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ON COEQUALITY RELATIONS ON SET WITH APARTNESS

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Abstract. This investigation is in the mathematics based on the Intuitionistic logic. A relation ρ is a coequality relation if it is consistent, symmetric and co-transitive. For a coequality relation ρ on a set X with apartness we analyze the family *Cop(X)* of all classes of the relation. Characteristics of this family allow us to introduce a new concept, 'co-partition' in set with apaerness - a specific family of proper subsets. In addition, a connection between the family of all coequality relations and the family of all co-partitions is given. At the end of this article, some examples and applications in the semigroups with apartness theory are given.

1. Introduction

This investigation is in Bishop's constructive mathematics in sense of well-known books [**1, 2, 3, 4, 5, 8, 9, 13, 14**] and our papers [**6, 7, 10, 11, 12**].

Bishop's constructive mathematics is develop on the Intuitionistic Logic ([**8, 13, 14**])- logic without the Law of Excluded Middle $P \vee \neg P$. Let us note that in the Intuitionistic Logic the 'Double Negation Law' $P \Longleftrightarrow \neg\neg P$ does not hold, but the following implication $P \Longrightarrow \neg\neg P$ holds even in the Minimal Logic. Since the Intuitionistic Logic is a part of the Classical Logic, these results in the Constructive mathematics are compatible with suitable results in the Classical mathematics. Let us recall that the following deduction principle $A \vee B, \neg B \vdash A$ is acceptable in the Intuitionistic Logic.

Let $(X, =, \neq)$ be a set, where the relation $\neq$ is a binary relation on X, called *diversity* on X, which satisfies the following properties:

$$\neg(x \neq x),\ x \neq y \Longrightarrow y \neq x,\ (x \neq y \wedge y = z) \Longrightarrow x \neq z.$$

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Following Heyting ([**8**]), if the following implication

$$(\forall x, y, z \in X)(x \neq z \Longrightarrow (x \neq y \vee y \neq z))$$

holds, the diversity $\neq$ is called *apartness*. Let x be an element of X and A a subset of X. We write $x \rhd A$ if and only if $(\forall a \in A)(x \neq a)$, and $A^{\rhd} = \{x \in X : x \rhd A\}$. In $X \times X$ the equality and diversity are defined by

$$(x, y) = (u, v) \Longleftrightarrow x = u \wedge y = v,$$
$$(x, y) \neq (u, v) \Longleftrightarrow x \neq u \vee y \neq v$$

and equality and diversity relations in power-set $\wp(X \times X)$ of $X \times X$ by

$$\alpha = \beta \Longleftrightarrow (\forall (x, y) \in X \times X)((x, y) \in \alpha \Longleftrightarrow (x, y) \in \beta),$$
$$\alpha \neq \beta \Longleftrightarrow (\exists x, y \in X)((x, y) \in \alpha \wedge (x, y) \rhd \beta) \vee (\exists x, y \in X)((x, y) \in \beta \wedge (x, y) \rhd \alpha).$$

Let us note that the diversity relation $\neq$ is not an apartness relation, in general case.

Example 1.1. (1) The relation $\neg(=)$ is an apartness on the set $\mathbf{Z}$ of integers.
(2) The relation q, defined on the set $\mathbf{Q}^N$ by

$$(f, g) \in q \Longleftrightarrow (\exists k \in \mathbf{N})(\exists n \in \mathbf{N})(m \geqslant n \Longrightarrow |f(m) - g(m)| > k^{-1}),$$

is an apartnerss relation.
(3) In the power-set $\wp(X)$ of set X we define diversity relation on the following way:

$$A \neq B \Longleftrightarrow (\exists a \in A)\neg(a \in B) \ \vee \ (\exists b \in B)\neg(b \in A). \ \diamondsuit$$

In this paper we analyze coequality relation q in set X with separation. Also, in Theorem 2.1 we analyze the family $X/q = \{aq : a \in X\}$ of all classes of the coequality relation q generated by elements of set X. This allows us to introduce a new concept - the specific family of subsets of X called 'co-partition' of set X (Theorem 2.2). Further on, we establishing a correspondence between the family of all coequality relations on set X and the family of all co-partitions in set X (Theorem 2.3). In addition, in section 3, we give some applications (Theorem 3.2 and Theorem 3.3) in the theory of semigroups with apartness.

