Properties of Product $T_hC_\phi$ Operators from $B^\alpha_{\log^p}$ to $Q_{K,\omega}(p, q)$ Spaces

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Abstract. In this paper, the authors introduce equivalent characterizations for the boundedness and compactness of the product composition operator and extended Cesáro operator from the weighted logarithmic $\alpha$-Bloch-type space $B^\alpha_{\log^p}$ to $Q_{K,\omega}(p, q)$ spaces on unit disk.

1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disc in the complex plane $\mathbb{C}$, $H(\mathbb{D})$ denote the class of all analytic functions in $\mathbb{D}$. Let $dA$ denote the Lebesgue measure on $\mathbb{D}$ normalized so that $A(\mathbb{D}) = 1$.

Following ([5]), for each $a \in \mathbb{D}$, $\varphi_a : \mathbb{D} \to \mathbb{D}$ denotes the Möbius transformations defined by

$$\varphi_a(z) := \frac{a - z}{1 - az} \quad \text{for } z \in \mathbb{D}.$$ 

Green’s function of $\mathbb{D}$ with logarithmic singularity at $a$, define as follows

$$g(z, a) := \log \left| \frac{1 - az}{z - a} \right| = \log \frac{1}{|\varphi_a(z)|}.$$ 

The study of composition operator $C_\phi$ acting on spaces of analytic functions has engaged many analysts for many years (see [3, 4, 11, 13] and others), readers interested in this topic can refer to (see [12]) the sources for the development of the theory of composition operators and function spaces.

Definition 1.1. For any analytic self-mapping $\phi$ of $\mathbb{D}$. The symbol $\phi$ induces a linear composition operator $C_\phi(f) := f \circ \phi$ from $H(\mathbb{D})$ into itself.

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The problem of boundedness and compactness of $C_\phi$ has been studied in many Banach spaces of analytic functions and the study of such operators has recently attracted the most attention (see [18, 19] and others).

The following definition was first introduced in ([9])

**Definition 1.2.** Let $h \in H(D)$, the extended Cesáro operator $T_h$ with symbol $h$ is the operator on $H(D)$,

$$T_h f(z) := \int_0^Z f(\xi)h'(\xi)d\xi, \quad f \in H(D), \ z \in D.$$  

This operator is called generalized Cesáro operator.

It has been studied in the following article [6, 7, 8] and other.

In our study, we consider the product of extended Cesáro operator $T_h$ and of composition operator $C_\phi$, which was first introduced and studied by the authors in ([2])

$$T_h C_\phi f(z) = \int_0^Z f(\phi(\xi))h'(\xi)d\xi, \quad f \in H(D), \ z \in D.$$  

The author in [16] introduced the definition of logarithmic Bloch-type space as follows

**Definition 1.3.** Let $\alpha > 0, \beta \geq 0$ and $f$ be an analytic function in $D$ the logarithmic Bloch-type space $B_{log}^\alpha$ is defined by

$$\|f\|_{B_{log}^\alpha} = \left\{ f \in H(D) : \|f\|_{B_{log}^\alpha} = \sup_{z \in D} (1 - |z|)^\alpha \ln (\frac{e^{\beta/\alpha}}{1 - |z|}) |f'(z)| < \infty \right\}.$$  

Case 1: $\beta = 0$ then $B_{log}^\alpha$ becomes the $\alpha$-Bloch space $B^\alpha$

Case 2: $\alpha = \beta = 1$ then $B_{log}^\alpha$ becomes the logarithmic Bloch space.

The authors in [14] introduced the definition of $Q_{K, \omega}(p, q)$ which has attracted a lot of attention in recent years. It defined as follows

**Definition 1.4.** Let $K : [0, \infty) \to [0, \infty)$ and $\omega : (0, 1] \to (0, \infty)$, are right-continuous and nondecreasing functions. If $0 < p < \infty$, $-2 < q < \infty$, then an analytic function $f$ in $D$ is said to belong to the space $Q_{K, \omega}(p, q)$ if

$$\|f\|_{Q_{K, \omega}(p, q)} := \sup_{a \in \overline{D}} \int_D |f'(z)|^p \frac{(1 - |z|)^q}{\omega^q(1 - |z|)} K(g(z, a))dA(z) < \infty.$$  

The notation $A \preceq B$ means that there is a positive constant $C$ such that $\frac{B}{C} \leq A \leq CB$.

