_s(G_4) = 4$. Hence we have $\gamma_s(G) \geq 3(k-1) + 4$. Similar to the above cases we now have $3n+1 = 13 + 12(k-2) + 12$, since $|V_k| = 12$. Hence $\left\lceil \frac{3n}{4} \right\rceil + 1 = 3(k-1) + 4$. Therefore $\gamma_s(G) \geq \left\lceil \frac{3n}{4} \right\rceil + 1$. Now $S = \{e_{4i-3}, e_{4j-2}, e_{4k-1} : 1 \leq i, j, k \leq$

$\left(\frac{n}{4}-1\right)\} \cup \{e_{n-1}, e_{n-2}, e_{n-3}, v\}$ is an SDS of cardinality $\left\lceil \frac{3n}{4} \right\rceil + 1$. Hence $\gamma_s(G) = \left\lceil \frac{3n}{4} \right\rceil + 1$ (Refer Figure 5). □

## 4. Middle Graphs of Complete Bipartite Graphs $K_{p,q}$

In this section we obtain $\gamma_s(M(K_{p,q}))$, for a complete bipartite graph $K_{p,q}$, $p \le q$.

For any graph G of order n, it is observed that M(G) contains n cliques with each vertex of G is contained in a unique clique. Thus for the complete bipartite graph $K_{p,q}$, the $M(K_{p,q})$ contains $pK_{q+1}$ cliques and $qK_{p+1}$ cliques. Further these cliques are disjoint. We now evaluate $\gamma_s(M(K_{p,q}))$.

**Theorem 4.1.** For the complete bipartite graph $K_{p,q}$, $p \le q$ and $p \le 2$

$$\gamma_s(M(K_{p,q})) = q+1.$$

*Proof.* Let (X, Y) be the bipartition of $K_{p,q}$ with $Y = \{y_1, y_2, ..., y_q\}$ and let $G = M(K_{p,q})$. When p = 1, G contains $K_{q+1}$ as an induced subgraph and q vertices of $K_{q+1}$ are respectively adjacent to exactly one leaf. Clearly $\gamma(G) = q$. Therefore, $\gamma_s(G) \ge q$. Also we see that no vertex in any $\gamma$-set D of G defends the vertex of degree q in G. Hence $\gamma_s(G) \ge q+1$.

Now $S = V(K_{1,q})$ is an SDS of cardinality q+1. Hence $\gamma_s(G) = q+1$. When p = 2, let $X = \{x_1, x_2\}$. Now $M(K_{2,q})$ contains q $K_3$'s. Since each $K_3$ is disjoint, any $\gamma$-set D of G contains q vertices, one from each of the q $K_3$'s. Hence $\gamma(G) = q$. Therefore $\gamma_s(G) \ge q$. None of the vertices in any $\gamma$-set will defend the vertices of degree q in G. Hence $\gamma_s(G) \ge q+1$. Now $S = \{w_{11}, w_{12}\} \cup \{w_{22}\} \cup \{y_3, y_4, ..., y_{q-1}, y_q\}$, where $w_{11}$, $w_{12}$ are the vertices introduced on the edges $x_1y_1$ and $x_1y_2$ respectively and $w_{22}$ is the vertex introduced on the edge $x_2y_2$ in $K_{2,q}$, is an SDS of cardinality q+1. Therefore $\gamma_s(G) = q+1$. □

