BULLETIN OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4874, ISSN (o) 2303-4955
www.imvibl.org /JOURNALS / BULLETIN
Vol. 9(2019), 159-168
DOI: 10.7251/BIMVI1901159H

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

PROPERTIES OF PRODUCT T_hC_ϕ OPERATORS FROM $\mathcal{B}^\alpha_{log^\beta}$ TO $Q_{K,\omega}(p,q)$ SPACES

Ibrahim Mohamed Hanafy, Alaa Kamal and M. Hamza Eissa

Abstract. In this paper, the authors introduce equivalent characterizations for the boundedness and compactness of the product composition operator and extended Cesáro operator from the weighted logarithmic α-Bloch-type space $\mathcal{B}^\alpha_{log^\beta}$ to $Q_{K,\omega}(p,q)$ spaces on unit disk.

1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disc in the complex plane $\mathbb{C}$, $H(\mathbb{D})$ denote the class of all analytic functions in $\mathbb{D}$. Let dA denote the Lebesegue measure on $\mathbb{D}$ normalized so that $A(\mathbb{D}) = 1$.

Following ([**5**]), for each $a \in \mathbb{D}$, $\varphi_a : \mathbb{D} \to \mathbb{D}$ denotes the Möbius transformations defined by

$$\varphi_a(z) := \frac{a-z}{1-\bar{a}z}, \quad \text{for } z \in \mathbb{D}.$$

Green's function of $\mathbb{D}$ with logarithmic singularity at a, define as follows

$$g(z,a) := \log\left|\frac{1-\bar{a}z}{z-a}\right| = \log\frac{1}{|\varphi_a(z)|}\ .$$

The study of composition operator C_ϕ acting on spaces of analytic functions has engaged many analysts for many years (see [**3, 4, 11, 13**] and others), readers interested in this topic can refer to (see [**12**]) the sources for the development of the theory of composition operators and function spaces.

Definition 1.1. For any analytic self-mapping ϕ of $\mathbb{D}$. The symbol ϕ induces a linear composition operator $C_\phi(f) := f \circ \phi$ from $H(\mathbb{D})$ into itself.

2010 *Mathematics Subject Classification.* Primary 47B38; Secondary 46E15.
Key words and phrases. Composition operator, extended Cesáro operator, compactness.

The problem of boundedness and compactness of C_ϕ has been studied in many Banach spaces of analytic functions and the study of such operators has recently attracted the most attention (see [**18, 19**] and others).

The following definition was first introduced in ([**9**])

DEFINITION 1.2. Let $h \in H(\mathbb{D})$, the extended Cesáro operator T_h with symbol h is the operator on $H(\mathbb{D})$,

$$T_h f(z) := \int_0^Z f(\xi)h'(\xi)d\xi, \qquad f \in H(\mathbb{D}),\ z \in \mathbb{D}.$$

This operator is called generalized Cesáro operator.

It has been studied in the following article [**6, 7, 8**] and other.

In our study, we consider the product of extended Cesáro operator T_h and of composition operator C_ϕ, which was first introduced and studied by the authors in ([**2**])

$$T_h C_\phi f(z) = \int_0^Z f(\phi(\xi))h'(\xi)d\xi, \qquad f \in H(\mathbb{D}),\ z \in \mathbb{D}.$$

The author in [**16**] introduced the definition of logarithmic Bloch-type space as follows

DEFINITION 1.3. Let $\alpha > 0$,$\beta \geqslant 0$ and f be an analytic function in $\mathbb{D}$ the logarithmic Bloch-type space $\mathcal{B}^\alpha_{log^\beta}$ is defined by

$$||f||_{\mathcal{B}^\alpha_{log^\beta}} = \left\{ f \in H(\mathbb{D}) : ||f||_{\mathcal{B}^\alpha_{log^\beta}} = \sup_{z\in\mathbb{D}} (1-|z|)^\alpha (\ln \frac{e^{\beta/\alpha}}{(1-|z|)})|f'(z)| < \infty \right\}.$$

Case 1: $\beta = 0$ then $\mathcal{B}^\alpha_{log^\beta}$ becomes the α-Bloch space $\mathcal{B}^\alpha$

Case 2: $\alpha = \beta = 1$ then $\mathcal{B}^\alpha_{log^\beta}$ becomes the logarithmic Bloch space.