2. **Auxiliary results**

In this section we state several results, which are used in the main result proofs.

Now, we will introduce the definition of boundedness and compactness of the operator $T_h C_\phi : B_{log}^\alpha \to Q_{K, \omega}(p, q)$.
Definition 2.1. The operator \( T_hC_\phi : B_{\log^\alpha}^\alpha \rightarrow Q_{K,w}(p,q) \) is said to be bounded, if there is a positive constant \( C \) such that \( ||T_hC_\phi f||_{Q_{K,w}(p,q)} \leq CB_{\log^\alpha}^\alpha \) for all \( f \in B_{\log^\alpha}^\alpha \).

Definition 2.2. The operator \( T_hC_\phi : B_{\log^\alpha}^\alpha \rightarrow Q_{K,w}(p,q) \) is said to be compact, if it maps any unit disk in \( B_{\log^\alpha}^\alpha \) onto a pre-compact set in \( Q_{K,w}(p,q) \).

The following three lemmas were presented and proved by [16, 17].

Lemma 2.1. Let \( f \in B_{\log^\alpha}^\alpha \). Then, for any \( z \in \mathbb{D} \), we have

\[
|f(z)| \leq C \begin{cases} 
||f||_{B_{\log^\alpha}^\alpha} & \text{if } \alpha \in (0,1) \text{ or } \alpha = 1, \beta > 1 \\
||f||_{B_{\log^\alpha}^\alpha} \max \left(1, \ln \frac{e^{3/\alpha}}{1-|z|}\right) & \text{if } \alpha = \beta = 1 \\
||f||_{B_{\log^\alpha}^\alpha} \left(\ln \frac{e^{3/\alpha}}{1-|z|}\right)^{1-\beta} & \text{if } \alpha = 1, \beta \in (0,1) \\
\frac{||f||_{B_{\log^\alpha}^\alpha}}{(1-|z|)^{\alpha-1}\left(\ln \frac{e^{3/\alpha}}{1-|z|}\right)^\beta} & \text{if } \alpha > 1, \beta \geq 0.
\end{cases}
\]

Lemma 2.2. Assume \( \alpha > 1, \beta \geq 0 \). Then there exist \( M = M(n) \in \mathbb{N} \) and functions \( f_1, \ldots, f_n \in B_{\log^\alpha}^\alpha \) such that

\[
|f_1(z)| + \ldots + |f_n(z)| \geq \frac{C}{(1-|z|)^{\alpha-1}\left(\ln \frac{e^{3/\alpha}}{1-|z|}\right)^\beta}, \quad z \in \mathbb{D},
\]

where \( C \) is a positive constant.

Lemma 2.3. Assume that \( f, h \in H(\mathbb{D}) \). Then

\[
[T_hC_\phi f(z)]' = f(\phi(z))h'(z).
\]

The next lemma was obtained in [10].

Lemma 2.4. If \( x > 0, y > 0 \), then the elementary inequality holds,

\[
(x+y)^p \leq \begin{cases} 
x^p + y^p & \text{for } 0 < p < 1, \\
2^{p-1}(x^p + y^p) & \text{for } p \geq 1.
\end{cases}
\]

This lemma still holds for sum of finite number \( n \), that is

\[
(x_1 + x_2 + \ldots + x_n)^p \leq C(x_1^p + x_2^p + \ldots + x_n^p),
\]

where \( x_1, x_2, \ldots, x_n > 0 \), and \( C > 0 \). Now, we will introduce and prove the following lemma which give the condition to the operator \( T_hC_\phi \) be compact.

