BULLETIN OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4874, ISSN (o) 2303-4955
www.imvibl.org /JOURNALS / BULLETIN
Vol. 9(2019), 169-175
DOI: 10.7251/BIMVI1901169B

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

MONOTONICITY AND INEQUALITIES RELATED TO GAMMA FUNCTION

Ravi Bhukya and Venkata Lakshmi Akavaram

Abstract. In this paper, the authors establish some monotonicity and inequalities related Gamma function by using the method of analysis and theory of inequality.

1. Introduction

The Euler gamma function $\Gamma(x)$ is defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} \mathrm{dt} \tag{1.1}$$

The logarithmic derivative of the gamma function $\Gamma(x)$ is denoted by

$$\Psi(x) = \frac{d}{dx}(\log(\Gamma(x))) = \frac{\Gamma'(x)}{\Gamma(x)}$$

is called the digamma or psi function. The gamma and digamma or psi function has been investigated intensively by many authors even recent years. In particular, many authors have published numerous interesting inequalities for this important function (see [**8**] **-** [**10**] and references therein). In this paper, we are interested to approximate for gamma function $\Gamma(x+1)$ on the interval $(2, \infty)$. In [**2**, Theorem. 8], Alzer proved that the function

$$R_n(x) = (-1)^n \left[\ln \Gamma(x) - \left(x - \frac{1}{2} \right) \ln x - \ln \sqrt{2\pi} - \sum_{j=1}^{n} \frac{B_{2j}}{2j(2j-1)x^{2j-1}} \right]$$

2010 *Mathematics Subject Classification.* Primary 33B15; Secondary 26D07.
Key words and phrases. Inequalities, Gamma function, Monotonicity.

is completely monotonic on $(0,\infty)$. Here B_n denote the Bernoulli numbers. This gives the following inequalities:

$$\sqrt{2\pi x}\left(\frac{x}{e}\right)^x \exp\left(\sum_{j=1}^{2n}\frac{B_{2j}}{2j(2j-1)x^{2j-1}}\right)$$

$$< \Gamma(x+1) < \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \exp\left(\sum_{j=1}^{2n+1}\frac{B_{2j}}{2j(2j-1)x^{2j-1}}\right), x>0 \tag{1.2}$$

In particular, we have, by (1.2),

$$\Gamma(x+1) < \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \exp\left(\frac{1}{12x}\right), x>0 \tag{1.3}$$

In this paper, we are going to improve the right hand side of the double inequality (1.2) on the interval $(2,\infty)$ we also establish some inequalities related to the gamma function and generalise following results from [**11**, Page. No. 193].

If $n \geqslant 2$ is an integer, then

$$\left[\frac{n(n+1)^3}{8}\right]^n > (n!)^4. \tag{1.4}$$

If $n \geqslant 2$ is an integer, then

$$2!4!\cdots(2n)! > ((n+1)!)^n. \tag{1.5}$$

If $n > 2$ is an integer, then

$$n! < 2^{\frac{n(n-1)}{2}}. \tag{1.6}$$

Our main results are stated as follows

(1) $\left[\dfrac{x(x+1)^3}{8}\right]^x > \Gamma(x+1)^4, x>1.$
(2) $\dfrac{\Gamma(1+e)}{e^{e/2}}x^{x/2} \leqslant \Gamma(x+1) \leqslant \left(\dfrac{\Gamma(1+e)}{(\frac{1+e}{2})^e}\right)\left(\dfrac{x+1}{2}\right)^x, x \geqslant e.$
(3) $\Gamma(x+1) < 2^{\frac{x(x-1)}{2}}, x>2.$
(4) $\dfrac{2^x}{x} < \dfrac{x^x}{\Gamma(x+1)} < 3^{x-1}, x>2.$
(5) $\Gamma(x+3)\Gamma(x+5)\Gamma(x+7)\ldots\Gamma(x+2n+1) > \Gamma(x+n+2)^n.$

The following lemma ([**2**], [**7**]) is used in proofs

LEMMA 1.1. *For $x>0$ we have*

(1) $\log(x) - \frac{1}{x} < \Psi(x) < \log(x) - \frac{1}{2x}$
(2) *For* $\beta \geqslant \frac{1}{2}$, *we have* $\Psi(x+\beta) \geqslant \log(x)$

2. The Main Results

THEOREM 2.1. *For $x > 2$, we have*

$$\frac{x^{x-1}}{3^{x-1}} < \Gamma(x+1) < \frac{x^x}{2^x}. \tag{2.1}$$

PROOF. Let $f(x) = x^x/\Gamma(x+1)2^x$. Define $g(x) = \log(f(x))$. Then

$$g(x) = x\log(x) - x\log(2) - \log(\Gamma(x+1)).$$

Differentiating with respect to x, we have

$$g'(x) = 1 - \log(2) + \log(x) - \Psi(x+1)$$

since, for $x > 0$, $\log(x) - \frac{1}{x} < \Psi(x) < \log(x) - \frac{1}{2x}$

$$\begin{aligned} g'(x) >& \frac{1}{2(x+1)} - \log(x+1) + \log(x) + 1 - \log(2) \\ =& \frac{1}{2(x+1)} + \log\left(\frac{x}{x+1}\right) + 1 - \log(2) \end{aligned}$$

The range of $\frac{1}{2(x+1)} + \log\left(\frac{x}{x+1}\right) + 1 - \log(2)$ is $(\frac{7}{6} - \ln(3), 1 - \ln(2))$, for $x > 2$. And $\frac{7}{6} - \ln(3) > 0$. Thus, g is increasing on $(2, \infty)$, which implies that f is increasing on $(2, \infty)$. For $2 < x$, we have $1 < f(2) < f(x)$, implies right side the inequality (2.1).

