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# A STUDY OF EULERIAN, HAMILTONIAN, REGULAR AND COMPLETE ALGEBRAIC GRAPHS

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Abstract. The main objective of this paper is to connect algebra and graph theory with functions. In this paper we introduce the notion of algebraic graph, Eulerian, Hamiltonian,regular and complete algebraic graphs and prove the results in graph theorey using definition of algebraic graphs. We study the properties of Hamiltonian, Eulerian algebraic graphs. In this paper,we prove that the necessary and sufficient condition for a graph to be Hamiltonian using definition of algebraic graphs.

## 1. Introduction

In 1735, Euler introduced graph theory to solve Konigsberg bridge problem. Graph theory serves as a mathematical model for any system involving a binary relation. A graph is a convenient way of representing information involving relationship between objects. The objects are represented by vertices and relations by edges. Many problems that occur in the field of Computer Science, Information Technology, Electrical Engineering and many other areas can be analyzed by using techniques described in graph theory.

**Ore's Theorem**: Let $G(V, E)$ be a graph of order $n(n \geqslant 3)$. If the sum of degrees of every pair of nonadjacent vertices greater than or equal to n, then $G(V, E)$ is a Hamiltonian graph.

**Dirac's Theorem**: Let $G(V, E)$ be a graph of order $n(n \geqslant 3)$. If $deg(v) \geqslant \frac{n}{2}$ for all $v \in V$, then $G(V, E)$ is a Hamiltonian graph .

Ore's Theorem and Dirac's Theorem provide only sufficient conditions for a graph to be Hamiltonian. We have a characterization of Eulerian graph whereas

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the characterization of Hamiltonian graph is a major unsolved problem in graph theory. In this paper,we prove that the necessary and sufficient condition for a graph to be Hamiltonian using definition of algebraic graphs.

## 2. Preliminaries.

A bijection mapping of a finite set $V$ onto itself is called a *permutation.* If $V = \{v_1, v_2, \dots, v_n\}$ is a finite set and $f$ is a bijection on $V$, then we write

$$f = \begin{pmatrix} v_1 & v_2 & \dots & v_n \\ f(v_1) & f(v_2) & \dots & f(v_n) \end{pmatrix}.$$

If $f : V \to V$ is a bijection, then the number of elements of $V$ is called the *degree* of $f$ and it is denoted by d(f). Let $V = \{v_1, v_2, \dots, v_n\}$. A permutation

$$f = \begin{pmatrix} v_1 & v_2 & \dots & v_k & v_{k+1} & \dots & v_n \\ v_2 & v_3 & \dots & v_{k+1} & v_{k+2} & \dots & v_1 \end{pmatrix}$$

is called a *cyclic permutation* of degree $n$. It is represented as $(v_1, v_2, \dots, v_n)$ which is a cycle of length $n$. If $f$, $g$ be two cycles on $V$ such that they have no common element, then $f$, $g$ are *disjoint cycles.*

A graph is a pair $(V, E)$ where $V$ is a non-empty set and $E$ is a set of unordered pairs of elements of $V$. The graph $(V, E)$ is denoted by $G(V, E)$. The number of vertices in $G(V, E)$ is called the order of $G$ and it is denoted by $|V|$. The number of edges in $G(V, E)$ is called the size of $G(V, E)$ and it is denoted by $|E|$ . Two vertices $x$ and $y$ in $G(V, E)$ are said to be *adjacent* or *neighbors* if $\{x, y\}$ is an edge of $G$. The neighbor set of a vertex x of $G(V, E)$ is the set of all elements in $V$ which are adjacent to x and it is denoted by $N(x)$. The *degree* of vertex $x$ is defined as the number of edges incident on $x$ and it is denoted by $d(x)$ or equivalently $deg(x) = |N(x)|$.

A closed path in a graph $G(V, E)$ is called an Euler circuit if it includes every edge exactly once.

A connected graph $G(V, E)$ that contains an Euler circuit is called an Euler graph or Eulerian graph.

A closed path in a graph $G(V, E)$ is called a Hamiltonian cycle if it includes every vertex of $G(V, E)$ exactly once.

A graph that contains a Hamiltonian cycle is called a Hamiltonian graph.