Lemma 2.5. Assume that \( \phi \) is an analytic self-map of \( \mathbb{D} \) and \( h \in H(\mathbb{D}) \). Then \( T_hC_\phi : B_{\log^\alpha}^\alpha \rightarrow Q_{K,w}(p,q) \) is compact if and only if \( T_hC_\phi : B_{\log^\alpha}^\alpha \rightarrow Q_{K,w}(p,q) \) is bounded and for any bounded sequence \( \{f_i\}_{i \in \mathbb{N}} \in B_{\log^\alpha}^\alpha \) which converges to zero uniformly on compact subsets of \( \mathbb{D} \) as \( i \to \infty \) we have \( \lim_{i \to \infty} ||T_hC_\phi f_i||_{Q_{K,w}(p,q)} = 0 \).
Thus the sequence converges to zero uniformly on compact subsets of $\mathbb{D}$ as $i \to \infty$. Then $\{T_h C_\phi f_i\}$ has a subsequence $\{T_h C_\phi f_{i_k}\}$ that converges to $h \in Q_{K,w}(p,q)$ thus by Lemma 2.1 for all compact subsets $T \subset \mathbb{D}$, there is a positive constant $C_T$ independent of $f_i$ such that

$$|T_h C_\phi f_{i_k}(z) - h(z)| \leq C_T \|T_h C_\phi f_{i_k} - h\|_{Q_{K,w}(p,q)}$$

for all $z \in \mathbb{D}$. Therefore, $\{T_h C_\phi f_{i_k}(z) - h(z)\}$ converges to zero uniformly on $T$. Notice that, there is a constant $C > 0$ such that $|h \circ \phi| < C$ for all $z \in T$. Also $\phi(T)$ is compact in $\mathbb{D}$ and so we have $\{f_{i_k}(\phi(z))\}$ converges to zero for each $z$ in $\mathbb{D}$. Therefore, $\|T_h C_\phi f_{i_k}(z) - h(z)\| \to 0$ uniformly on $T$. Thus for the arbitrariness of $T$, we have $h \equiv 0$. Since it is true for arbitrary subsequence of $\{f_i\}$, we see that $T_h C_\phi f_{i_k}(z) \to 0$ in $Q_{K,w}(p,q)$, when $i \to \infty$.

Conversely, let $\{h_t\}$ be a bounded sequence in $B^\alpha_{\log^\beta}$. Since $\|f\|_{B^\alpha_{\log^\beta}} = M < \infty$, the sequence $\{h_t\}$ is uniformly bounded on compact subsets of $\mathbb{D}$ and hence a normal family. Hence we may extract a subsequence $\{h_{t_j}\}$ which converges uniformly on compact subsets of $\mathbb{D}$ to some $h \in H(\mathbb{D})$. Moreover, $h \in B^\alpha_{\log^\beta}$ and $\|h\|_{B^\alpha_{\log^\beta}} \leq M$. Thus the sequence $\{h_{t_j} - h\}$ is such that $\|h_{t_j} - h\|_{B^\alpha_{\log^\beta}} \leq M$ and converges to zero on compact subsets of $\mathbb{D}$. By hypothesis, we have $T_h C_\phi h_{t_j} \to T_h C_\phi h$ in $Q_{K,w}(p,q)$ Thus $T_h C_\phi : B^\alpha_{\log^\beta} \to Q_{K,w}(p,q)$ is compact as desired. \hfill $\Box$

3. The properties of the operator $T_h C_\phi : B^\alpha_{\log^\beta} \to Q_{K,w}(p,q)$

In this section we characterize the operator $T_h C_\phi$ from weighted logarithmic $\alpha$-Bloch to $Q_{K,w}(p,q)$ in four different cases dependent on the value of $\alpha$ and $\beta$. Moreover, we give the conditions which prove the boundedness and compactness of the operator $T_h C_\phi$.