So, what is the specificity of this text?

Firstly, it is the using of the Intuitionistic logic instead of the Classical logic. In Intuitionistic logic formula 'the Law of Excluded Middle' is neither an axiom nor a valid formula. Therefore, in this case, a set looks like as the relational system $(X, =, \neq)$ where the $' \neq '$ is an apartness relation (extensive to the equality on the set in the following sense: $= \circ \neq \subseteq \neq$).

Secondly, the duality of the relationship, which appears with this aspect of observation on concepts and processes in mathematics based on the Intuitionistic logic, opens possibilities for us to analyze the specific relationships that do not appear in classical mathematics as coequality relation, for example. So, we are interested to study some specific relations that appear on sets with the apartness. In addition, we are also interested to analyze structures based on those specific relations.

2. Coequality relation and its co-partition

DEFINITION 2.1. A relation q on X is a *coequality relation* on X if and only if it is consistent, symmetric and cotransitive:

$$q \subseteq \neq, \quad q = q^{-1}, \quad q \subseteq q * q,$$

where "$*$" the operation of relations $\alpha \subseteq X \times Y$ and $\subseteq Y \times Z$, called *filled product* of relations α and β, are relation on $X \times Z$ defined by

$$(a, c) \in \beta * \alpha \Longleftrightarrow (\forall b \in X)((a, b) \in \alpha \vee (b, c) \in \beta).$$

For an equivalence e and a coequalence q on set X we say that they are *associated* if $e \circ q \subseteq q$ holds.

Put on $C(x) = \{y \in X : y \neq x\}$. The subset $C(x)$ satisfies the following implication:

$$y \in C(x) \Longrightarrow (\forall z \in X)(y \neq z \vee z \in C(x)).$$

It is called a *principal strongly extensional subset* of X such that $x \triangleright C(x)$. Following this special case, for a subset A of X, we say that it is a *strongly extensional subset* of X if and only if the following implication holds

$$x \in A \Longrightarrow (\forall y \in X)(x \neq y \vee y \in A).$$

For a subset A of set X we say that it is *detachable* if $(\forall x \in X)(x \in A \vee x \triangleright A)$ holds.

For a coequality relation q on a set X we can form the family $A_q = \{qx\}_{x \in X}$ of classes of the relation q generated by elements of X. It is clear that $xq = qx$ because the relation q is symmetric. Since q is a consistent relation, we have $x \triangleright qx$. Besides, since q is a cotransitive relation any qx is a strongly extensional subset of X. Indeed, for any elements $x, y, z \in X$ such that $(x, y) \in q$, holds $(x, z) \in q \vee (z, y) \in q$. Thus, by consistency of q, $z \in qx \vee y \neq z$. So, the family $\{qx\}_{x \in X}$ is a subfamily of strongly extensional subsets of X. Suppose that for two classes xq and yq is true $xq \neq yq$. It means $(\exists u \in X)(u \in xq \wedge \neg(u \in yq))$ or $(\exists v \in X)(\neg(v \in xq) \wedge v \in yq)$. From $(x, u) \in q$ follows $(x, y) \in q \vee (y, u) \in q$. Hence, we have $xq \cup yq = X$ because the second case is impossible. From $(v, y) \in q$ we analogously again got $xq \cup yq = X$. Therefore, for the family $\{qx\}_{x \in X}$ is true:

(i) $x \triangleright xq$; (ii) $xq = qx$; and (iii) $xq \neq yq \Longrightarrow xq \cup yq = X$.

Now, suppose that a family $\{A_t\}_{t \in X}$ of strongly extensional proper subsets of X satisfies the following conditions:

(a) For any $t \in X$ there exists a strongly extensional subset A_t such that $t \triangleright A_t$;

(b) $A_t \neq A_s \Longrightarrow A_t \cup A_s = X$ for any $t, s \in X$.