The authors in [**14**] introduced the definition of $Q_{K,\omega}(p,q)$ which has attracted a lot of attention in recent years. It defined as follows

DEFINITION 1.4. Let $K : [0,\infty) \to [0,\infty)$ and $\omega : (0,1] \to (0,\infty)$, are right-continuous and nondecreasing functions. If $0 < p < \infty$, $-2 < q < \infty$, then an analytic function f in $\mathbb{D}$ is said to belong to the space $Q_{K,\omega}(p,q)$ if

$$||f||_{Q_{K,\omega}(p,q)} := \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p \frac{(1-|z|)^q}{\omega^p(1-|z|)}\ K(g(z,a))dA(z) < \infty.$$

The notation $A \asymp B$ means that there is a positive constant C such that $\frac{B}{C} \leqslant A \leqslant CB$.

2. Auxiliary results

In this section we state several results, which are used in the main result proofs. Now, we will introduce the definition of boundedness and compactness of the operator $T_h C_\phi : \mathcal{B}^\alpha_{log^\beta} \to Q_{K,w}(p,q)$.

DEFINITION 2.1. The operator $T_hC_\phi : \mathcal{B}^\alpha_{log^\beta} \to Q_{K,w}(p,q)$ is said to be bounded, if there is a positive constant C such that $||T_hC_\phi f||_{Q_{K,\omega}(p,q)} \leqslant C\mathcal{B}^\alpha_{log^\beta}$ for all $f \in \mathcal{B}^\alpha_{log^\beta}$.

DEFINITION 2.2. The operator $T_hC_\phi : \mathcal{B}^\alpha_{log^\beta} \to Q_{K,w}(p,q)$ is said to be compact, if it maps any unit disk in $\mathcal{B}^\alpha_{log^\beta}$ onto a pre-compact set in $Q_{K,w}(p,q)$.

The following three lemmas were presented and proved by [**16, 17**].

LEMMA 2.1. *Let $f \in \mathcal{B}^\alpha_{log^\beta}$. Then, for any $z \in \mathbb{D}$, we have*

$$|f(z)| \leqslant C \begin{cases} \|f\|_{\mathcal{B}^\alpha_{log^\beta}} & \text{if} \quad \alpha \in (0,1) \text{ or } \alpha = 1, \beta > 1 \\ \|f\|_{\mathcal{B}^\alpha_{log^\beta}} \max\left(1, \ln\ln \frac{e^{\beta/\alpha}}{(1-|z|)}\right) & \text{if} \quad \alpha = \beta = 1 \\ \|f\|_{\mathcal{B}^\alpha_{log^\beta}} (\ln \frac{e^{\beta/\alpha}}{(1-|z|)})^{1-\beta} & \text{if} \quad \alpha = 1, \beta \in (0,1) \\ \frac{\|f\|_{\mathcal{B}^\alpha_{log^\beta}}}{(1-|z|)^{\alpha-1}(\ln \frac{e^{\beta/\alpha}}{(1-|z|)})^\beta} & \text{if} \quad \alpha > 1, \beta \geqslant 0. \end{cases}$$

LEMMA 2.2. *Assume $\alpha > 1, \beta \geqslant 0$. Then there exist $M = M(n) \in \mathbb{N}$ and functions $f_1, ...f_n \in \mathcal{B}^\alpha_{log^\beta}$ such that*

$$|f_1(z)| + ... + |f_n(z)| \geqslant \frac{C}{(1-|z|)^{\alpha-1} \frac{e^{\beta/\alpha}}{(1-|z|)^\beta}}, \qquad z \in \mathbb{D}, \tag{2.1}$$

where C is a positive constant.

LEMMA 2.3. *Assume that $f, h \in H(\mathbb{D})$. Then*

$$[T_hC_\phi f(z)]' = f(\phi(z))h'(z).$$

The next lemma was obtained in [**10**].

LEMMA 2.4. *If $x > 0,\ y > 0$, then the elementary inequality holds,*

$$(x+y)^p \leqslant \begin{cases} x^p + y^p & \text{for} \quad 0 < p < 1, \\ 2^{y-1}(x^p + y^p) & \text{for} \quad p \geqslant 1. \end{cases}$$

This lemma still holds for sum of finite number n, that is

$$(x_1 + x_2 + ... + x_n)^p \leqslant C(x_1^p + x_2^p + ... + x_n^p), \tag{2.2}$$

where $x_1, x_2, ...x_n > 0,$ *and* $C > 0$. Now, we will introduce and prove the following lemma which give the condition to the operator T_hC_ϕ be compact.