For the other part, let $f(x) = \Gamma(x+1)3^{x-1}/x^{x-1}$. Define $g(x) = \log(f(x))$. Then

$$g(x) = (x-1)\log(3) + \log\Gamma(x+1) - (x-1)\log(x).$$

Differentiating with respect to x, we have

$$g'(x) = \log(3) + \Psi(x+1) - 1 + \frac{1}{x} - \log(x)$$

since, for $x > 0$, $\log(x) - \frac{1}{x} < \Psi(x) < \log(x) - \frac{1}{2x}$

$$\begin{aligned} g'(x) >& \log(x+1) - \frac{1}{x+1} + \log(3) - \log(x) - 1 + \frac{1}{x} \\ =& \log\left(\frac{x+1}{x}\right) + \frac{1}{x(x+1)} + \log(3) - 1 \end{aligned}$$

The range of $\log\left(\frac{x+1}{x}\right) + \frac{1}{x(x+1)} + \log(3) - 1$ is $\left(\ln(3) - 1, \ln\left(\frac{9}{2}\right) - \frac{5}{6}\right)$, for $x > 2$. And $\ln(3) - 1) > 0$. Thus, g is increasing on $(2, \infty)$, which implies that f is increasing on $(2, \infty)$. For $2 < x$, we have $f(2) < f(x)$, implies left side of the inequality (2.1). $\square$

REMARK 2.1. From the inequality (1.3), the right hand side of the inequality (2.1) is shaper than the right hand side of the inequality (1.2) for all $x > 2$.

REMARK 2.2. The left hand side of the inequality (2.1) is holds for all $x > 0$.

THEOREM 2.2. *For $x > 1$, we have*

$$\left[\frac{x(x+1)^3}{8}\right]^x > \Gamma(x+1)^4. \tag{2.2}$$

PROOF. Let $f(x) = \left[\frac{x(x+1)^3}{8}\right]^x / \Gamma(x+1)^4$. Define $g(x) := \log(f(x))$. We have

$$g(x) = x\big[\log(x) + 3\log(x+1) - \log(8)\big] - 4\Gamma(x+1).$$

Then

$$\begin{aligned} g'(x) &= \log(x) + 3\log(x+1) - \log(8) + 1 + \frac{3}{x+1} - 4\Psi(x+1) \\ &= \log(\frac{x}{8}) + 3\log(x+1) + 1 + \frac{3x}{x+1} - 4\Psi(x+1) \end{aligned}$$

since, $\log(x) - \frac{1}{x} < \Psi(x) < \log(x) - \frac{1}{2x}$,

$$\begin{aligned} g'(x) &> \log(\frac{x}{8}) + 3\log(x+1) + 1 + \frac{3x}{x+1} + \frac{4}{2(x+1)} - 4\log(x+1) \\ &> \log\left(\frac{x}{8(x+1)}\right) + 1 + \frac{3x+2}{x+1} >, 0 \quad \text{for } x > 1. \end{aligned}$$

Thus, g is increasing on $(1, \infty)$, which implies that f is increasing on $(1, \infty)$. For $1 < x$, we have $f(1) < f(x)$, using (1.4) we have the inequality (2.2). □

REMARK 2.3. After letting $x = n$ the inequality (2.2) becomes

$$\left[\frac{n(n+1)^3}{8}\right]^n > (n!)^4, n > 2.$$

which is same as (1.4).

THEOREM 2.3. *For $x \geqslant e$, we have*

$$\frac{\Gamma(1+e)}{e^{e/2}} x^{x/2} \leqslant \Gamma(x+1) \leqslant \left(\frac{\Gamma(1+e)}{(\frac{1+e}{2})^e}\right)\left(\frac{x+1}{2}\right)^x. \tag{2.3}$$

PROOF. Let $f(x) = \left(\frac{x+1}{2}\right)^x / \Gamma(x+1)$. Define $g(x) := \log(f(x))$, we have

$$g(x) = x\log(\frac{x+1}{2}) - \log\Gamma(x+1).$$

Then

$$\begin{aligned} g'(x) &= \log(\frac{x+1}{2}) + \frac{x}{x+1} - \Psi(x+1) \\ &= \log(x+1) - \log(2) + \frac{x}{x+1} - \Psi(x+1) \end{aligned}$$

since, $\log(x) - \frac{1}{x} < \Psi(x) < \log(x) - \frac{1}{2x}$,

$$\begin{aligned} g'(x) >& \frac{1}{2(x+1)} - \log(x+1) - \log(2) + \log(x+1) + \frac{x}{x+1} \\ >& 1 - \log(2) - \frac{1}{2(x+1)} > 0 \quad \text{for } x \geqslant e. \end{aligned}$$

Thus, g is increasing on $[e, \infty)$, which implies that f is increasing on $[e, \infty)$. For $e < x$, we have $f(e) \leqslant f(x)$. We have the righside inequality of (2.3).