Let $G(V, E)$ be a graph. Then twice the number of edges of graph $G(V, E)$ is sum of the degrees of all vertices belong to $V$.

A graph is an Eulerian if and only if it is connected and degree of every vertex is even.

## 3. Eulerian and Hamiltonian algebraic graphs

In this section, we introduce the notion of algebraic graph, Eulerian, Hamiltonian and complete algebraic graphs and study the properties of these graphs.

Definition 3.1. A graph $G(V, E)$ is said to be *algebraic graph* if there are bijective functions $f_i : V_i \to M_i$, $(i = 1, 2, \ldots, n)$ where $V_i$ and $M_i$ are subsets of $V$ and functions satisfying the following conditions.

(i) $d(f_1) \geqslant d(f_2) \geqslant \ldots \geqslant d(f_n)$ and $\bigcup_{i=1}^{n} V_i \cup M_i = V$
and there is no function defined on $M$ which is a subset of $V$ if $o(M) > o(V_1)$,
(ii) If $\{a, b\} \in E$, then there exists a unique function $f_i$ such that $f_i(a) = b$ .
(iii) If there is a path $\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \ldots \{v_{n-1}, v_n\} \in E$, then there exists a function $f_i$, such that $f_i(v_1) = v_2, f_i(v_2) = v_3, f_i(v_3) = v_4,$
$\ldots, f_i(v_{n-1}) = v_n$.

The number of elements in a set $M$ is denoted by $o(M)$.
An algebraic graph of graph $G(V, E)$ is denoted by $G(V, E, F)$.

Definition 3.2. Let $G(V, E, F)$ be an algebraic graph where $F = \{f_i \,|\, i = 1, 2, \ldots, n\}$. The *degree* of function $f \in F$ is defined as the number of elements in the domain of $f$.

Definition 3.3. Let $G(V, E, F)$ be an algebraic graph where $F = \{f_i \,|\, i = 1, 2, \ldots, n\}$. *Size* of an algebraic graph is defined as $n$ if $|E| = n$ and *order* of an algebraic graph is defined as $|V|$.

Definition 3.4. *Diameter* of an algebraic graph $G(V, E, F)$ where

$$F = \{f_i \,|\, i = 1, 2, \ldots, n\}, \ d(f_1) \geqslant d(f_2) \geqslant \ldots \geqslant d(f_n),$$

is defined as degree of $f_1$.

Example 3.1. Define graph

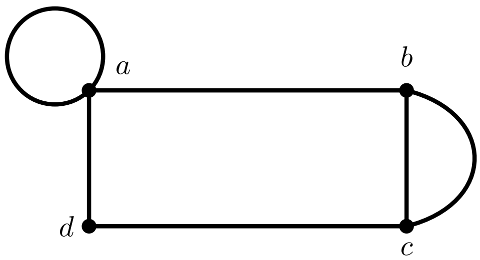


Figure 1. Graph

$f_1 = \begin{pmatrix} a & b & c & d \\ b & c & d & a \end{pmatrix}$ and $f_2 = \begin{pmatrix} a & b \\ a & c \end{pmatrix}$. $d(f_1) + d(f_2) = 6$. Here $F = d\{f_1, f_2\}$. Therefore $\sum d(F) = |E|$.

Example 3.2. Let $G(V, E)$ be a graph with vertices $V = \{a, b, c, d, e\}$ and edges $E = \{\{a, c\}, \{a, d\}, \{d, b\}, \{b, e\}, \{c, b\}\}$. Define a function $f_1$ on $V_1 = \{c, a, d, b\}$ such that $f_1(c) = a, f_1(a) = d, f_1(d) = b, f_1(b) = e$ and a function $f_2$ on $V_2 = \{c\}$

by $f_2(c) = b$. Therefore degree of $f_1 = 4$ and degree of os $f_2 = 1$. Therefore number of edges = 4+1=5. $G(V, E, F)$ where $F = \{f_1, f_2\}$ is an algebraic graph and diameter of an algebraic graph $G(V, E, F)$ is 4.

