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A COMMON FIXED POINT THEOREM FOR FOUR MAPS SATISFYING GENERALIZED α - WEAKLY CONTRACTIVE CONDITION IN ORDERED PARTIAL METRIC SPACES

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Abstract. In this paper we obtain a common fixed point theorem for four maps satisfying generalized α - weakly contractive condition and we give an example to illustrate our main theorem. Our result generalize and improve the theorem of Seonghoon Cho [**9**].

1. Introduction and Preliminaries

There are many generalizations of the concept of metric spaces in the literature. One of them is a partial metric space introduced by Matthews [**18**] as a part of study of denotational semantics of data flow networks.After that fixed and common fixed point results in partial metric spaces were studied by many other authors, for example [**26, 13, 14, 15, 6, 7, 22, 29, 12, 25, 3**].

Throughout this paper, $\mathbb{R}^+$ and $\mathbb{N}$ denote the set of all non-negative real numbers and the set of all positive integers respectively.

First we recall some basic definitions and lemmas which play crucial role in the theory of partial metric spaces .

Definition 1.1. ([**18**]) A partial metric on a non empty set X is a function $p : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$,

(p_1) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$,
(p_2) $p(x, x) \leqslant p(x, y), p(y, y) \leqslant p(x, y)$,

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(p_3) $p(x,y) = p(y,x)$,
(p_4) $p(x,y) \leqslant p(x,z) + p(z,y) - p(z,z)$.
The pair (X,p) is called a partial metric space(PMS).

If p is a partial metric on X, then the function $d_p : X \times X \to \mathbb{R}^+$ given by

$$d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y)$$

is a metric on X. It is clear that
(i) $p(x,y) = 0 \Rightarrow x = y$,
(ii) $x \neq y \Rightarrow p(x,y) > 0$ and
(iii) $p(x,x)$ may not be 0.

Example 1.1. (See e.g. [**14, 18, 3**]) Consider $X = \mathbb{R}^+$ with $p(x,y) = max\{x,y\}$. Then (X,p) is a partial metric space. It is clear that p is not a (usual) metric. Note that in this case $d_p(x,y) = |x-y|$.

We now state some basic topological notations(such as convergence, completeness, continuity) on partial metric spaces (See e.g. [**14, 15, 6, 18, 3**]).

Definition 1.2. Let (X,p) be a partial metric space.

(i) A sequence $\{x_n\}$ in (X,p) is said to be convergent to $x \in X$ if and only if $p(x,x) = \lim_{n\to\infty} p(x,x_n)$.
(ii) A sequence $\{x_n\}$ in (X,p) is said to be Cauchy sequence if $\lim_{n,m\to\infty} p(x_n,x_m)$ exists and is finite.
(iii) (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges with respect to τ_p, to a point $x \in X$ such that $p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m)$.

The following lemma is one of the basic results in PMS ([**14, 15, 6, 18, 3**])

Lemma 1.1. *Let (X,p) be a partial metric space.*

(i) *A sequence $\{x_n\}$ in (X,p) is said to be Cauchy sequence if and only if it is a Cauchy sequence in the metric space (X,d_p).*
(ii) *(X,p) is complete if and only if the metric space (X,d_p) is complete. Moreover $\lim_{n\to\infty} d_p(x,x_n) = 0$ if and only if $p(x,x) = \lim_{n\to\infty} p(x,x_n) = \lim_{n,m\to\infty} p(x_n,x_m)$.*

Next we give a simple lemma which will be used in the proof of our main result. For the proof we refer to [**3**].

Lemma 1.2 ([**3**]). *If $\{x_n\}$ converges to z in a partial metric space (X,p) and $p(z,z) = 0$ then $\lim_{n\to\infty} p(x_n,y) = p(z,y)$ for all $y \in X$.*

Samet et al. [**27**] introduced the notion of α - admissible mappings associated with a single map. Later Karapinar et al. [**16**] , Shahi et al. [**28**], Abdeljawad [**4**] and Rao et al. [**23**] extended α - admissible mappings associated with two and four mappings and proved fixed and common fixed point theorems for mappings on various spaces.

Definition 1.3. Let X be a non empty set and $\alpha : X \times X \to \mathbb{R}^+$

(*i*) ([**27**]): A mapping of $T : X \to X$ is called α - admissible if $\alpha(x, y) \geqslant 1$ implies $\alpha(Tx, Ty) \geqslant 1$ for all $x, y \in X$.