LEMMA 2.5. *Assume that ϕ is an analytic self-map of $\mathbb{D}$ and $h \in H(\mathbb{D})$. Then $T_hC_\phi : \mathcal{B}^\alpha_{log^\beta} \to Q_{K,w}(p,q)$ is compact if and only if $T_hC_\phi : \mathcal{B}^\alpha_{log^\beta} \to Q_{K,w}(p,q)$ is bounded and for any bounded sequence $\{f_i\}_{i\in N} \in \mathcal{B}^\alpha_{log^\beta}$ which converges to zero uniformly on compact subsets of $\mathbb{D}$ as $i \to \infty$ we have $\lim\limits_{i\to\infty} ||T_hC_\phi f_i||_{Q_{K,w}(p,q)} = 0$.*

Proof. Assume that $T_hC_\phi : \mathcal{B}^\alpha_{log^\beta} \to Q_{K,w}(p,q)$ compact and $\{f_i\} \in \mathcal{B}^\alpha_{log^\beta}$ with $\sup_{a\in\mathbb{D}} ||f||_{\mathcal{B}^\alpha_{log^\beta}} = M < \infty$ and which converges to zero locally uniformly on $\mathbb{D}$ as $i \to \infty$. Then $\{T_hC_\phi f_i\}$ has a subsequence $\{T_hC_\phi f_{i_t}\}$ that converges to $h \in Q_{K,w}(p,q)$ thus by Lemma 2.1 for all compact subsets T $\subset \mathbb{D}$, there is a positive constant C_T independent of f_i such that

$$|T_hC_\phi f_{i_t}(z) - h(z)| \leqslant C_T ||T_hC_\phi f_{i_t} - h||_{Q_{K,w}(p,q)}$$

for all $z \in \mathbb{D}$. Therefore,$\{T_hC_\phi f_{i_t}(z) - h(z)\}$ converges to zero uniformly on T. Notice that, there is a constant C > 0 such that $|h \circ \phi| < C$ for all $z \in T$. Also $\phi(T)$ is copmpact in $\mathbb{D}$ and so we have $\{f_{i_t}(\phi(z))\}$ converges to zero for each z in $\mathbb{D}$ .Therefore,$|T_hC_\phi f_{i_t}(z) - h(z)| \to 0$ uniformly on T.Thus for the arbitrariness of T, we have h $\equiv 0$. Since it is true for arbitrary subsequence of $\{f_i\}$, we see that $T_hC_\phi f_{i_t}(z) \to 0$ in $Q_{K,w}(p,q)$, when i $\to \infty$.

Conversely, let $\{h_t\}$ be a bounded sequence in $\mathcal{B}^\alpha_{log^\beta}$. Since $||f||_{\mathcal{B}^\alpha_{log^\beta}} = M < \infty$, the sequence $\{h_t\}$ is uniformly bounded on compact subsets of $\mathbb{D}$ and hence a normal family. Hence we may extract a subsequence $\{h_{j_t}\}$ which converges uniformly on compact subsets of $\mathbb{D}$ to some h $\in H(\mathbb{D})$. Moreover, h $\in \mathcal{B}^\alpha_{log^\beta}$ and $||h||_{\mathcal{B}^\alpha_{log^\beta}} \leqslant M$. Thus the sequence $\{h_{j_t} - h\}$ is such that $||\{h_{j_t} - h\}||_{\mathcal{B}^\alpha_{log^\beta}} \leqslant M$ and converges to zero on compact subsets of $\mathbb{D}$. By hypothesis, we have , $T_hC_\phi h_{j_t} \to T_hC_\phi h$ in $Q_{K,w}(p,q)$ Thus $T_hC_\phi : \mathcal{B}^\alpha_{log^\beta} \to Q_{K,w}(p,q)$ is compact as desired. □

3. The properties of the operator $T_hC_\phi : \mathcal{B}^\alpha_{log^\beta} \to Q_{K,w}(p,q)$

In this section we characterize the operator T_hC_ϕ from weighted logarithmic α-Bloch to $Q_{K,w}(p,q)$ in four different cases dependent on the value of α and β. Moreover, we give the conditions which prove the boundedness and compactness of the operator T_hC_ϕ .

3.1. The case $\alpha > 1$ and $\beta \geqslant 0$.

Theorem 3.1. *Let $\alpha > 1$, $\beta \geqslant 0$, and $h \in H(\mathbb{D})$. let $\phi \in \mathbb{D}$ be an analytic self mapping. Then $T_hC_\phi : \mathcal{B}^\alpha_{log^\beta} \to Q_{K,w}(p,q)$ is bounded if and only if*

$$\text{(3.1)}\ M_1 := \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{|h'(z)|^p(1-|z|)^qK(g(z,a))}{\omega^p(1-|z|)(1-|\phi(z)|)^{(\alpha-1)p}(ln\frac{e^{\beta/\alpha}}{1-\phi(z)})^{p\beta}}dA(z) < \infty.$$