EXAMPLE 3.3. Let $G(V, E)$ be a graph shown in the figure 2, where $V = \{a, b, c, d, e\}$ and $E = \{\{a, e\}, \{a, d\}, \{e, d\}, \{a, c\}, \{e, b\}, \{d, c\}, \{d, b\}\}$. Define functions $f_1 : \{a, e, b, d, c\} \to V$ by $f_1 = \begin{pmatrix} a & e & b & d & c \\ e & b & d & c & a \end{pmatrix}$ and $f_2 : \{a, d\} \to V$ by $f_2 = \begin{pmatrix} a & d \\ d & e \end{pmatrix}$. Define functions $g_1 : \{a, e, d, c\} \to V$ by $g_1 = \begin{pmatrix} a & e & d & c \\ e & d & c & a \end{pmatrix}$ and $g_2 : \{e, b, d\} \to V$ by $g_2 = \begin{pmatrix} e & b & d \\ b & d & a \end{pmatrix}$.

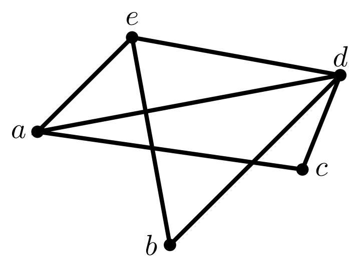


FIGURE 2. Diagram of graph $G(V, E)$

We observe that $G(V, E, F_1)$ where $F_1 = \{f_1, f_2\}$ is an algebraic graph and $G(V, E, F_2)$ where $F_2 = \{g_1, g_2\}$ is not an algebraic graph,since domain of $g_1$ is a proper subset of $f_1$.Hence $F_2$ is not satisfying the condition (i) in Definition 3.1

EXAMPLE 3.4. Let $G(V, E)$ be a bipartite graph, shown in the figure 2, with vertices $V = \{a, b, c, d, e\}$ and $E = \{\{a, d\}, \{a, e\}, \{b, d\}, \{b, e\}, \{c, d\}, \{c, e\}\}$. We define functions by $f_1 = \begin{pmatrix} a & d & b & e & c \\ d & b & e & c & d \end{pmatrix}$ and $f_2 = \begin{pmatrix} a \\ e \end{pmatrix}$.We observe that $G(V, E, F)$ where $F = \{f_1, f_2\}$ is an algebraic graph

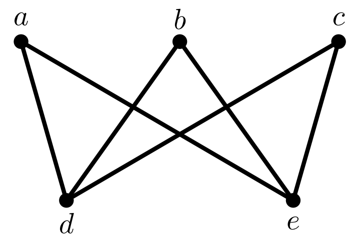


FIGURE 3. Bipartite graph $G(V, E)$.

EXAMPLE 3.5. Let $G(V,E)$ be a graph with $V = \{a,b,c,d\}$ and $E = \{\{a,b\}, \{b,c\}\}$. We cannot define a function on any subset of $V$ which contains $d$. Hence $G(V,E)$ is not an algebraic graph.

THEOREM 3.1 (**The first theorem of algebraic graph theory**).
*Let $G(V,E,F)$ be an algebraic graph. Then the number of edges of an algebraic graph $G(V,E,F)$ is the sum of the degrees of all functions belong to $F$.*

PROOF. Let $G(V,E,F)$ be an algebraic graph and $F = \{f_i \mid i = 1,2,\ldots,n\}$. Suppose the degree of each function $f_i$ is $d_i$. By the Definition [3.3],if the *degree* of function $f \in F$ is n, then there are n edges $\in E$. Therefore the number of edges in $E = d_1 + d_2 + \ldots + d_n$. Hence number of edges of algebraic graph is the sum of degrees of all functions belong to $F$. Hence the theorem. □

DEFINITION 3.5. Let $G(V,E,F)$ where $F = \{f_i \mid i = 1,2,\ldots,n\}$ be an algebraic graph. *Size* of an algebraic graph is defined as $n$ if $|E| = n$ and *order* is defined as $|V|$.

DEFINITION 3.6. Let $G(V,E,F)$ where $F = \{f_i \mid i = 1,2,\ldots,n\}$ be an algebraic graph. The *degree* of function $f \in F$ is defined as the number of elements in the domain of $f$and it is denoted by d(f).

DEFINITION 3.7. *Diameter* of an algebraic graph $G(V,E,F)$ where

$$F = \{f_i \mid i = 1,2,\ldots,n\} \text{ and } d(f_1) \geqslant d(f_2) \geqslant \ldots \geqslant d(f_n),$$

is defined as degree of $f_1$ .