Let us define a relation R on X by

$$(x, y) \in R \text{ if and only if } (\exists u \in X)(x \in A_u \wedge y \triangleright A_u).$$

It is clear that relation R is consistent. Besides, for elements x, y there exist subsets A_x and A_y such that $x \triangleright A_x$ and $y \triangleright A_y$. So, since $x \in A_u \wedge x \triangleright A_x$ we have $A_u \cup A_x = X$. Hence, $y \in A_x$. Thus, we have $x \in A_y$. Finally, we have $x \in A_y \wedge y \triangleright A_y \wedge x \triangleright A_x \wedge y \in A_x$. So, the relation R is symmetric.

Assume $(x, z) \in R$ and $y \in X$. Then there exist subsets A_x and A_z such that $x \triangleright A_x$, $z \triangleright A_z$, $x \in A_z$ and $z \in A_x$. By (b), we have $A_x \cup A_z = X$ and $y \in A_x$ or $y \in A_z$. Therefore, we have $x \triangleright A_x \wedge y \in A_x$ or $y \in A_z \wedge z \triangleright A_z$. We conclude that $(x, y) \in R$ or $(z, y) \in R$. So, the relation R is a cotransitive relation on X. Finally, we have that the relation R is a coequality relation on X.

In the following assertion we describe the connection between a coequality relation R and the corresponding family $\{A_t\}_{t\in X}$.

Theorem 2.1. *For a coequality relation R on a set X there exists the unique family $\{A_t\}_{t\in X}$ of strongly extensional subsets of X which satisfies the condition (a) and (b).*

Opposite, if the family $\{A_t\}_{t\in X}$ satisfies condition (a) and (b), then the relation q_A, defined by $(x, y) \in q_A \Longleftrightarrow (\exists u \in X)(x \in A_u \wedge y \triangleright A_u)$, is a coeguality relation on set X.

Therefore, we can construct the family $\{aq : a \in X\}$ of all classes $aq = \{x \in X : (a, x) \in q\}$ of q generated by the elements $a \in X$, with

$$aq = bq \Longleftrightarrow (a, b) \triangleright q,\ aq \neq bq \Longleftrightarrow (a, b) \in q.$$

It is clear that the mapping $\vartheta_q : X \longrightarrow X/q$, defined by $\vartheta_q(x) = xq$, is a strongly extensional surjective function.

If q is a coequality relation on set X, then the relation $q^{\triangleright} = \{(x, y) \in S \times S : (x, y) \triangleright q\}$ is an equivalence on X associated with q ([**6**], Theorem 2.3) in the following way $q^{\triangleright} \circ q \subseteq q$, and we can ([**6**], Theorem 2.4) construct the factor-set $X/(q^{\triangleright}, q) = \{aq^{\triangleright} : a \in X\}$, where $aq^{\triangleright} = \{x \in X : (x, a) \in q^{\triangleright}\}$ is a class of $q^{\triangleright}$ generated by the element a, with:

$$aq^{\triangleright} = bq^{\triangleright} \Longleftrightarrow (a, b) \triangleright q,\ aq^{\triangleright} \neq bq^{\triangleright} \Longleftrightarrow (a, b) \in q.$$

It is easily to check that there exists the strongly extensional, surjective, injective and embedding mapping $\phi : X/q \longrightarrow X/(q^{\triangleright}, q)$. That mapping is defined by $\phi(aq^{\triangleright}) = aq$.

Definition 2.2. A *copartition* of a set X is a nonempty collection of nonempty subsets of X whose satisfy conditions

(a) For any $t \in X$ there exists a strongly extensional subset A_t such that $t \triangleright A_t$,

(b) $A_t \neq A_s \Longrightarrow A_t \cup A_s = X$ for any $t, s \in X$,

and it is written as *Cop(X)*.

The next theorem shows that if we generate a copartition by means of a coequality relation q, then the coequality relation generated by the copartition is simply q again; and similarly if we begin with the coequality relation generated by a copartition, this relation generates the given copartition.

Theorem 2.2. *Let $(X, =, \neq)$ be a set with an apartness. Let $Coeq(X)$ be the family of all coequaity relations on set X, and let $Copart(X)$ be the family of all copartitions on set X. Then*

(1) *$q_{X/c} = c$ for every $c \in Coeq(X)$;*

(2) *$X/q_{Cop(X)} = Cop(X)$ for every $Cop(X) \in Copart(X)$.*

PROOF. (1) Let c be a coequality relation on X. Then

$(x, y) \in c \Longleftrightarrow y \in A_x$
$\Longrightarrow x \rhd A_x \wedge y \in A_x$
$\Longleftrightarrow (x, y) \in q_{X/c}$.