(*ii*) ([**16**]): A mapping of $T : X \to X$ is called triangular α - admissible if $\alpha(x, y) \geqslant 1 \Rightarrow \alpha(Tx, Ty) \geqslant 1$ for all $x, y \in X$ and $\alpha(x, z) \geqslant 1$ and $\alpha(z, y) \geqslant 1 \Rightarrow \alpha(x, y) \geqslant 1$ for all $x, y, z \in X$.

(*iii*) ([**28**]): Let $f, g : X \to X$. Then f is said to be α- admissible with respect to g if $\alpha(gx, gy) \geqslant 1$ implies $\alpha(fx, fy) \geqslant 1$ for all $x, y \in X$.

(*iv*) ([**4**]): Let $f, g : X \to X$. Then the pair (f, g) is said to be α-admissible if $\alpha(x, y) \geqslant 1$ implies $\alpha(fx, gy) \geqslant 1$ and $\alpha(gx, fy) \geqslant 1$ for all $x, y \in X$.

(*v*) ([**23**]): Let $f, g, S, T : X \to X$. Then the pair (f, g) is said to be α-admissible w.r.to the pair (S, T) if $\alpha(Sx, Ty) \geqslant 1$ implies $\alpha(fx, gy) \geqslant 1$ and $\alpha(Tx, Sy) \geqslant 1$ implies $\alpha(gx, fy) \geqslant 1$ for all $x, y \in X$.Furthermore, we say that the pair (f, g) is triangular α-admissible with respect to the pair (S, T) if (f, g) is α- admissible w.r.to the pair (S, T) and $\alpha(x, z) \geqslant 1, \alpha(z, y) \geqslant 1 \Rightarrow \alpha(x, y) \geqslant 1$ for all $x, y, z \in X$.

Recently Abbas et al. [**1, 2**] introduced the new concepts in a partially ordered set as follows.

Definition 1.4. ([**1, 2**]) Let $(X, \preceq)$ be a partially ordered set and $f : X \to X$.

(*i*) f is said to be a dominating map if $x \preceq fx, \forall x \in X$.

(*ii*) f is said to be dominated if $fx \preceq x, \forall x \in X$.

In 1977, Alber et al. [**5**] generalized the Banach contraction principle by introducing the concept weak contraction mappings in Hilbert space and proved that every weak contraction mapping on a Hilbert space has a unique fixed point.

Rhodes [**24**] extended weak contraction principle in Hilbert spaces to metric spaces. Later many authors, for example, [**10, 11, 8, 17, 20, 21, 19**] obtained generalizations and extensions of the weak contraction principle to obtain fixed, common fixed, coupled and common coupled fixed point theorems in various spaces.

Definition 1.5. Let X be a non-empty set and $f : X \to \mathbb{R}^+$. Then f is called lower semi continuous at $x \in X$ if $f(x) \leqslant \lim_{n\to\infty} inf\ f(x_n)$ whenever $\{x_n\} \subset X$ with $\lim_{n\to\infty} x_n = x$. Let

$\Psi = \{\psi : \mathbb{R}^+ \to \mathbb{R}^+ \textit{ such that } \psi \textit{ is continuous and } \psi(t) = 0 \Leftrightarrow t = 0\}$, and

$$\Phi = \left\{ \begin{array}{c} \phi : \mathbb{R}^+ \to \mathbb{R}^+ \textit{ such that } \phi \textit{ is lower semi continuous and} \\ \phi(t) = 0 \Leftrightarrow t = 0 \end{array} \right\}.$$

Recently Seonghoon Cho [**9**] proved the following theorem.

Theorem 1.1 (Theorem 2.1 of [**9**]). *Let* (X, d) *be a complete metric space and* $T : X \to X$ *be a mapping satisfying*

$$\psi(d(Tx, Ty) + \varphi(Tx) + \varphi(Ty)) \leqslant \psi(m(x, y, d, T, \varphi)) - \phi(l(x, y, d, T, \varphi))$$

for all $x, y \in X$*, where* $\psi \in \Psi$*,* $\phi \in \Phi$*,* $\varphi : X \to \mathbb{R}^+$ *is a lower semi continuous function and*

$$m(x,y,d,T,\varphi)) = \max \left\{ \begin{array}{l} d(x,y) + \varphi(x) + \varphi(y), \\ d(x,Tx) + \varphi(x) + \varphi(Tx), \\ d(y,Ty) + \varphi(y) + \varphi(Ty), \\ \frac{1}{2} \left[\begin{array}{l} d(x,Ty) + \varphi(x) + \varphi(Ty) + \\ d(y,Tx) + \varphi(y) + \varphi(Tx) \end{array} \right] \end{array} \right\}$$

and

$$l(x,y,d,T,\varphi) = \max \left\{ d(x,y) + \varphi(x) + \varphi(y), d(y,Ty) + \varphi(y) + \varphi(Ty) \right\}.$$