Proof. First direction, we assume that (3.1) is holds and let $f \in \mathcal{B}^\alpha_{log^\beta}$, by Lemma 2.1 and Lemma 2.3 we obtain

$$\begin{aligned} ||T_hC_\phi f||^p_{Q_{K,w}(p,q)} &= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |(T_hC_\phi f)'(z)|^p \frac{(1-|z|)^qK(g(z,a))}{\omega^p(1-|z|)}dA(z) \\ &= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |(T_hf(\phi)(z))'(z)|^p \frac{(1-|z|)^qK(g(z,a))}{\omega^p(1-|z|)}dA(z) \\ &= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |(f(\phi(z))h'(z))|^p \frac{(1-|z|)^qK(g(z,a))}{\omega^p(1-|z|)}dA(z) \end{aligned}$$

and

$$\begin{aligned}
&\leqslant C||f||^p_{\mathcal{B}^\alpha_{log^\beta}} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{|h'(z)|^p\big(1-|z|\big)^q K(g(z,a))}{\omega^p(1-|z|)\,(1-|\phi(z)|)^{(\alpha-1)p}(ln\frac{e^{\beta/\alpha}}{1-\phi(z)})^{p\beta}} dA(z)\\
&= C||f||^p_{\mathcal{B}^\alpha_{log^\beta}} M_1.\\
&< \infty.
\end{aligned}$$

It follows that $T_hC_\phi : \mathcal{B}^\alpha_{log\beta} \to Q_{K,w}(p,q)$ is bounded.

Now, we proof the other direction, we assume that $T_hC_\phi : \mathcal{B}^\alpha_{log\beta} \to Q_{K,w}(p,q)$ is bounded. Let any two $f,g \in \mathcal{B}^\alpha_{log^\beta}$, then using Lemma 2.4 and Lemma 2.2, we have

$$\begin{aligned}
&\Big\{||T_hC_\phi f||^p_{Q_{K,\omega}(p,q)} + ||T_hC_\phi g||^p_{Q_{K,\omega}(p,q)}\Big\}\\
&= \Big\{\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \Big[\big|(T_hC_\phi f)'(z)\big|^p + \big|(T_hC_\phi g)'(z)\big|^p\Big] \frac{\big(1-|z|^2\big)^q K(g(z,a))}{\omega^p(1-|z|)} dA(z)\Big\}\\
&\geqslant \Big\{\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \Big[\big|(T_hC_\phi f)'(z)\big| + \big|(T_hC_\phi g)'(z)\big|\Big]^p \frac{\big(1-|z|^2\big)^q K(g(z,a))}{\omega^p(1-|z|)} dA(z)\Big\}\\
&\geqslant \Big\{\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \Big[\big|f(\phi(z))\big| + \big|g(\phi(z))\big|\Big]^p \big|h'(z)\big|^p \frac{\big(1-|z|^2\big)^q K(g(z,a))}{\omega^p(1-|z|)} dA(z)\Big\}\\
&\geqslant C\Big\{\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{\big|h'(z)\big|^p\big(1-|z|^2\big)^q K(g(z,a))}{\omega^p(1-|z|)(1-|\phi(z)|)^{(\alpha-1)p}(ln\frac{e^{\beta/\alpha}}{1-\phi(z)})^{p\beta}} dA(z)\Big\}\\
&= CM_1.
\end{aligned}$$

Form this and the boundedness of T_hC_ϕ, it follows that(3.1) holds. The proof of this theorem is completed. □

THEOREM 3.2. *Let $\alpha > 1, \beta \geqslant 1$ and $h \in H(\mathbb{D})$. Let ϕ is an analytic mapping from $\mathbb{D}$ into itself. Then $T_hC_\phi : \mathcal{B}^\alpha_{log^\beta} \to Q_{K,\omega}(p,q)$ is compact if and only if (3.1) holds.*

PROOF. First direction, we assume that $T_hC_\phi : \mathcal{B}^\alpha_{log^\beta} \to Q_{K,\omega}(p,q)$ is compact. Then it is bounded and (3.1) holds from Theorem 3.1.

Now, we proof the other direction We assume that (3.1) holds then, form (3.1) we obtain

$$K_1 := \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{|h'(z)|^p(1-|z|)^q K(g(z,a))}{\omega^p(1-|z|)} < \infty. \tag{3.2}$$

Since $\sup\limits_{y\in[0,1)} (1-y^2)^{(\alpha-1)}(\ln\frac{e^{\beta/\alpha}}{(1-|y|)^\beta}) > 0$.