3.1. The case $\alpha > 1$ and $\beta \geq 0$.

**Theorem 3.1.** Let $\alpha > 1$, $\beta \geq 0$, and $h \in H(\mathbb{D})$. Let $\phi \in \mathbb{D}$ be an analytic self mapping. Then $T_h C_\phi : B^\alpha_{\log^\beta} \to Q_{K,w}(p,q)$ is bounded if and only if

$$M_1 := \sup_{\Delta} \int_{\Delta} |h'(z)|^p (1 - |z|)^{\beta} (1 - |\phi(z)|)^{(\alpha - 1)p}|h_{\log^\beta}|(1 - |\phi(z)|)^{\beta} dA(z) < \infty.$$  

**Proof.** First direction, we assume that (3.1) is holds and let $f \in B^\alpha_{\log^\beta}$, by Lemma 2.1 and Lemma 2.3 we obtain

$$\|T_h C_\phi f\|_{Q_{K,w}(p,q)}^p = \sup_{\Delta} \int_{\Delta} |(T_h C_\phi f)'(z)|^p \frac{(1 - |z|)^{\beta} K(g(z,a))}{\omega^p(1 - |z|)} dA(z)$$

$$= \sup_{\Delta} \int_{\Delta} |(T_h f)'(\phi(z))|^p \frac{(1 - |z|)^{\beta} K(g(z,a))}{\omega^p(1 - |z|)} dA(z)$$

$$= \sup_{\Delta} \int_{\Delta} |(f(\phi(z))h'(z))|^p \frac{(1 - |z|)^{\beta} K(g(z,a))}{\omega^p(1 - |z|)} dA(z)$$

$$\leq C \|f\|_{B^\alpha_{\log^\beta}} < \infty.$$
and

\[
C||f||_{L^{\infty}}^p \sup_{a \in D} \int_D |h'(z)|^p (1 - |z|)^{q}K(g(z,a)) \omega^p(1 - |z|) (1 - |\phi(z)|)^{(\alpha - 1)p} \left( \ln \frac{e^{\frac{\alpha}{|z|}}} {1 - \phi(z)} \right)^{p \alpha} dA(z) \\
= C||f||_{L^{\infty}}^p \ M_1 \\
< \infty.
\]

It follows that \(T_h C_\phi : B_{log^\alpha}^o \to Q_{K,w}(p,q)\) is bounded.

Now, we proof the other direction, we assume that \(T_h C_\phi : B_{log^\alpha}^o \to Q_{K,w}(p,q)\) is bounded. Let any two \(f, g \in B_{log^\alpha}^o\), then using Lemma 2.4 and Lemma 2.2, we have

\[
\begin{aligned}
&\left\{ ||T_h C_\phi f||_{Q_{K,w}(p,q)}^p + ||T_h C_\phi g||_{Q_{K,w}(p,q)}^p \right\} \\
= & \left\{ \sup_{a \in D} \int_D \left[ (|T_h C_\phi f)'(z)|^p + |(T_h C_\phi g)'(z)|^p \right] \frac{(1 - |z|^2)^qK(g(z,a))}{\omega^p(1 - |z|)} dA(z) \right\} \\
\geq & \left\{ \sup_{a \in D} \int_D \left[ |f(\phi(z))| + |g(\phi(z))| \right] |h'(z)|^p \frac{1 - |z|^2}{\omega^p(1 - |z|)} dA(z) \right\} \\
\geq & C \left\{ \sup_{a \in D} \int_D \left[ |f(\phi(z))| + |g(\phi(z))| \right] \frac{|h'(z)|^p (1 - |z|^2)^qK(g(z,a))}{\omega^p(1 - |z|) (1 - |\phi(z)|)^{(\alpha - 1)p} \left( \ln \frac{e^{\frac{\alpha}{|z|}}} {1 - \phi(z)} \right)^{p \alpha}} dA(z) \right\} \\
= & CM_1.
\end{aligned}
\]

Form this and the boundedness of \(T_h C_\phi\), it follows that (3.1) holds. The proof of this theorem is completed. \(\square\)

**Theorem 3.2.** Let \(\alpha > 1, \beta > 1\) and \(h \in H(D)\). Let \(\phi\) is an analytic mapping from \(D\) into itself. Then \(T_h C_\phi : B_{log^\alpha}^o \to Q_{K,w}(p,q)\) is compact if and only if (3.1) holds.

**Proof.** First direction, we assume that \(T_h C_\phi : B_{log^\alpha}^o \to Q_{K,w}(p,q)\) is compact. Then it is bounded and (3.1) holds from Theorem 3.1.