Then there exists a unique $z \in X$ *such that* $Tz = z$ *and* $\varphi(z) = 0$*.*

Using these concepts, we prove one common fixed point theorem for four maps in partially ordered partial metric spaces. Our theorem generalize and extend the Theorem 2.1 of Seonghoon Cho [**9**]. We also give an example to illustate our theorem. We call the condition (2.1.3) as generalized α-weakly contractive condition associated with four maps involved in it.
Now we give our main result.

2. The Main Result

Theorem 2.1. *Let* $(X, p, \preceq)$ *be a partially ordered partial metric space,* $\alpha : X \times X \to \mathbb{R}^+$ *be an admissible function and* $f, g, S, T : X \to X$ *be mappings satisfying*

(2.1.1) f *and* g *are dominated and* S*,* T *are dominating mappings respectively,*

(2.1.2) $f(X) \subseteq T(X)$ *and* $g(X) \subseteq S(X)$ *,*

(2.1.3) $\alpha(Sx,Ty)\psi(p(fx,gy)+\varphi(fx)+\varphi(gy)) \leqslant \psi(M(x,y)) - \phi(M(x,y))$ *for all comparable elements* $x, y \in X$*, where* $\psi \in \Psi$*,* $\phi \in \Phi$*,*
$\varphi : X \to \mathbb{R}^+$ *is a lower semi continuous function and*

$$M(x,y) = \max \left\{ \begin{array}{l} p(Sx,Ty) + \varphi(Sx) + \varphi(Ty), \\ p(Sx,fx) + \varphi(Sx) + \varphi(fx), \\ p(Ty,gy) + \varphi(Ty) + \varphi(gy), \\ \frac{1}{2} \left[\begin{array}{l} p(Sx,gy) + \varphi(Sx) + \varphi(gy) + \\ p(Ty,fx) + \varphi(Ty) + \varphi(fx) \end{array} \right] \end{array} \right\}$$

(2.1.4) *the pair* (f, g) *is triangular* α*-admissible with respect to the pair* (S, T)*,*

(2.1.5) $\alpha(Sx_1, fx_1) \geqslant 1$ *and* $\alpha(fx_1, Sx_1) \geqslant 1$ *for some* $x_1 \in X$*,*

(2.1.6) *If for a non-increasing sequence* $\{x_n\}$ *in* X *with* $y_n \preceq x_n$ *for all* $n \in \mathbb{N}$ *and* $y_n \to u$ *for some* $u \in X$ *implies* $u \preceq x_n$ *for all* $n \in \mathbb{N}$*,*

(2.1.7)(a) *Suppose* $S(X)$ *is a complete sub space of* X*. Further assume that* $\alpha(\theta, y_{2n-1}) \geqslant 1$*, for all* $n \in \mathbb{N}$ *and* $\alpha(\theta,\theta) \geqslant 1$ *whenever there exists a sequence* $\{y_n\}$ *in* X *such that* $\alpha(y_n, y_{n+1}) \geqslant 1$*,* $\alpha(y_{n+1}, y_n) \geqslant 1$ *for all* $n \in \mathbb{N}$ *and* $y_n \to \theta$ *for some* $\theta \in X$*.*

(or)

(2.1.7)(b) *Suppose $T(X)$ is a complete sub space of X. Further assume that $\alpha(y_{2n},\theta) \geqslant 1$, for all $n \in \mathbb{N}$ and $\alpha(\theta,\theta) \geqslant 1$ whenever there exists a sequence $\{y_n\}$ in X such that $\alpha(y_n, y_{n+1}) \geqslant 1$, $\alpha(y_{n+1}, y_n) \geqslant 1$ for all $n \in \mathbb{N}$ and $y_n \to \theta$ for some $\theta \in X$.*

Then f, g, S and T have a common fixed point in X.