$(x, y) \in q_{X/c} \Longleftrightarrow (\exists Y_u \in A_q)(x \rhd Y_u \wedge y \in Y_u)$
$\Longleftrightarrow (\exists Y_u \in A_q)(x \rhd Y_u \wedge (y, u) \in c)$
$\Longrightarrow (\exists Y_u \in A_q)(x \rhd Y_u \wedge ((y, x) \in q \vee (x, u) \in c))$
$\Longrightarrow (\exists Y_u \in A_q)(x \rhd Y_u \wedge (y, x) \in c)$
$\Longrightarrow (y, x) \in c$.

(2) Let $Cop(X) \in Copart(X)$ be a copartition on set X. If $Y \in Cop(X)$, then $Y \subset X$ and $(\exists x \in X)(x \rhd Y)$. So, for every $y \in Y$, we have $(x, y) \in q_{Cop(X)}$. Therefore $y \rhd Y_x$. Thus $Y \subseteq Y_x$. At the other hand, $u \in Y_x$ implies $(x, u) \in q_{Cop(x)}$, i.e. $x \rhd Y$ and $u \in Y$. So, $Y = Y_x$. We have $Cop(X) \subseteq X/q_{Cop(x)}$. Let $Y_x \in X/q_{Cop(X)}$. Then for every element y of Y_x we have $(x, y) \in q_{Cop(X)})$. Thus, we conclude that there exists $Y \in Cop(X)$ such that $x \rhd Y$ and $y \in Y$. So $Y_x \subseteq Y$. Therefore $Y_x \in Cop(X)$, and $X/q_{Cop(X)} \subseteq Cop(X)$. □

So, there is a natural correspondence between the family $Coeq(X)$ of all coequality relations on X and the family $Copart(X)$ of all copartitions of X.

THEOREM 2.3. *There exists the unique injective and surjective mapping*

$$\psi : Coeq(X) \longrightarrow Copart(X).$$

PROOF. The proof of this assertion is a compilation of Theorem 2.1 and Theorem 2.2. If we generate a copartition X/q by means of a coequality relation q, then the coequality relation $q_{X/q}$ generated by the copartition X/q is simply q again. Similarly, if we begin with the coequality relation $q_{Cop(X)}$ generated by an copartition $Cop(X)$, this relation generates the given copartition, i.e. holds $X/q_{Cop(X)} = Cop(X)$ again. □

In the following theorem we will give some basic properties of classes of associated a pair of an equality and a coequality relations.

THEOREM 2.4. *An equality relation e and a coequality relations q on a set $(X, =, \neq)$ are associated if and only if*

$$(\forall x, z \in X)(x \neq z \wedge xe \cap qz \neq \emptyset \Longrightarrow xe \subseteq yq)$$

PROOF. (1) Let e and q be associated relations on set X and let $xe \cap yq \neq \emptyset$ for each x, z in X such that $x \neq z$. Then $(\exists y \in X)(y \in xe \wedge y \in qz)$, i.e. $(\exists y \in X)((x, y) \in e \wedge (y, z) \in q)$. Thus $(x, z) \in q$ because relation e and q are associated. Further on, we have

$u \in ex \Longleftrightarrow (x, u) \in e$
$\Longrightarrow (x, u) \in e \wedge (x, z) \in q$
$\Longrightarrow (u, z) \in q$
$\Longleftrightarrow u \in zq$.

(2) Let $(\forall x, z \in X)(x \neq z \wedge xe \cap qz \neq \emptyset \Longrightarrow xe \subseteq yq)$ holds. Then

$(x, y) \in e \wedge (y, z) \in q \Longleftrightarrow$

$(\exists ex \in X/e)(x \in ex \wedge y \in ex) \wedge (\exists zq \in X/q)(y \rhd zq \wedge z \rhd zq) \Longrightarrow$
$(\exists xe \in X/e)(\exists zq \in X/q)(x \in xe \wedge y \in xe \cap zq \wedge z \rhd zq) \Longrightarrow$
$(\exists zq \in X/q)(x \in xe \subseteq zq \wedge z \rhd zq) \Longleftrightarrow (x, z) \in q.$ □