NOTE 1. An algebraic graph is a graph but a graph need not be an algebraic graph.

NOTE 2. Let $G(V,E,F)$ be an algebraic graph. If $f \in F$ and

$$f = \begin{pmatrix} a & b & c & d & e \\ b & c & d & e & a \end{pmatrix}$$

then by definition of algebraic graph there is a closed path from $a-b-c-d-e-a$.

NOTE 3. Let $G(V,E,F)$ be an algebraic graph. Suppose the functions $f,g \in F$ such that $f = \begin{pmatrix} a & b & c & d & e \\ b & c & d & e & b \end{pmatrix}$ and $g = \begin{pmatrix} b & d \\ d & a \end{pmatrix}$. By definition of algebraic graph there is a closed path from $a-b-c-d-e-b-d-a$.

NOTE 4. Let $G(V,E,F)$ be an algebraic graph. If all the degrees of vertices are two then there is a only one cyclic permutation on $V$.

NOTE 5. Let $G(V,E,F)$ be an algebraic graph. If all the degrees of vertices are two except one vertex of degree is 3 then there are only two functions in $F$ and one function is a cyclic permutation on $V$.

NOTE 6. Let $G(V,E,F)$ be an algebraic graph.If the *degree* of function $f \in F$ is n then there are n edges $\in E$

EXAMPLE 3.6. Let $G(V,E)$ be a graph with $V = \{a,b,c,d,e,f\}$ and $E = \{\{a,b\},\{b,c\},\{c,d\},\{d,e\},\{e,f\},\{f,a\}\}$. We define a cyclic permutation $f_1 : V \rightarrow V$ such that $f_1(a) = b$, $f_1(b) = c$, $f_1(c) = d$, $f_1(d) = e$, $f_1(e) = f$, $f_1(f) = a$. Hence $G(V,E,F)$ where $F = \{f_1\}$ is an algebraic graph and *diameter* of the algebraic graph $G(V,E,F)$ is 6.

EXAMPLE 3.7. Let $G(V,E)$ be a graph with $V = \{a,b,c,d,e\}$ and $E = \{\{a,c\},\{a,d\},\{d,b\},\{b,e\},\{c,b\}\}$. Define a function $f_1$ on $V_1 = \{c,a,d,b\}$ such that $f_1(c) = a, f_1(a) = d, f_1(d) = b, f_1(b) = e$.and a function $f_2$ on $V_2 = \{c\}$ by $f_2(c) = b$. Obviously degree of $f_1 = 4$ *and degree of* $f_2 = 1$. Therefore $G(V,E,F)$ where $F = \{f_1, f_2\}$ is an algebraic graph and diameter of an algebraic graph $G(V,E,F)$ is 4 and number of edges = 4+1=5.

DEFINITION 3.8. An algebraic graph $G(V,E,F)$ is said to be Hamiltonian algebraic graph if there is a function $f \in F$ such that $f$ is a cyclic permutation on V.

DEFINITION 3.9. An algebraic graph $G(V,E,F)$ is said to be Eulerian algebraic graph if there exists a subset $F_1$ of F and each $f_i \in F_1$ is a cyclic permutation on $V_i$, $\cup V_i = V$ and $\cap V_i \neq \phi$

DEFINITION 3.10. An algebraic graph $G(V,E,F)$ is said to be $k - regular$ algebraic graph if $deg(v) = k$ for all $v \in V$.

DEFINITION 3.11. An algebraic graph $G(V,E,F)$ is said to be complete if there exists a subset $F_1$ of F such that $|F_1|$= k where $k$ is a natural number such that $|E| - k|V| < |V|$ and each $f \in F_1$ is a cyclic permutation on $V$

THEOREM 3.2. *A graph $G(V,E)$ is an Eulerian if and only if algebraic graph $G(V,E,F)$ is an Eulerian algebraic graph*

PROOF. Suppose $G(V,E)$ is an Eulerian, then there exists a closed path

$$v_1 - v_2 - v_3 - v_4 - v_1 - v_5 - v_6 \cdot \cdots - v_n - v_1.$$

Define functions

$$f_1 = \begin{pmatrix} v_1 & v_5 & \cdots & v_n \\ v_5 & v_6 & \cdots & v_1 \end{pmatrix} \text{ and } f_2 = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 \\ v_2 & v_3 & v_4 & v_1 \end{pmatrix}.$$