Assume that $\{f_i\}_{i\in N}$ is bounded sequence in $\mathcal{B}^\alpha_{log^\beta}$, such that $f_i \to 0$ uniformly on the compact subsets of $\mathcal{B}^\alpha_{log^\beta}$, as $i \to \infty$. Suppose that $\sup\limits_{i\in N} ||f_i||_{\mathcal{B}^\alpha_{log^\beta}} \leqslant L$. It follows from (3.1) that for any $\epsilon > 0$, there exist a constant $\delta \in (0,1)$, such that

$$\sup_{a\in\mathbb{D}} \int_{|\phi(z)|>\delta} \frac{|h'(z)|^p(1-|z|)^q K(g(z,a))}{\omega^p(1-|\phi(z)|)^{(\alpha-1)p}(ln\frac{e^{\beta/\alpha}}{1-\phi(z)})^{p\beta}} dA(z) < \epsilon^p. \tag{3.3}$$

Let $T_1 = \{\omega \in \mathbb{D}, |\omega| \leqslant \delta\}$, then T_1 is compact subset of $\mathbb{D}$. Since $f_i \to 0$ uniformly on the compact subsets of $\mathbb{D}$ as $j \to \infty$. and $h \in Q_{K,w}(p,q)$, we have

$$\begin{aligned}
||T_hC_\phi f||^p_{Q_{K,w}(p,q)} &= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |(T_hC_\phi f)'(z)|^p \frac{(1-|z|)^q K(g(z,a))}{\omega^p(1-|z|)} dA(z)\\
&= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |(T_h f(\phi)(z))'(z)|^p \frac{(1-|z|)^q K(g(z,a))}{\omega^p(1-|z|)} dA(z)\\
&= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |(f(\phi(z))h'(z))|^p \frac{(1-|z|)^q K(g(z,a))}{\omega^p(1-|z|)} dA(z)\\
&= \sup_{a\in\mathbb{D}} \int_{|\phi(z)|\leqslant\delta} |(f(\phi(z))h'(z))|^p \frac{(1-|z|)^q K(g(z,a))}{\omega^p(1-|z|)} dA(z)\\
&\quad + \sup_{a\in\mathbb{D}} \int_{|\phi(z)|>\delta} |(f(\phi(z))h'(z))|^p \frac{(1-|z|)^q K(g(z,a))}{\omega^p(1-|z|)} dA(z)\\
&= I_1 + I_2.
\end{aligned}$$

Since T_1 is compact subset of $\mathbb{D}$ and from (3.2) we have

$$\begin{aligned}
I_1 : &= \sup_{a\in\mathbb{D}} \int_{|\phi(z)|\leqslant\delta} |(f(\phi(z))h'(z))|^p \frac{(1-|z|)^q K(g(z,a))}{\omega^p(1-|z|)} dA(z)\\
&\leqslant \sup_{\omega\in T_1} |f_i(\omega)|^p \int_{|\phi(z)|\leqslant\delta} |(h'(z))|^p \frac{(1-|z|)^q K(g(z,a))}{\omega^p(1-|z|)} dA(z)\\
&\leqslant K_1 \sup_{\omega\in T_1} |f_i(\omega)|^p \to 0, \qquad i \to \infty. \qquad (3.4)
\end{aligned}$$

On other hand, by Lemma 2.4 and from (3.3), we have

$$\begin{aligned}
I_2 : &= \sup_{a\in\mathbb{D}} \int_{|\phi(z)|>\delta} |(f(\phi(z))h'(z))|^p \frac{(1-|z|)^q K(g(z,a))}{\omega^p(1-|z|)} dA(z)\\
&\leqslant C||f||^p_{\mathcal{B}^\alpha_{log^\beta}} \sup_{a\in\mathbb{D}} \int_{|\phi(z)|>\delta} \frac{|h'(z)|^p(1-|z|^2)^q K(g(z,a))}{(1-|\phi(z)|^2)^{(\alpha-1)p}} dA(z)\\
&\leqslant CL^p\epsilon^p. \qquad (3.5)
\end{aligned}$$

From 3.4,3.5 and since ϵ is an arbitrary positive number, we get

$$\lim_{i\to\infty} ||T_hC_\phi f_i||^p_{Q_{K,\omega}(p,q)} = 0. \tag{3.6}$$

Hence by (3.6) and Lemma 2.1, we get $T_hC_\phi : \mathcal{B}^\alpha \to Q_{K,\omega}(p,q)$ is compact. This completes the proof of this theorem. $\square$