Now, we proof the other direction We assume that (3.1) holds then, form (3.1) we obtain

\[
(3.2) \quad K_1 := \sup_{a \in D} \int_D \frac{|h'(z)|^p (1 - |z|)^qK(g(z,a))}{\omega^p(1 - |z|)} < \infty.
\]

Since \(\sup_{y \in [0,1]} (1 - y^2)^{(\alpha - 1)}(\ln \frac{e^{\frac{\alpha}{|z|}}} {1 - \phi(z)}) > 0\).

Assume that \(\{f_i\}_{i \in N}\) is bounded sequence in \(B_{log^\alpha}^o\), such that \(f_i \to 0\) uniformly on the compact subsets of \(B_{log^\alpha}^o\), as \(i \to \infty\). Suppose that \(\sup_{i \in N} ||f_i||_{log^\alpha}^o \leq L\). It
follows from (3.1) that for any \( \epsilon > 0 \), there exist a constant \( \delta \in (0, 1) \), such that

\[
\text{(3.3)} \quad \sup_{a \in \mathbb{D}} \int_{|\phi(z)| > \delta} \frac{|h'(z)|^p (1 - |z|)^q K(g(z, a))}{\omega^p (1 - |\phi(z)|)^{(\alpha-1)p} (1 - |z|)^{\alpha p} dA(z) < \epsilon^p.}
\]

Let \( T_1 = \{ \omega \in \mathbb{D}, |\omega| \leq \delta \} \), then \( T_1 \) is compact subset of \( \mathbb{D} \). Since \( f_i \to 0 \) uniformly on the compact subsets of \( \mathbb{D} \) as \( j \to \infty \), and \( h \in Q_{K, w}(p, q) \), we have

\[
\|T_h C_\phi f\|_{Q_{K, w}(p, q)}^p = \sup_{a \in \mathbb{D}} \int_{|\phi(z)| \leq \delta} |(T_h C_\phi f)(z)|^p \frac{(1 - |z|)^q K(g(z, a))}{\omega^p (1 - |\phi(z)|)} dA(z)
\]

\[
= \sup_{a \in \mathbb{D}} \int_{|\phi(z)| \leq \delta} |(f(\phi(z)) h'(z))|^p \frac{(1 - |z|)^q K(g(z, a))}{\omega^p (1 - |\phi(z)|)} dA(z)
\]

\[
= \sup_{a \in \mathbb{D}} \int_{|\phi(z)| \leq \delta} |(f(\phi(z)) h'(z))|^p \frac{(1 - |z|)^q K(g(z, a))}{\omega^p (1 - |\phi(z)|)} dA(z)
\]

\[
+ \sup_{a \in \mathbb{D}} \int_{|\phi(z)| > \delta} |(f(\phi(z)) h'(z))|^p \frac{(1 - |z|)^q K(g(z, a))}{\omega^p (1 - |\phi(z)|)} dA(z)
\]

\[
= I_1 + I_2.
\]

Since \( T_1 \) is compact subset of \( \mathbb{D} \) and from (3.2) we have

\[
I_1 : = \sup_{a \in \mathbb{D}} \int_{|\phi(z)| \leq \delta} |(f(\phi(z)) h'(z))|^p \frac{(1 - |z|)^q K(g(z, a))}{\omega^p (1 - |\phi(z)|)} dA(z)
\]

\[
\leq \sup_{\omega \in T_1} |f_i(\omega)|^p \int_{|\phi(z)| \leq \delta} |h'(z)|^p \frac{(1 - |z|)^q K(g(z, a))}{\omega^p (1 - |\phi(z)|)} dA(z)
\]

\[
\leq K_1 \sup_{\omega \in T_1} |f_i(\omega)|^p \to 0, \quad i \to \infty.
\]

On other hand, by Lemma 2.4 and from (3.3), we have

\[
I_2 : = \sup_{a \in \mathbb{D}} \int_{|\phi(z)| > \delta} |(f(\phi(z)) h'(z))|^p \frac{(1 - |z|)^q K(g(z, a))}{\omega^p (1 - |\phi(z)|)} dA(z)
\]

\[
\leq C \int_{|\phi(z)| > \delta} \sup_{a \in \mathbb{D}} \int \frac{|h'(z)|^p (1 - |z|)^q K(g(z, a))}{\omega^p (1 - |\phi(z)|)^{(\alpha-1)p} (1 - |z|)^{\alpha p}} dA(z)
\]

\[
\leq CL^p \epsilon^p.
\]