Then algebraic graph $G(V,E,F)$ where $F = \{f_1, f_2\}$ and $f_1, f_2$ are cyclic permutations. Therefore by Definition [3.9], $G(V,E,F)$ is an Eulerian algebraic graph . Converse is obvious. □

COROLLARY 3.1. *A graph $G(V,E)$ is a Hamitonian if and only if algebraic graph $G(V,E,F)$ is a Hamitonian*

THEOREM 3.3. *If a graph $G(V,E)$ is complete, then algebraic graph $G(V,E,F)$ is a complete algebraic graph*

THEOREM 3.4. *If an algebraic graph $G(V,E,F)$ is $2n - regular$, then there are $n$ Hamiltonian disjoint cycles.*

Proof. Suppose $G(V,E,F)$ is $2n-regular$. Then $\sum d(f_i) = 2n\frac{|V|}{2}$. Thus $\sum d(f_i) = n|V|$ and $d(f_i) = |V|$, for all $i$. Therefore there are $n$ Hamiltonian disjoint cycles. □

Theorem 3.5. *Let $G(V,E,F)$ be a complete algebraic graph with n vertices each $f \in F$ is a cyclic permutation on $V$. Then the number of functions in $F$ is $(n-1)/2$ if n is odd.*

Proof. We know that maximum number of edges in any simple graph with $n$ vertices is $n(n-1)/2$ and let $F = \{f_i, i = 1, 2, \cdots, k\}$. Then by algebraic graph first theorem we have $\sum d(f_i) = n(n-1)/2$. We know that maximum degree of each function is $n$

$$\Rightarrow n + n + \cdots k\ terms = n(n-1)/2$$
$$\Rightarrow kn = n(n-1)/2$$
$$\Rightarrow k = (n-1)/2.$$

Therefore maximum number of functions in $F$ is
$(n-1)/2$ if n is odd □

Theorem 3.6. *Let $G(V,E)$ be a complete graph and order of $V$ be $n$ ($n$ is odd). Then there are $(n-1)/2$ edge disjoint Hamiltonian cycles.*

Proof. Suppose $G(V,E)$ is a complete graph then $G(V,E,F)$ is an algebraic complete graph. Therefore number of edges in the algebraic graph is $n(n-1)/2$. Since $G(V,E,F)$ is a complete algebraic graph,we have $F = \{f_i\}$ $i = 1, 2, \cdots, k$ and each $f_i$ is a cyclic permutation on $V$. Hence there are $k$ cyclic permutations on $V$. We know that every cyclic permutation represents a Hamiltonian cycle. Therefore

$$o(f_1) + o(f_2) + \cdots + o(f_k) = n(n-1)/2$$
$$\Rightarrow nk = n(n-1)/2$$
$$Therefore\ k = (n-1)/2.$$

Hence there are $(n-1)/2$ edge disjoint Hamiltonian cycles. □

Theorem 3.7. *The $G(V,E,F)$ be a complete algebraic graph with n vertices ($n \geqslant 3$) is an Eulerian if n is odd.*

Proof. Let $G(V,E,F)$ be a complete algebraic graph of order $n$, $n \geqslant$ and $n$ is odd. We know that the maximum number of edges in complete graph $G(V,E) = n(n-1)/2$. By Theorem [3.6],there are $(n-1)/2$ edge disjoint Hamiltonian cycles. Therefore every complete algebraic graph of order n is an Eulerian □

Example 3.8. Let $G(V,E)$ be a graph with vertices $V = \{a,b,c,d,e,f\}$ and edges $E = \{\{a,b\},\{b,c\},\{c,d\},\{d,e\},\{e,f\},\{f,a\}\}$. We define a cyclic perumutation $f_1 : V \to V$ such that $f_1(a) = b, f_1(b) = c, f_1(c) = d, f_1(d) = e, f_1(e) = f, f_1(f) = a$. Hence $G(V,E,F)$ where $F = \{f_1\}$ is Hamiltonian and Eulerian algebraic graph and *diameter* of an algebraic graph $G(V,E,F)$ is 6.