3. Examples and applications

EXAMPLE 3.1. For set $X = \{1, 2, 3, 4\}$ and coequality relation

$$R = \{(1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (4, 1), (4, 2)\}$$

the corresponding family of strongly extensional subsets contains the following subsets: $1R = \{2, 3, 4\}$, $2R = \{1, 3, 4\}$, $3R = \{1, 2\}$ and $4R = \{1, 2\}$. ♢

EXAMPLE 3.2. (1) ([**11**]) Let T be a set and $\Im$ be a subfamily of $\wp(T)$ such that $\emptyset \subseteq \Im$, $A \subseteq B \wedge B \in \Im \Longrightarrow A \in \Im$, $A \cap B \in \Im \Longrightarrow A \in \Im \vee B \in \Im$. If $(X_t)_{t\in T}$ is a family of sets, then the relation q on $\prod_{t\in T} X_t$ $(\neq \emptyset)$ defined by $(f, g) \in q \Longleftrightarrow \{s \in T : f(s) = g(s)\} \in \Im$, is a coequality relation on the Cartesian product $\prod_t X_t$.
(2) ([**9**]) A ring R is a local ring if for each $r \in R$, either r or $1 - r$ is a unit, and let M be a module over R. The relation q on M, defined by $(x, y) \in q$ if there exists a homomorphism $f : M \longrightarrow R$ such that $f(x - y)$ is a unit, is a coequality relation on M.
(3) Let T be a strongly extensional subset of semigroup S such that $(\forall x, y \in S)(xy \in T \Longrightarrow x \in T \wedge y \in T)$. Then relation q on semigroup S, defined by $(a, b) \in q$ if and only if $a \neq b \wedge (a \in T \vee b \in T)$, is a coequality relation on S and it is compatible with semigroup operation in the following sense $(\forall x, y, a, b \in S)((xay, xby) \in q \Longrightarrow (a, b) \in q)$. In this case, such coequality we call *cocongruence* on semigroup S.
(4) Let $(R, =, \neq, +, 0, \cdot, 1)$ be a commutative ring. A subset Q of R is a *coideal* of R if and only if $0 \rhd Q$, $-x \in Q \Longrightarrow x \in Q$, $x + y \in Q \Longrightarrow x \in Q \vee y \in Q$, $xy \in Q \Longrightarrow x \in Q \wedge y \in Q$.

Coideals of commutative ring with apartness were first defined and studied by Ruitenburg 1982 in his dissertation ([**13**]). After that, coideals (anti-ideals) studied by A.S. Troelstra and D. van Dalen in their monograph [**14**] (Vol. II; Section: Algebra). This author proved, in his paper [**10**], if Q is a coideal of a ring R, then the relation q on R, defined by $(x, y) \in q \Longleftrightarrow x - y \in Q$, satisfies the following properties:
(a) *q is a coequality relation on R*;
(b) $(\forall x, y, u, v \in R)((x + u, y + v) \in q \Longrightarrow (x, y) \in q \vee (u, v) \in q)$;
(c) $(\forall x, y, u, v \in R)((xu, yv) \in q \Longrightarrow (x, y) \in q \vee (u, v) \in q)$.
A relation q on R, which satisfies the property (a)-(c), is called *cocongruence* on R ([**10**]) or coequality relation compatible with ring operations. If q is a cocongruence on a ring R, then the set $Q = \{x \in R : (x, 0) \in q\}$ is a coideal of R ([**10**]). ♢

Let q be a coequality relation on a set X and let $f : X \times X \longrightarrow X$ be a strongly extensional mapping. We say that f is *compatible* with the coequality relation q if

$$(\forall x, y, u, v \in X)((f(x, y), f(u, v)) \in q \Longrightarrow (x, u) \in q \vee (y, v) \in q)$$

holds.

In the following theorem we give a result on compatibility of function $f : X^2 \longrightarrow X$ with the given coequality relation q on the set X.