3.2. The case $\alpha \in (0,1)$ *or* $\alpha = 1, \beta > 1$**.**

THEOREM 3.3. *Let* $\alpha \in (0,1)$ *or* $\alpha = 1, \beta > 1$ *and* ϕ *is an analytic mapping from* $\mathbb{D}$ *into itself. Then* $T_hC_\phi : \mathcal{B}^\alpha_{log^\beta} \to Q_{K,\omega}(p,q)$ *is bounded if and only if* $h \in Q_{K,\omega}(p,q)$*. Moreover, if* $T_hC_\phi : \mathcal{B}^\alpha_{log^\beta} \to Q_{K,\omega}(p,q)$ *is bounded. Then*

$$||T_hC_\phi f||_{\mathcal{B}^\alpha_{log^\beta}} \asymp ||h||_{Q_{K,\omega}(p,q)}. \tag{3.7}$$

PROOF. First direction, we assume that $h \in Q_{K,\omega}(p,q)$. For any $f \in \mathcal{B}^\alpha_{log^\beta}$, by Lemma 2.1 and Lemma 2.3 we have

$$\begin{aligned}
||T_hC_\phi f||^p_{Q_{K,w}(p,q)} &= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |(T_hC_\phi f)'(z)|^p \frac{(1-|z|)^q K(g(z,a))}{\omega^p(1-|z|)} dA(z) \\
&= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |(T_h f(\phi)(z))'(z)|^p \frac{(1-|z|)^q K(g(z,a))}{\omega^p(1-|z|)} dA(z) \\
&= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |(f(\phi(z))h'(z))|^p \frac{(1-|z|)^q K(g(z,a))}{\omega^p(1-|z|)} dA(z) \\
&\leqslant C||f||^p_{\mathcal{B}^\alpha_{log^\beta}} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{|h'(z)|^p \big(1-|z|\big)^q K(g(z,a))}{\omega^p(1-|z|)} dA(z),
\end{aligned}$$

that is

$$||T_hC_\phi f||_{\mathcal{B}^\alpha_{log^\beta}} \leqslant ||h||_{Q_{K,\omega}(p,q)} \tag{3.8}$$

Now, we proof the other direction, we assume that $T_hC_\phi : \mathcal{B}^\alpha_{log^\beta} \to Q_{K,w}(p,q)$ is bounded. By taking the function $f_0(z) = 1 \in \mathcal{B}^\alpha_{log^\beta}$ and $||f_0||_{\mathcal{B}^\alpha_{log^\beta}} = 1$, then we obtain

$$\begin{aligned}
||T_hC_\phi f_0||^p_{Q_{K,w}(p,q)} &= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |(T_hC_\phi f_0)'(z)|^p \frac{(1-|z|)^q K(g(z,a))}{\omega^p(1-|z|)} dA(z) \\
&= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |(T_h f_0(\phi)(z))'(z)|^p \frac{(1-|z|)^q K(g(z,a))}{\omega^p(1-|z|)} dA(z) \\
&= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |(f(\phi(z))h'(z))|^p \frac{(1-|z|)^q K(g(z,a))}{\omega^p(1-|z|)} dA(z) \\
&= ||h||^p_{Q_{K,w}(p,q)},
\end{aligned}$$

that is

$$||h||_{Q_{K,\omega}(p,q)} \leqslant ||T_hC_\phi f||_{\mathcal{B}^\alpha_{log^\beta}}. \tag{3.9}$$

Thus from (3.8)and (3.9) we have the relation in (3.7). The proof of this theorem is completed. $\square$

THEOREM 3.4. *Let* $\alpha \in (0,1)$ *or* $\alpha = 1, \beta > 1$*, and* ϕ *is an analytic mapping from* $\mathbb{D}$ *into itself. Then* $T_hC_\phi : \mathcal{B}^\alpha_{log^\beta} \to Q_{K,\omega}(p,q)$ *is compact if and only if* $h \in Q_{K,\omega}(p,q)$*. and (3.7) holds.*

PROOF. The proof of this theorem is similar to that of Theorem 3.2. $\square$

3.3. The case $\alpha = 1; \beta \in (0,1)$.

THEOREM 3.5. *Let $\alpha = 1$, $\beta \in (0,1)$ and ϕ is an analytic mapping from $\mathbb{D}$ into itself. Then $T_h C_\phi : \mathcal{B}^1_{log^\beta} \to Q_{K,\omega}(p,q)$ is bounded (compact) if*

$$M_2 := \sup_{a\in\mathbb{B}} \int_{\mathbb{B}} |h'(z)|^p \left(ln \frac{e^\beta}{1-|\phi(z)|} \right)^{(1-\beta)p} (1-|z|)^q K(g(z,a)) dA(z) < \infty. \tag{3.10}$$