EXAMPLE 3.9. Let $G(V, E)$ be a bipartite graph with vertices $V = \{a, b, c, d, e\}$ and $E = \{\{a, d\}, \{a, e\}, \{b, d\}, \{b, e\}, \{c, d\}, \{c, e\}\}$. We define functions by $f_1 = \begin{pmatrix} a & d & b & e & c \\ d & b & e & c & d \end{pmatrix}$ and $f_2 = \begin{pmatrix} a \\ e \end{pmatrix}$. We observe that $G(V, E, F)$ where $F = \{f_1, f_2\}$ is not a Hamiltonian algebraic graph

EXAMPLE 3.10. Let $G(V, E)$ be a graph where $V = \{a, b, c, d, e\}$ and $E = \{\{a, e\}, \{a, d\}, \{e, d\}, \{a, c\}, \{e, b\}, \{d, c\}, \{d, b\}\}$. The graph shown in the figure 4. Define $f_1 : \{a, e, b, d, c\} \to V$ by $f_1 = \begin{pmatrix} a & e & b & d & c \\ e & b & d & c & a \end{pmatrix}$ and $f_2 : \{a, d\} \to V$ by $f_2 = \begin{pmatrix} a & d \\ d & e \end{pmatrix}$. Define $g_1 : \{a, e, d, c\} \to V$ by $g_1 = \begin{pmatrix} a & e & d & c \\ e & d & c & a \end{pmatrix}$ and $g_2 : \{e, b, d\} \to V$ by $g_2 = \begin{pmatrix} e & b & d \\ b & d & a \end{pmatrix}$.

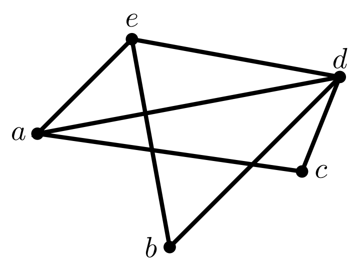


FIGURE 4. Diagram of graph $G(V, E)$

We observe that $G(V, E, F_1)$ where $F_1 = \{f_1, f_2\}$ is a Hamiltonian algebraic graph and $G(V, E, F_2)$ where $F_2 = \{g_1, g_2\}$ is not an algebraic graph.

EXAMPLE 3.11. Consider the graph shown in the figure 5. Define a function

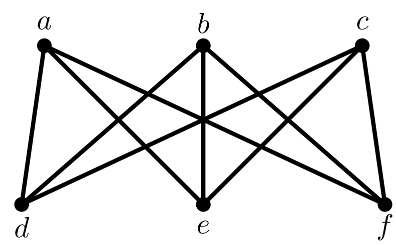


FIGURE 5. Graph

$\lambda : \{a, b, c, d, e, f\} \to V$ by $\lambda = \begin{pmatrix} a & d & b & e & c & f \\ d & b & e & c & f & a \end{pmatrix}$ and $\mu : \{a, b, c\} \to V$ by

$\mu = \begin{pmatrix} a & c & b \\ e & d & f \end{pmatrix}$. Therefore degree($\lambda$) = 6 and degree($\mu$) = 3 . Here $\lambda$ is a cyclic permutation of degree 6. Hence complete bipartite graph $k_{3,3}$ is a Hamiltonian.

THEOREM 3.8. *Every complete algebraic graph $G(V, E, F)(n \geqslant 3)$ is a Hamiltonian algebraic graph .*

PROOF. Suppose $G(V, E, F)$ is a complete algebraic graph of order $n$ and $n \geqslant 3$. By the definition of complete algebraic graph there exists a $f_1 \in F$ such that $f_1$ is a cyclic permutation on $V$ and $d(f_1) = n$. Hence by Theorem [ 2.7], $G(V, E, F)$ is a Hamiltonian graph. □

COROLLARY 3.2. *Every complete algebraic graph $G(V, E, F)(n \geqslant 3$ and $n$ is odd) is Eulerian and Hamiltonian.*

THEOREM 3.9. *A graph $G(V, E)$ is an Eulerian graph if and only if its algebraic graph $G(V, E, F)$ has only cyclic permutations whose domains are not disjoint and union of domains is $V$.*

PROOF. Suppose $G(V, E, F)$ has only two cyclic permutations $f$ and $g$ whose domains $A$ and $B$ are not disjoint and union of domains is V. Let $v \in A \cap B$. Since $f$ and $g$ are only cyclic permutations on $A$ and $B$ respectively, there exist closed paths $v - v_1 - v_2 - v_3 - \cdots - v$ and $v - v_1' - v_2' - v_3' - \cdots - v$.Therefore $v - v_1 - v_2 - v_3 - \cdots - v - v_1' - v_2' - v_3' - \cdots - v$ is a circuit. Hence $G(V, E)$ is an Eulerian graph.