PROOF. From (2.1.5), there exists $x_1 \in X$ such that $\alpha(Sx_1, fx_1) \geqslant 1$ and $\alpha(fx_1, Sx_1) \geqslant 1$. From (2.1.2), there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that $y_{2n+1} = fx_{2n+1} = Tx_{2n+2}$, $n = 0, 1, 2, \ldots$ and $y_{2n} = gx_{2n} = Sx_{2n+1}$, $n = 1, 2, \ldots$.

Now we have

$$\begin{aligned}
\alpha\left(Sx_1, fx_1\right) \geqslant 1 &\Rightarrow \alpha\left(Sx_1, Tx_2\right) \geqslant 1,\ from\ the\ definition\ of\ \{y_n\} \\
&\Rightarrow \alpha\left(fx_1, gx_2\right) \geqslant 1,\ from\ (2.1.4), i.e., \alpha\left(y_1, y_2\right) \geqslant 1 \\
&\Rightarrow \alpha\left(Tx_2, Sx_3\right) \geqslant 1,\ from\ the\ definition\ of\ \{y_n\} \\
&\Rightarrow \alpha\left(gx_2, fx_3\right) \geqslant 1,\ from\ (2.1.4), i.e., \alpha\left(y_2, y_3\right) \geqslant 1 \\
&\Rightarrow \alpha\left(Sx_3, Tx_4\right) \geqslant 1,\ from\ the\ definition\ of\ \{y_n\} \\
&\Rightarrow \alpha\left(fx_3, gx_4\right) \geqslant 1,\ from\ (2.1.4), i.e., \alpha\left(y_3, y_4\right) \geqslant 1.
\end{aligned}$$

Continuing in this way, we have

$$\alpha(y_n, y_{n+1}) \geqslant 1, \forall n \in \mathbb{N}. \tag{2.1}$$

Similarly by using $\alpha(fx_1, Sx_1) \geqslant 1$ we can show that

$$\alpha(y_{n+1}, y_n) \geqslant 1, \forall n \in \mathbb{N}. \tag{2.2}$$

From triangular α - admissible condition (2.1.4), we have

$$\alpha(y_m, y_n) \geqslant 1,\ \forall\ m,\ n\ \in\ \mathbb{N},\ m \geqslant n \tag{2.3}$$

From (2.1.1), we have

$$x_{2n+1} \preceq Sx_{2n+1} = gx_{2n} \preceq x_{2n} \preceq Tx_{2n} = fx_{2n-1} \preceq x_{2n-1}$$

Thus

$$x_{n+1} \preceq x_n, \forall n \in \mathbb{N}. \tag{2.4}$$

Case(i): Suppose $p(y_n, y_{n+1}) + \varphi(y_n) + \varphi(y_{n+1}) = 0$ for some n. Let $n = 2m$. Then $y_{2m} = y_{2m+1}$ and $\varphi(y_{2m}) = \varphi(y_{2m+1}) = 0$. From (2.1.3) and (2.4) and (2.1), we have

$$\alpha(Sx_{2m+1}, Tx_{2m+2}) = \alpha(y_{2m}, y_{2m+1}) \geqslant 1.$$

From (2.1.3) and (2.4), we have

(2.5)

$$\begin{aligned}
&\psi\left(p(y_{2m+1}, y_{2m+2}) + \varphi(y_{2m+1}) + \varphi(y_{2m+2})\right) \\
&\quad = \psi(p(fx_{2m+1}, gx_{2m+2}) + \varphi(fx_{2m+1}) + \varphi(gx_{2m+2})) \\
&\quad \leqslant \alpha(Sx_{2m+1}, Tx_{2m+2})\psi(p(fx_{2m+1}, gx_{2m+2}) + \varphi(fx_{2m+1}) + \varphi(gx_{2m+2})) \\
&\quad \leqslant \psi(M(x_{2m+1}, x_{2m+2})) - \phi(M(x_{2m+1}, x_{2m+2}))
\end{aligned}$$

where

$$M(x_{2m+1}, x_{2m+2}) = max \left\{ \begin{array}{l} p(y_{2m}, y_{2m+1}) + \varphi(y_{2m}) + \varphi(y_{2m+1}), \\ p(y_{2m}, y_{2m+1}) + \varphi(y_{2m}) + \varphi(y_{2m+1}), \\ p(y_{2m+1}, y_{2m+2}) + \varphi(y_{2m+1}) + \varphi(y_{2m+2}), \\ \frac{1}{2} \left[\begin{array}{l} p(y_{2m}, y_{2m+2}) + \varphi(y_{2m}) + \varphi(y_{2m+2}) + \\ p(y_{2m+1}, y_{2m+1}) + \varphi(y_{2m+1}) + \varphi(y_{2m+1}) \end{array} \right] \end{array} \right\}$$