PROOF. Assume that (3.10) holds. For any $f \in \mathcal{B}^1_{log^\beta}$, by Lemma 2.1 and Lemma 2.3 we have

$$\begin{aligned}
||T_h C_\phi f||^p_{Q_{K,w}(p,q)} &= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |(T_h C_\phi f)'(z)|^p \frac{(1-|z|)^q K(g(z,a))}{\omega^p(1-|z|)} dA(z) \\
&= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |(T_h f(\phi)(z))'(z)|^p \frac{(1-|z|)^q K(g(z,a))}{\omega^p(1-|z|)} dA(z) \\
&= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |(f(\phi(z))h'(z))|^p \frac{(1-|z|)^q K(g(z,a))}{\omega^p(1-|z|)} dA(z) \\
&\leqslant C||f||^p_{\mathcal{B}^\alpha_{log^\beta}} \sup_{a\in\mathbb{B}} \int_{\mathbb{D}} |h'(z)|^p \left(ln \frac{e^\beta}{1-|\phi(z)|} \right)^{(1-\beta)p} \\
&\quad (1-|z|)^q K(g(z,a)) dA(z) \\
&= C||f||^p_{\mathcal{B}^1_{log^\beta}} M_2 \leqslant \infty.
\end{aligned}$$

So, $T_h C_\phi : \mathcal{B}^\alpha_{log^\beta} \to Q_{K,\omega}(p,q)$ is bounded. The proof of compactness is similar to the corresponding part of Theorem 3.2. □

3.4. The case $\alpha = 1; \beta \in (0,1)$.

THEOREM 3.6. *Let $\alpha = \beta = 1$, $h \in Q_{K,\omega}(p,q)$. and ϕ ,is an analytic mapping from $\mathbb{D}$ into itself. Then $T_h C_\phi : \mathcal{B}^\alpha_{log^\beta} \to Q_{K,\omega}(p,q)$ is bounded (compact) if*

$$M_3 := \sup_{a\in\mathbb{D}} \int_{\mathbb{B}} |h'(z)|^p \max(1, \ln\ln \frac{e^{(\beta/\alpha)}}{(1-|z|)}) K(g(z,a)) dA(z) < \infty. \tag{3.11}$$

PROOF. First direction, we assume that 3.11 holds. For any $f \in \mathcal{B}^1_{log^1}$, by Lemma 2.1 and Lemma 2.3 we have

$$\begin{aligned}
||T_h C_\phi f||^p_{Q_{K,w}(p,q)} &= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |(T_h C_\phi f)'(z)|^p \frac{(1-|z|)^q K(g(z,a))}{\omega^p(1-|z|)} dA(z) \\
&= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |(T_h f(\phi)(z))'(z)|^p \frac{(1-|z|)^q K(g(z,a))}{\omega^p(1-|z|)} dA(z) \\
&= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |(f(\phi(z))h'(z))|^p \frac{(1-|z|)^q K(g(z,a))}{\omega^p(1-|z|)} dA(z)
\end{aligned}$$

and

$$\begin{aligned} &\leqslant C||f||^p_{\mathcal{B}^1_{log^1}} \sup_{a\in\mathbb{D}} \int_{\mathbb{B}} |h'(z)|^p \max(1, \ln\ln \frac{e^{(\beta/\alpha)}}{(1-|z|)}) \\ &\quad K(g(z,a))dA(z)dA(z) \\ &= C||f||^p_{\mathcal{B}^1_{log^1}} M_3 \leqslant \infty. \end{aligned}$$

So $T_hC_\phi : \mathcal{B}^1_{log^1} \to Q_{K,\omega}(p,q)$ is bounded. The proof of compactness is similar to the corresponding part of Theorem 3.2. □

4. Conclusion

In this paper, we proved the boundedness and compactness property of product of composition operator and extended Cesáro operator from the weighted logarithmic α-Bloch-type space $\mathcal{B}^\alpha_{log^\beta}$ to $Q_{K,\omega}(p,q)$ spaces spaces in some cases on unit disk.

References

[1] J. H. Shapiro. *Composition operators and Classical function theory.* Springer Science abd Business Media, 2012.

[2] S. Li and S. Stević. Products of Volterra type operator and composition operator from H^∞ and Bloch spaces to Zygmund spaces, *J. Math. Anal. Appl.*, **345**1(2008), 40–52.

[3] A. El-Sayed Ahmed and M. A. Bakhit. Composition operators on some holomorphic Banach function spaces. *Math. Scandinavica*, **104**(2)(2009), 275–295.