Theorem 3.1. *If the strongly extensional mapping $f : X^2 \longrightarrow X$ is compatible with the coequality relation q on X, then there is a strongly extensional mapping $F : X/q \times X/q \longrightarrow X/q$ such that $\vartheta_X \circ f = F \circ (\vartheta_X, \vartheta_X)$.*

Proof. Let us define mapping F by $F(uq, vq) = f(u, v)q$. Then:
(1) Let $(xq, yq) = (uq, vq)$. It means $xq = uq$ and $yq = vq$. Suppose that $s \in f(x, y)q$, i.e. suppose that $(f(x, y), s) \in q$. Thus, by cotransitivity of q, we have $(f(x, y), f(u, v)) \in q$ or $(f(u, v), s) \in q$ Hence, by compatibility f and q follows $(x, u) \in q$ or $(y, v) \in q$ or $s \in f(u, v)q$. So, $s \in f(u, v)q$ because $(x, u) \triangleright q$ and $(y, v) \triangleright q$. Finally, we have $f(x, y)q \subseteq f(u, v)q$. We also have $f(u, v)q \subseteq f(x, y)q$ by analogy. Finally, we have $f(u, v)q = f(x, y)q$. Therefore, the correspondence F is a mapping.
(2) Let $F(uq, vq) \neq F(xq, yq)$ be holds for $uq, vq, xq, yq \in X/q$. It means $f(u, v)q \neq f(x, y)q$ and $(f(u, v), f(x, y)) \in q$. Since the mapping f is compatible with q, follows $(u, x) \in q$ or $(v, y) \in q$. Finally, we have $uq \neq xq$ or $vq \neq yq$. So, the mapping F is a strongly extensional.
(3) Let (x, y) be an arbitrary pair of elements of $X \times X$. We have

$$(\vartheta_X \circ f)(x, y) = \vartheta_X(f(x, y)) = f(x, y)q = F(xq, yq) = F(\vartheta_X(x), \vartheta_X(y)) = F((\vartheta_X, \vartheta_X)(x, y)) = (F \circ (\vartheta_X, \vartheta_X))(x, y).$$

Therefore, seeking equality is valid. □

In the next statement we give a proposition: If q is a coequality relation on a semigroup S compatible with the semigroup operation, then the copartition S/q is a semigroup. (On semigroup with apartness reader can find in our articles [**6, 7**].)

Theorem 3.2. *Let $(S, =, \neq, \cdot)$ be a semigroup where the semigroup operation is compatible with the apartness $\neq$. If q is a coequality relation on S, then the family S/q is a semigroup and semigroup operation in S/q, defined by $aq \cdot bq = (ab)q$ $(a, b \in S)$ is compatible with the apartness in S/q.*

Proof. Let be x, y, u, v be arbitrary elements in S such that $xq = uq$ and $yq = vq$. Let s be an arbitrary element of $(ab)q$. Thus, $((ab), s) \in q$. Follows $(ab, uv) \in q \vee (uv, s) \in q$. By compatibility of the semigroup operation with the coequality relation q, we have $(a, u) \in q \vee (b, v) \in q \vee s \in (uv)q$. So, $s \in (uv)q$ because $(a, u) \triangleright q \wedge (b, v) \triangleright q$. Therefore, we have $(ab)q \subseteq (uv)q$. Similarly, we can conclude $(uv)q \subseteq (ab)q$. Finally, the operation in S/q is well defined.

Let $aq \cdot bq = (ab)q \neq (xy)q = xq \cdot yq$ be holds for elements a, b, x, y of S. Then $(ab, xy) \in q$. Thus $(a, x) \in q$ or $(b, y) \in q$. So, $aq \neq xq$ or $bq \neq yq$. Therefore, the operation in S/q is a strongly extensional mapping.

Let a, b, c be arbitrary elements of S. We have

$$aq \cdot (bq \cdot cq) = aq \cdot (bc)q = a(bc)q = (ab)cq = (ab)q \cdot cq = (aq \cdot bq) \cdot cq.$$

Finally, the operation in S/q is associative and therefore, S/q is a semigroup. □

Let q be a coequality relation on a semigroup S with apartness. In the following theorem we give a construction of coequality relation $q^\star$ compatible with the semigroup operation such that $q^\star$ is the minimal extension of q.