Converse is obvious. □

COROLLARY 3.3. *A regular algebraic graph $G(V, E, F)$ has only cyclic permutations on $V$ if and only if all vertices of $G$ are of even degree.*

THEOREM 3.10. *If an algebraic graph $G(V, E, F)$ is a $2-regular$ algebraic graph then $F = \{f\}$ and $d(f) = |V|$.*

PROOF. Suppose $G(V, E, F)$ is a $2-$*regular* algebraic graph. Then there exists only one cyclic permutation $f$ on $V$. Hence by the first theorem on algebraic graph theory, $d(f) = |E| = |V|$. □

NOTE 7. Let $G(V, E, F)$ be an algebraic graph. If all the degrees of vertices are two except one vertex of degree 3 then there are only two functions in $F$ and one is a cyclic permutation and other is a function.

EXAMPLE 3.12. Consider the graphs $(G(V_1, E_1))$, $(G(V_2, E_2))$ shown in the figure 6. Define functions

$$f_1 = \begin{pmatrix} a & b & c & d & e & f & g \\ b & c & d & e & f & g & a \end{pmatrix} \text{ and } f_2 = \begin{pmatrix} a & c & e & g & b & d & f \\ c & e & g & b & d & f & a \end{pmatrix}.$$

Here $f_1$ and $f_2$ are cyclic permutations. Therefore there are two disjoint Hamiltonian cycles.

Define functions

$$g_1 = \begin{pmatrix} p & q & r & s & t & u & v \\ q & r & s & t & u & v & p \end{pmatrix} \text{ and } g_2 = \begin{pmatrix} p & s & v & r & u & q & t \\ s & v & r & u & q & t & p \end{pmatrix}.$$

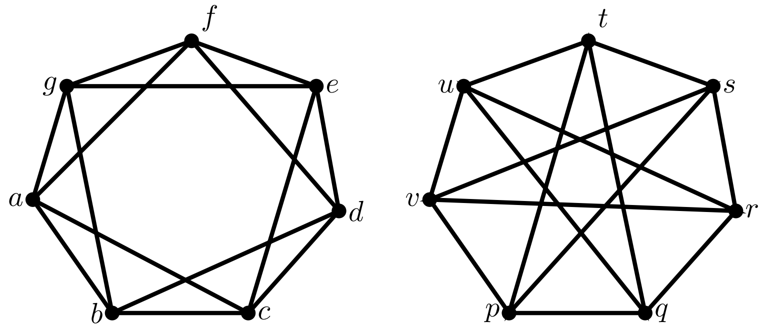


FIGURE 6. Graphs $(G\,(V_1, E_1))$,$(G\,(V_2, E_2))$

Here $g_1$ and $g_2$ are cyclic permutations. Therefore there are two disjoint Hamiltonian cycles.

THEOREM 3.11. *An algebraic graph $G(V, E, F)$ is an Eulerian, $|V| = n$ and $|E| = n+3$ if and only if there are at least two cyclic permutations of degrees nand 3.*

PROOF. Suppose $G(V, E, F)$ is an algebraic graph with $V = n$ , $|E| = n+3$ and there exist functions $f, g \in F$ are cyclic permutations of degree $n$ and 3 respectively, that is $f = \begin{pmatrix} v_1 & v_2 & v_3 \cdots & v_{n-1} & v_n \\ v_2 & v_3 & v_4 \cdots & v_n & v_1 \end{pmatrix}$ and $g = \begin{pmatrix} v_2 & v_4 & v_6 \\ v_4 & v_6 & v_2 \end{pmatrix}$. Therefore $G(V, E, F)$ where $F = \{f, g\}$ is an Eulerian algebraic graph.