But

$$\begin{aligned} \frac{1}{2} \left[\begin{array}{l} p(y_{2m}, y_{2m+2}) + \varphi(y_{2m}) + \varphi(y_{2m+2}) \\ p(y_{2m+1}, y_{2m+1}) + \varphi(y_{2m+1}) + \varphi(y_{2m+1}) \end{array} \right] & \\ & \leqslant \frac{1}{2} \left[\begin{array}{l} p(y_{2m}, y_{2m+1}) + p(y_{2m+1}, y_{2m+2}) - \\ p(y_{2m+1}, y_{2m+1}) + \varphi(y_{2m}) + \varphi(y_{2m+2}) + \\ p(y_{2m+1}, y_{2m+1}) + \varphi(y_{2m+1}) + \varphi(y_{2m+1}) \end{array} \right] \\ & \leqslant \max \left\{ \begin{array}{l} p(y_{2m}, y_{2m+1}) + \varphi(y_{2m}) + \varphi(y_{2m+1}), \\ p(y_{2m+1}, y_{2m+2}) + \varphi(y_{2m+1}) + \varphi(y_{2m+2})\} \end{array} \right\} \end{aligned}$$

Thus

$$\begin{aligned} M(x_{2m+1}, x_{2m+2}) &= \max \left\{ \begin{array}{l} p(y_{2m}, y_{2m+1}) + \varphi(y_{2m}) + \varphi(y_{2m+1}), \\ p(y_{2m+1}, y_{2m+2}) + \varphi(y_{2m+1}) + \varphi(y_{2m+2}) \end{array} \right\}. \\ &= p(y_{2m+1}, y_{2m+2}) + \varphi(y_{2m+1}) + \varphi(y_{2m+2}) \text{ from case (i).} \end{aligned}$$

Now (2.5) becomes

$$\psi \left(\begin{array}{l} p(y_{2m+1}, y_{2m+2}) \\ \varphi(y_{2m+1}) + \varphi(y_{2m+2}) \end{array} \right) \leqslant \psi \left(\begin{array}{l} p(y_{2m+1}, y_{2m+2}) \\ \varphi(y_{2m+1}) + \varphi(y_{2m+2}) \end{array} \right) - \phi \left(\begin{array}{l} p(y_{2m+1}, y_{2m+2}) \\ \varphi(y_{2m+1}) + \varphi(y_{2m+2}) \end{array} \right)$$

which in turn yields that

$$\phi(p(y_{2m+1}, y_{2m+2}) + \varphi(y_{2m+1}) + \varphi(y_{2m+2})) = 0.$$

Hence

$$p(y_{2m+1}, y_{2m+2}) + \varphi(y_{2m+1}) + \varphi(y_{2m+2}) = 0.$$

Thus

$$y_{2m+1} = y_{2m+2}, \ \ \varphi(y_{2m+1}) = \varphi(y_{2m+2}) = 0.$$

Continuing in this way, we get $y_{2m} = y_{2m+1} = y_{2m+2} = \cdots$. Thus $\{y_n\}$ is a constant Cauchy sequence.

Case(ii): Assume that $p(y_n, y_{n+1}) + \varphi(y_n) + \varphi(y_{n+1}) \neq 0$ for all n. Then as in Case (i) and (2.5) we have

$$\psi \left(\ p(y_{2n+1}, y_{2n+2}) + \varphi(y_{2n+1}) + \varphi(y_{2n+2}) \ \right) \leqslant \psi\left(M(x_{2n+1}, x_{2n+2})\right) - \phi\left(M(x_{2n+1}, x_{2n+2})\right) \tag{2.6}$$

where

$$M(x_{2n+1}, x_{2n+2}) = max \left\{ \begin{array}{l} p(y_{2n}, y_{2n+1}) + \varphi(y_{2n}) + \varphi(y_{2n+1}), \\ p(y_{2n+1}, y_{2n+2}) + \varphi(y_{2n+1}) + \varphi(y_{2n+2}) \end{array} \right\}$$