[4] A. El-Sayed Ahmed and M. A. Bakhit. Composition operators acting between some weighted Möbius invariant spaces. *Annals Functional Analysis*, **2**(2)(2011), 138–152.

[5] M. Essén, H. Wulan and J. Xiao. Several function-theoretic characterizations of Möbius invariant Q_K spaces. *J. Functional Anal.*, **230**(1)(2006), 78–115.

[6] Z.-S. Fang and Z.-H. Zhou. Extended Cesáro operators from generally weighted Bloch spaces to Zygmund space. *J. Math. Anal. Appl.*, **359**(2)(2009), 499–507.

[7] Z.-S. Fang and Z.-H. Zhou. Extended Cesáro operators on $BMOA$ spaces in the unit ball. *J. Math. Ineq.*, **4**(1)(2010), 27–36.

[8] D. Gu. Extended Cesáro operators from logarithmic-type spaces to Bloch-type spaces. *Abstract and Applied Analysis*, **2009**(2009), Article ID 246521 (9 pages)

[9] Z. Hu. Extended Cesáro operators on mixed norm spaces. *Proc. Amer. Math. Soc.* , **131**(7)(2003), 2171–2179.

[10] Ji-Huai,Shi, Inequalities for the integral means of holomorphic functions and their derivatives in the unit ball of C^n. *Trans. Am. Math. Soc.*, **328**(2)(1991), 619–637.

[11] X. Li, F. Pérez-González and J. Rattya., Composition operators in hyperbolic Q-classes. *Annales Academiæ Scientiarum Fennicæ Mathematica Dissertationes*, **31**(2006), 391–404.

[12] Y.-x. Liang, and Z.-H. Zhou and R.-Y. Chen. Product of extended Cesáro operator and composition operator from the logarithmic Bloch-type space to $F(p,q,s)$ space on the unit ball. *J. Comput. Anal. Appl.*, **15**(3)(2013), 432–440.

[13] R. A. Rashwan, A. El-Sayed Ahmed and M. A. Bakhit. Composition operators on some general families of function spaces. *J. Appl. Funcl Anal.*, **10**(1-2)(2015), 164–175.

[14] R. A. Rashwan, A. El-Sayed Ahmed and A. Kamal. Integral characterizations of weighted Bloch space and $Q_{K,\omega}(p,q)$ Spaces. *Mathematica (Cluj)*, **51(74)**(1)(2009), 63–76.

[15] S. Stević. On a new operator from the logarithmic Bloch space to the Bloch-type space on the unit ball. *Appl. Math. Comput.*, **206**(1)(2008), 313–320.

[16] S. Stević. On an integral-type operator from logarithmic Bloch-type and mixed-norm spaces to Bloch-type spaces. *Nonlinear Analysis: Theory, Methods and Applications*, **71**(12)(2009), 6323–6342.

[17] S. Stević. On operator P_φ^g from the logarithmic Bloch-type space to the mixed-norm space on the unit ball, *Appl. Math. Comput.*, **215**(12)(2010), 4248–4255.

[18] M. Tjani. Compact composition operators on Besov spaces, *Trans. Am. Math. Soc.*, **355**(11)(2003), 4683–4698.

[19] H. Wulan, D. Zheng and K. Zhu. Compact composition operators on $BMOA$ and the Bloch space. *Proc. Am. Math. Soc.*, **137**(11)(2009), 3861–3868.

[20] H.-Ggggg. Zeng and Z.-H. Zhou. An estimate of the essential norm of a composition operator from $F(p, q, s)$ to B^α in the unit ball. *J. Inequal. Appl.*, **2010**(2010):132970

[21] Z.-H. Zhou and R.-Y. Chen. Weighted composition operators from $F(p, q, s)$ to Bloch type spaces on the unit ball. *Int. J. Math.*, **19**(8)(2008), 899–926.

[22] Z.-H. Zhou and J. Shi. Compactness of composition operators on the Bloch space in classical bounded symmetric domains. *The Michigan Math. J.*, **50**(2)(2002), 381–405.

Receibed by editors 26.03.2018; Revised version 21.11.2018; Available online 03.12.2018.

I. M. Hanafy: Department of Mathematics, Faculty of science, Port Said University, Port Said, Egypt

A. Kamal: Department of Mathematics, Faculty of science, Port Said University, Port Said, Egypt, Current addres :Department of Mathematics, Qassim University, College of Sciences and Arts in Muthnib, Kingdom of Saudi Arabia.

E-mail address: alaa_mohamed1@yahoo.com

M. Hamza Eissa: Department of Mathematics, Faculty of science, Port Said University, Port Said, Egypt