**Theorem 4.2.** For a complete bipartite graph $K_{p,q}$, $3 \le p \le q$,

$$\gamma_s(M(K_{p,q})) = q + \left\lceil \frac{p}{2} \right\rceil.$$

*Proof.* Let (X, Y) be a bipartition of $K_{p,q}$, where |X| = p, |Y| = q. Let $X = \{x_1, x_2, ..., x_p\}$ and $Y = \{y_1, y_2, ..., y_q\}$ and let $G = M(K_{p,q})$. Let $A_1, A_2, ..., A_p$ be the p $K_{q+1}$ cliques and $B_1, B_2, ..., B_q$ be the q $K_{p+1}$ cliques in $M(K_{p,q})$. Further $A_i$ and $B_j$ have exactly one common vertex. Let $w_{ij}$ be the vertex common to both $A_i$ and $B_j$. Let S be a $\gamma_s$-set of G. Since there are q disjoint cliques, $\gamma(G) = q$ and $\gamma_s(G) \ge q$. Now we claim that $\gamma_s(G) \geq q + \left\lceil \frac{p}{2} \right\rceil$. Assume that $\gamma_s(G) < \gamma(G) + \left\lceil \frac{p}{2} \right\rceil$. Then there exists at least two cliques $A_r$ and $A_s$ such that both $A_r$ and $A_s$ contains exactly one vertex each in S and clearly let $w_{rr}$ and $w_{ss}$ be in S. But $P_n(w_{rr}, S) \cup \{x_r, y_r\}$ and $P_n(w_{ss}, S) = \{x_s, y_s\}$ and $G[P_n(w_{rr}, S) \cup \{x_r, y_r\}]$ and $G[P_n(w_{ss}, S) \cup \{x_s, y_s\}]$ are not cliques. Hence by Proposition 1.1, $w_{rr}$ can neither S-defend $x_r$ nor $y_r$. Similarly, $w_{ss}$ can neither S-defend $x_s$ nor $y_s$, which is a contradiction. Therefore, $\gamma_s(G) \geq \gamma(G) + \left\lceil \frac{p}{2} \right\rceil$. Now let $S = \{w_{ij}, w_{i(j+1)} : i, j = 1, 3, 5, ..., p-1\} \cup \{w_{kk} : k = 2, 4, 6, ..., p\} \cup \{y_q, y_{q-1}, ..., y_{p+1}\}$, if p is even and $S = \{w_{ij}, w_{i(j+1)} : i, j = 1, 3, 5, ..., p\} \cup \{w_{kk} : k = 2, 4, 6, ..., p-1\} \cup \{y_q, y_{q-1}, ..., y_{p+2}\}$, if p is odd and it is of cardinality $q + \left\lceil \frac{p}{2} \right\rceil$. Therefore, $\gamma_s(G) = q + \left\lceil \frac{p}{2} \right\rceil$. (Refer Figure 6 and Figure 7). □