Converse is obvious. □

EXAMPLE 3.13. Consider the graphs $(G\,(V_3, E_3))$,$(G\,(V_4, E_4))$ shown in the figures 7. Define a function $\lambda = \begin{pmatrix} a & b & c & d & e & f \\ b & c & d & e & f & a \end{pmatrix}$. Here $F_1 = \{\lambda\}$ and $\lambda$ is a cyclic permutation. Therefore $(G\,(V_1, E_1, F_1))$ is an algebraic graph and it is a Hamiltonian algebraic graph and Eulerian algebraic graph.

Define a function $\mu = \begin{pmatrix} p & s & q & t & r & u \\ s & q & t & r & u & p \end{pmatrix}$. Here $F_2 = \{\mu\}$ and $\mu$ is a cyclic permutation. Therefore $(G\,(V_2, E_2, F_2))$ is an algebraic graph and it is a Hamiltonian algebraic graph and Eulerian algebraic graph.

THEOREM 3.12. *If an algebraic graph $G(V, E, F)$, $F = \{f\}$ and $f$ is a cyclic permutation on $V$, then the graph $G(V, E)$ is Hamiltonian and Eulerian.*

PROOF. Suppose $G(V, E, F)$ is an algebraic graph, $o(V) \geqslant 3$ and $F = \{f\}$ where $f$ is a cyclic permutation on $V$. Let $V = \{v_1, v_2, \cdots, v_n\}$ and

$$f = \begin{pmatrix} v_1 & v_2 & v_3 \cdots & v_n \\ v_2 & v_3 & v_4 \cdots & v_1 \end{pmatrix}.$$

Therefore there exists a closed path. Hence $G(V, E, F)$ is Eulerian and Hamiltonian algebraic graph. □

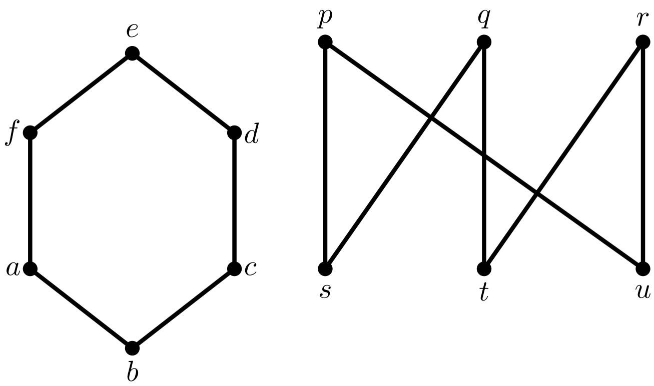


FIGURE 7. Graph $(G(V_3, E_3))$,$(G(V_4, E_4))$

COROLLARY 3.4. *If an algebraic graph $G(V, E, F)$ where $F$ does not contains cyclic permutation on $V$ or cyclic permutation on subsets of $V$ and whose union is $V$, then the graph $G(V, E)$ is neither Hamiltonian nor Eulerian.*

THEOREM 3.13. *Let $G(V, E, F)$ be an algebraic graph of order $n$ $(n \geqslant 3)$. Then graph $G(V, E)$ of order $n$ is a Hamiltonian if and only if a function $f \in F$ such that $f$ is a cyclic permutation on $V$.*

PROOF. Suppose $G(V, E)$ is a Hamiltonian graph of order $n$.Therefore there is a closed path $v_1 - v_2 - v_3 - v_4 \cdots v_{n-1} - v_n - v_1$ in $G$. Then by the definition of algebraic graph there exists a function $f \in F$ such that $f$ is a cyclic permutation on $V$ and $f = \begin{pmatrix} v_1 & v_2 & v_3 \cdots & v_n \\ v_2 & v_3 & v_4 \cdots & v_1 \end{pmatrix}$.

Converse is obvious. Hence the theorem. □

## 4. Conclusion

In this paper, we introduced the notion of Eulerian algebraic graph and the notion of Hamiltonian algebraic graph. We studied the properties of Hamiltonian and Eulerian algebraic graphs and we proved that a graph $G(V, E)$ is a Hamiltonian, $|V| = n$ if and only if there is a cyclic permutation of degree $n$. In continuation of this paper we introduce subgraphs and planar graphs of algebraic graph theory

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