THEOREM 3.3. *Let q be a coequality relation on a semigroup $(S,=,\neq,\cdot,1)$. Then the relation $q^\star = \{(x,y) \in S \times S : (\exists a,b \in S)((axb,ayb) \in q)\}$ is a coequality relation compatible with the semigroup operation such thet $q \subseteq q^\star$. If r is a coequality relation on S compatible with the semigroup operation such that $q \subseteq r$, then $q^\star \subseteq r$.*

PROOF. $(x,y) \in q^\star \Longleftrightarrow (\exists a,b \in S)((axb,ayb) \in q)$
$\Longrightarrow (\exists a,b \in S)(\forall u \in S)((axb,ayb) \neq (aub,aub))$
$\Longrightarrow (\exists a,b \in S)(\forall u \in S)(axb \neq aub \vee ayb \neq aub)$
$\Longrightarrow (\forall u \in S)(x \neq u \vee y \neq u)$
$\Longleftrightarrow (\forall u \in S)(x,y) \neq (u,u))$.

Let $(x,y) \in q^\star$ be an arbitrary element. Then there exist elements a,b in S such that $(axb,ayb) \in q$. Thus $(ayb,axb) \in q$ because q is symmetric. So, $(y,x) \in q^\star$.

Let (x,z) be an element of $q^\star$ and let y be arbitrary element of S. Then there exist elements a,b in S such that $(axb,azb) \in q$. Thus we have $(axb,ayb) \in q$ or $(ayb,azb) \in q$ because q is cotransitive. Therefore, $(x,y) \in q^\star$ or $(y,z) \in q^\star$.

Let (az,ys) be an element of $q^\star$. Thus there exist elements a and b in S such that $(axzb,aysb) \in q$. Further on, we have $(axzb,ayzb) \in q$ or $(ayzb,aysb) \in q$. Therefore $(\exists a,zb \in S)((ax(zb),ay(zb)) \in q)$ or $(\exists ay,b \in S)(((ay)zb,(ay)sb) \in q)$. So, $(x,y) \in q^\star$ or $(z,s) \in q^\star$.

Let (x,y) be an arbitrary element of $q^\star$. Then there exist elements $a,b \in S$ such that $(axb,ayb) \in q \subseteq r$. Thus, $(x,y) \in r$ since r is a coequality relation on S compatible with the semigroup operation in S. So, the relation $q^\star$ is the minimal extension of q. □

References

[1] E.Bishop. *Foundations of Constructive Analysis*; McGraw-Hill, New York 1967
[2] E.Bishop and D.Bridges. *Constructive Analysis*, Grundlehren der mathematischen Wissenschaften 279, Springer, Berlin, 1985.
[3] D.S.Bridges and F.Richman. *Varieties of Constructive Mathematics*, London Mathematical Society Lecture Notes 97, Cambridge University Press, Cambridge, 1987
[4] D.S.Bridges and L.S.Vita. *Techniques of Constructive Analysis*, Universitext, Springer, 2006.
[5] D.S.Bridges and L.S.Vita. *Apartness and Uniformity A Constructive Development*, Theory and Applications of Computability, Springer-Verlag, Heidelberg, 2011.
[6] S.Crvenković, M.Mitrović and D.A.Romano. Semigroups with Apartness; *Math. Logic Quart.*, **59**(6)(2013), 407-414
[7] S.Crvenković, M.Mitrović and D.A.Romano. Notions of (Constructive) Semigroups with Apartness, *Semigroup Forum*, **92**(3)(2016), 659-674.
[8] A. Heyting. *Intuitionism. An Introduction.* North-Holland, Amsterdam 1956.
[9] R.Mines, F.Richman and W.Ruitenburg. *A Course of Constructive Algebra*; Springer, New York 1988.
[10] D.A.Romano. Rings and fields, a constructive view; *Math. Logic Quart.*, **34**(1)(1988), 25-40
[11] D.A.Romano. Equality and coequality relations on the Cartesian product of sets; *Math. Logic Quart*, **34**(5)(1988), 471-480
[12] D.A.Romano. *Coequality relations, a survey*; Bull.Soc.Math.Banja Luka, **3**(1996), 1-36.

[13] W.Ruitenburg: *Intuitionistic Algebra*; Ph.D. Thesis, University of Utrecht, Utrecht 1982
[14] A.S.Troelstra and D. van Dalen. *Constructivism in Mathematics, An Introduction, Vol. I; Vol. II*; North-Holland, Amsterdam 1988

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