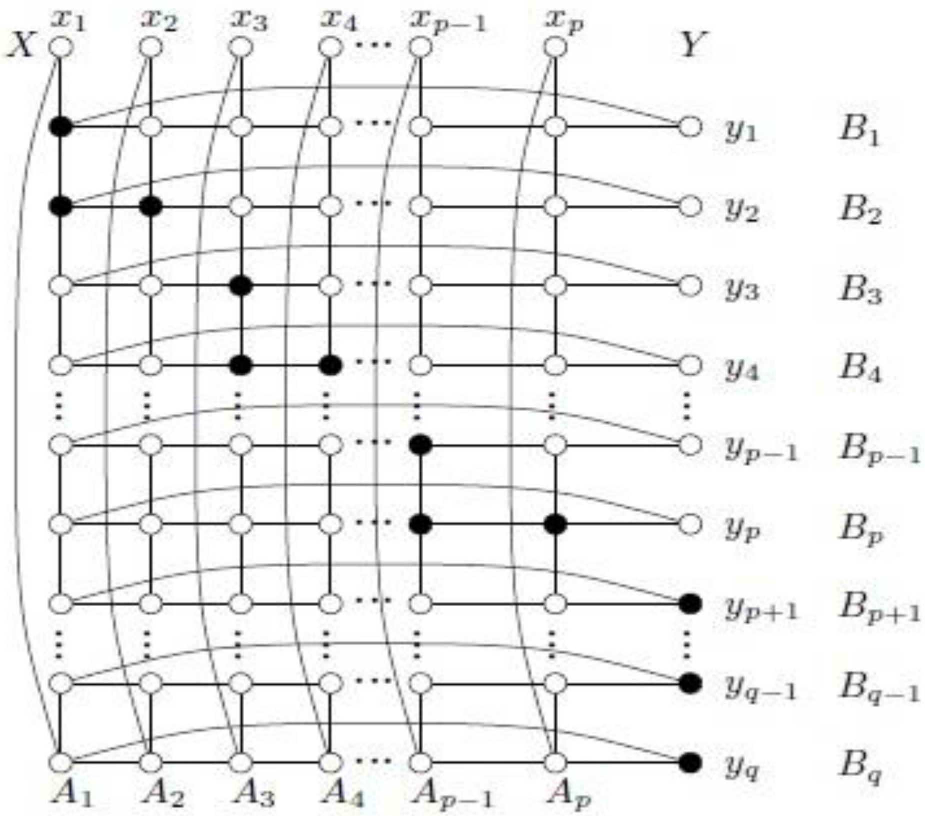


**Figure 6**. Shaded vertices indicate a $\gamma_s$-set of $M(K_{p,q})$, when p is even.

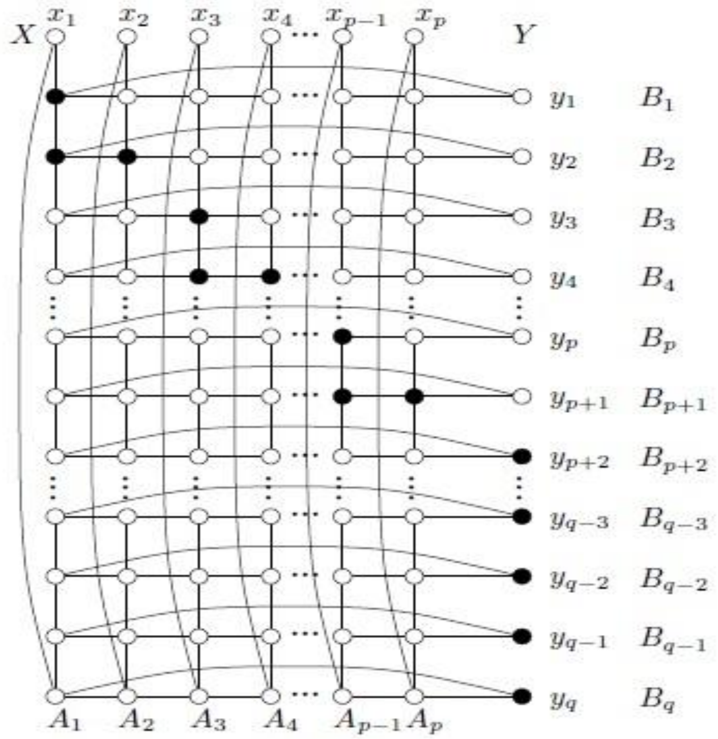


**Figure 7**. Shaded vertices indicate a $\gamma_s$-set of $M(K_{p,q})$, when p is odd.

## 5. Middle Graph of Trees

In this section we obtain a lower bound for middle graph of trees.

**Theorem 5.1.** For a tree T of order n, $\gamma_s(M(T)) \geq \ell + \left\lceil \frac{n-\ell}{2} \right\rceil$, where $\ell$ is the number of leaf vertices. Further the bound is sharp and equality is attained for a star graph $K_{1,n-1}$.

*Proof.* Let G = M(T). As every leaf vertex is contained in a unique clique $K_1$, without loss of generality, we assume that all the non-leaf vertices adjacent to these leaf vertices belong to any $\gamma_s$-set, say S of G. By the definition of middle graph, there are n cliques in G. Each of the $\ell$ leaf vertices is in a unique clique $K_1$. Then each of the remaining n-$\ell$ vertices of T is in a unique clique. Hence for every pair of adjacent cliques at least one vertex belongs to S.

Hence $\gamma_s(G) \geq \ell + \left\lceil \frac{n-\ell}{2} \right\rceil$ and it is easy to verify that for a star graph $K_{1,n-1}$ $\gamma_s(M(K_{1,n-1})) = \ell + \left\lceil \frac{n-\ell}{2} \right\rceil$. □

## Conclusion

In this paper, secure domination number for middle graphs of certain graph families such as paths, cycles, wheels and complete bipartite graphs are determined. Further a sharp lower bound for secure domination number for middle graph of trees is obtained.

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