From 3.4.3.5 and since \( \epsilon \) is an arbitrary positive number, we get

\[
\lim_{i \to \infty} \|T_h C_\phi f_i\|_{Q_{K, w}(p, q)}^p = 0.
\]

Hence by (3.6) and Lemma 2.1, we get \( T_h C_\phi : B^\alpha \to Q_{K, w}(p, q) \) is compact. This completes the proof of this theorem. □
3.2. The case $\alpha \in (0, 1)$ or $\alpha = 1, \beta > 1$.

Theorem 3.3. Let $\alpha \in (0, 1)$ or $\alpha = 1, \beta > 1$ and $\phi$ is an analytic mapping from $\mathbb{D}$ into itself. Then $T_h C_\phi : B_{log}^\alpha \to Q_{K,\omega}(p, q)$ is bounded if and only if $h \in Q_{K,\omega}(p, q)$. Moreover, if $T_h C_\phi : B_{log}^\alpha \to Q_{K,\omega}(p, q)$ is bounded. Then

$$||T_h C_\phi f||_{B_{log}^\alpha} \approx ||h||_{Q_{K,\omega}(p, q)}.$$  \hspace{1cm} (3.7)

Proof. First direction, we assume that $h \in Q_{K,\omega}(p, q)$. For any $f \in B_{log}^\alpha$, by Lemma 2.1 and Lemma 2.3 we have

$$||T_h C_\phi f||_{Q_{K,\omega}(p, q)}^p = \sup_{a \in B} \int |(T_h C_\phi f)'(z)|^p \frac{(1 - |z|)^{\eta} K(g(z, a))}{\omega^p(1 - |z|)} dA(z)$$

$$= \sup_{a \in B} \int |(T_h f)(\phi(z))'g(z)|^p \frac{(1 - |z|)^{\eta} K(g(z, a))}{\omega^p(1 - |z|)} dA(z)$$

$$\leq C ||f||_{B_{log}^\alpha} \sup_{a \in B} \int |h'(z)|^p \frac{(1 - |z|)^{\eta} K(g(z, a))}{\omega^p(1 - |z|)} dA(z),$$

that is

$$||T_h C_\phi f||_{B_{log}^\alpha} \leq ||h||_{Q_{K,\omega}(p, q)}.$$  \hspace{1cm} (3.8)

Now, we proof the other direction, we assume that $T_h C_\phi : B_{log}^\alpha \to Q_{K,\omega}(p, q)$ is bounded. By taking the function $f_0(z) = 1 \in B_{log}^\alpha$ and $||f_0||_{B_{log}^\alpha} = 1$, then we obtain

$$||T_h C_\phi f_0||_{Q_{K,\omega}(p, q)}^p = \sup_{a \in B} \int |(T_h C_\phi f_0)'(z)|^p \frac{(1 - |z|)^{\eta} K(g(z, a))}{\omega^p(1 - |z|)} dA(z)$$

$$= \sup_{a \in B} \int |(T_h f_0)(\phi(z))'g(z)|^p \frac{(1 - |z|)^{\eta} K(g(z, a))}{\omega^p(1 - |z|)} dA(z)$$

$$= \sup_{a \in B} \int |h'(z)|^p \frac{(1 - |z|)^{\eta} K(g(z, a))}{\omega^p(1 - |z|)} dA(z)$$

$$= ||h||_{Q_{K,\omega}(p, q)}^p.$$  \hspace{1cm} (3.9)

Thus from (3.8) and (3.9) we have the relation in (3.7). The proof of this theorem is completed. \hfill \square

Theorem 3.4. Let $\alpha \in (0, 1)$ or $\alpha = 1, \beta > 1$, and $\phi$ is an analytic mapping from $\mathbb{D}$ into itself. Then $T_h C_\phi : B_{log}^\alpha \to Q_{K,\omega}(p, q)$ is compact if and only if $h \in Q_{K,\omega}(p, q)$, and (3.7) holds.