For the other part, let $f(x) = \Gamma(x+1)/x^{x/2}$. Define $g(x) := \log(f(x))$, we have $g(x) = \log\Gamma(x+1) - \frac{x}{2}\log(x)$. Then

$$\begin{aligned} g'(x) =& \Psi(x+1) - \frac{x}{2}\frac{1}{x} - \frac{1}{2}\log(x) \\ =& \Psi(x+1) - \frac{1}{2} - \frac{1}{2}\log(x) \end{aligned}$$

since, for $x > 0$ and $\beta \geqslant \frac{1}{2}$, then $\Psi(x+\beta) \geqslant \log(x)$,

$$\begin{aligned} g'(x) \geqslant& \log(x) - \frac{1}{2} - \frac{1}{2}\log(x) \\ \geqslant& \frac{1}{2}\left[\log(x) - 1\right] \geqslant 0 \quad \text{for } x \geqslant e. \end{aligned}$$

Thus, g is increasing on $[e, \infty)$, which implies that f is increasing on $[e, \infty)$. For $e \leqslant x$, we have $f(e) \leqslant f(x)$,we have the leftside inequality of (2.3). $\square$

THEOREM 2.4. *For $x > 0$ and $n \geqslant 2$, we have*

$$\Gamma(x+3)\Gamma(x+5)\Gamma(x+7)\ldots\Gamma(x+2n+1) > \Gamma(x+n+2)^n. \tag{2.4}$$

PROOF. Let $f(x) = \Gamma(x+3)\Gamma(x+5)\Gamma(x+7)\ldots\Gamma(x+2n+1)/\Gamma(x+n+2)^n$. Define $g(x) = \log(f(x))$. Then

$$g(x) = \sum_{k=1}^{n} \log\Gamma(x+2k+1) - n\log\Gamma(x+n+2).$$

Differentiating with respect to x, we have

$$g'(x) = \sum_{k=1}^{n} \Psi(x+2k+1) - n\Psi(x+n+2)$$

since, for $x > 0$ and $\beta \geqslant \frac{1}{2}$, then $\Psi(x+\beta) \geqslant \log(x)$,

$$g'(x) \geqslant \sum_{k=1}^{n} \log(x) - n\log(x) = \sum_{k=1}^{n}\Big(\log(x) - \log(x)\Big) \geqslant 0.$$

Thus, g is increasing on $(0, \infty)$, which implies that f is increasing on $(0, \infty)$. For $0 < x$, we have $f(0) < f(x)$, using (1.5) we have the inequality (2.4). $\square$

REMARK 2.4. After letting $x = 0$ the inequality (2.4) becomes

$$2!4!\ldots(2n)! > ((n+1)!)^n, n \geqslant 2.$$

which is same as (1.5).

THEOREM 2.5. *For $x > 2$, we have*

$$\Gamma(x+1) < 2^{\frac{x(x-1)}{2}}. \tag{2.5}$$

PROOF. Let $f(x) = 2^{\frac{x(x-1)}{2}}/\Gamma(x+1)$. Define $g(x) = \log(f(x))$. Then

$$g(x) = \frac{x(x-1)}{2}\log(2) - \log\Gamma(x+1).$$

Differentiating with respect to x, we have

$$g'(x) = (\frac{2x-1)}{2})\log(2)\Psi(x+1)$$

since, for $x > 0$, $\log(x) - \frac{1}{x} < \Psi(x) < \log(x) - \frac{1}{2x}$

$$g'(x) > \frac{1}{2(x+1)} - \log(x+1) + (\frac{2x-1}{2})\log(2)$$

The range of $\frac{1}{2(x+1)} - \log(x+1) + (\frac{2x-1}{2})\log(2)$ is $\left(\frac{1}{6}\left(1 - 3\ln\left(\frac{9}{8}\right)\right), \infty\right)$, for $x > 2$. And $\left(1 - 3\ln\left(\frac{9}{8}\right)\right) > 0$. Thus, g is increasing on $(2, \infty)$, which implies that f is increasing on $(2, \infty)$. For $2 < x$, we have $f(2) < f(x)$, implies the inequality (2.5). $\square$

REMARK 2.5. After letting $x = n > 2$ the inequality (2.5) becomes $n! < 2^{\frac{n(n-1)}{2}}$. which is same as (1.6).

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Receibed by editors 25.05.2018; Revised version 21.11.2018; Available online 03.12.2018.

Department of Mathematics, Government College for men, Kurnool, India
E-mail address: ravidevi19@gmail.com

Department of Mathematics, Osmania University, Hyderabad, India
E-mail address: akavaramvlr@gmail.com