Proof. The proof of this theorem is similar to that of Theorem 3.2. \hfill \square
Lemma 2.3 we have Lemma 2.1 and Lemma 2.3 we have

\[(3.10)\]

\[M_2 := \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |h'(z)|^p \left( \ln \frac{e^\beta}{1 - |\phi(z)|} \right)^{(1-\beta)p} (1 - |z|)^q K(g(z,a))dA(z) < \infty.\]

**Proof.** Assume that (3.10) holds. For any \( f \in \mathcal{B}_{\log}^{1}, \) by Lemma 2.1 and Lemma 2.3 we have

\[
||T_h C_f||_{Q_{K,\omega}(p,q)}^p = \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |(T_h C_f)(z)|^p (1 - |z|)^q K(g(z,a))\frac{1}{\omega^p(1 - |z|)}dA(z)
\]

\[
= \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |(T_h f)(z)|^p (1 - |z|)^q K(g(z,a))\frac{1}{\omega^p(1 - |z|)}dA(z)
\]

\[
= \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f(\phi(z))h'(z)|^p (1 - |z|)^q K(g(z,a))\frac{1}{\omega^p(1 - |z|)}dA(z)
\]

\[
\leq C||f||_{\mathcal{B}_{\log}^{1}}^p \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |h'(z)|^p \left( \ln \frac{e^\beta}{1 - |\phi(z)|} \right)^{(1-\beta)p} (1 - |z|)^q K(g(z,a))dA(z)
\]

\[
= C||f||_{\mathcal{B}_{\log}^{1}}^p M_2 \leq \infty.
\]

So, \( T_h C_f : \mathcal{B}_{\log}^{\alpha} \rightarrow Q_{K,\omega}(p,q) \) is bounded. The proof of compactness is similar to the corresponding part of Theorem 3.2.

**3.4. The case** \( \alpha = 1; \beta \in (0,1). \)

**Theorem 3.5.** Let \( \alpha = 1, \beta \in (0,1) \) and \( \phi \) is an analytic mapping from \( \mathbb{D} \) into itself. Then \( T_h C_f : \mathcal{B}_{\log}^{1} \rightarrow Q_{K,\omega}(p,q) \) is bounded (compact) if

\[(3.11)\]

\[M_3 := \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |h'(z)|^p \max(1,\ln \frac{e^{(\beta/\alpha)}}{1 - |z|})^p K(g(z,a))dA(z) < \infty.\]

**Proof.** First direction, we assume that (3.11) holds. For any \( f \in \mathcal{B}_{\log}^{1}, \) by Lemma 2.1 and Lemma 2.3 we have

\[
||T_h C_f||_{Q_{K,\omega}(p,q)}^p = \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |(T_h C_f)(z)|^p (1 - |z|)^q K(g(z,a))\frac{1}{\omega^p(1 - |z|)}dA(z)
\]

\[
= \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |(T_h f)(z)|^p (1 - |z|)^q K(g(z,a))\frac{1}{\omega^p(1 - |z|)}dA(z)
\]

\[
= \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f(\phi(z))h'(z)|^p (1 - |z|)^q K(g(z,a))\frac{1}{\omega^p(1 - |z|)}dA(z)
\]
and
\[
C\|f\|_{B_{\log}^1}^p \sup_{a \in \mathbb{D}} \int_{\mathbb{B}} |h'(z)|^p \max\{1, \ln \ln \left(\frac{e(\beta/\alpha)}{(1-|z|)}\right)\} K(g(z,a))dA(z)dA(z) = C\|f\|_{B_{\log}^1}^p M_3 \leq \infty.
\]

So $T_h C_\phi : B_{\log}^1 \to Q_{K,\omega}(p,q)$ is bounded. The proof of compactness is similar to the corresponding part of Theorem 3.2.

\[\square\]

4. Conclusion

In this paper, we proved the boundedness and compactness property of product of composition operator and extended Cesàro operator from the weighted logarithmic $\alpha$-Bloch-type space $B_{\log}^\alpha$ to $Q_{K,\omega}(p,q)$ spaces spaces in some cases on unit disk.

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