If

$$p(y_{2n}, y_{2n+1}) + \varphi(y_{2n}) + \varphi(y_{2n+1}) < p(y_{2n+1}, y_{2n+2}) + \varphi(y_{2n+1}) + \varphi(y_{2n+2}),$$

then

$$M(x_{2n+1}, x_{2n+2}) = p(y_{2n+1}, y_{2n+2}) + \varphi(y_{2n+1}) + \varphi(y_{2n+2})$$

Now (2.6) becomes

$$\psi\begin{pmatrix} p(y_{2n+1}, y_{2n+2}) + \\ \varphi(y_{2n+1}) + +\varphi(y_{2n+2}) \end{pmatrix} \leqslant \psi\begin{pmatrix} p(y_{2n+1}, y_{2n+2}) + \\ \varphi(y_{2n+1}) + \varphi(y_{2n+2}) \end{pmatrix} - \phi\begin{pmatrix} p(y_{2n+1}, y_{2n+2}) + \\ \varphi(y_{2n+1}) + \varphi(y_{2n+2}) \end{pmatrix}$$

which in turn yields that

$$\phi(p(y_{2n+1}, y_{2n+2}) + \varphi(y_{2n+1}) + \varphi(y_{2n+2})) = 0.$$

Thus

$$p(y_{2n+1}, y_{2n+2}) + \varphi(y_{2n+1}) + \varphi(y_{2n+2}) = 0$$

which is a contradiction to Case (ii). Hence

$$p(y_{2n+1}, y_{2n+2}) + \varphi(y_{2n+1}) + \varphi(y_{2n+2}) \leqslant p(y_{2n}, y_{2n+1}) + \varphi(y_{2n}) + \varphi(y_{2n+1}).$$

Similarly we can show that

$$p(y_{2n}, y_{2n+1}) + \varphi(y_{2n}) + \varphi(y_{2n+1}) \leqslant p(y_{2n-1}, y_{2n}) + \varphi(y_{2n-1}) + \varphi(y_{2n}).$$

Thus $\{p(y_n, y_{n+1}) + \varphi(y_n) + \varphi(y_{n+1})\}$ is non-increasing sequence of non-negative real numbers and hence converges to $r \geqslant 0$.

Now from (2.6), we have

$$\psi\begin{pmatrix} p(y_{2n+1}, y_{2n+2}) + \\ \varphi(y_{2n+1}) + \varphi(y_{2n+2}) \end{pmatrix} \leqslant \psi\left(p(y_{2n}, y_{2n+1}) + \varphi(y_{2n}) + \varphi(y_{2n+1})\right) - \phi\left(p(y_{2n}, y_{2n+1}) + \varphi(y_{2n}) + \varphi(y_{2n+1})\right) \tag{2.7}$$

Assume $r > 0$.

Letting $n \to \infty$ in (2.7) and using continuity of ψ and lower semi continuity of ϕ, we get $\psi(r) \leqslant \psi(r) - \phi(r)$ which in turn yields that $\phi(r) = 0$ so that $r = 0$. Thus $\lim_{n\to\infty} [p(y_n, y_{n+1}) + \varphi(y_n) + \varphi(y_{n+1})] = 0$. Hence

$$\lim_{n\to\infty} p(y_n, y_{n+1}) = 0 \tag{2.8}$$

and

$$\lim_{n\to\infty} \varphi(y_n) = 0 \tag{2.9}$$

Now we prove that $\{y_{2n}\}$ is Cauchy. On contrary, suppose that $\{y_{2n}\}$ is not Cauchy. Then there exist $\epsilon > 0$ and monotone increasing sequences of natural numbers $\{2m(k)\}$ and $\{2n(k)\}$ such that $n(k) > m(k)$,

$$d_p(y_{2m(k)}, y_{2n(k)}) \geqslant \epsilon \tag{2.10}$$

and

$$d_p(y_{2m(k)}, y_{2n(k)-2}) < \epsilon \tag{2.11}$$

From (2.10)

$$\begin{aligned}\epsilon &\leqslant d_p(y_{2m(k)}, y_{2n(k)})\\ &\leqslant d_p(y_{2m(k)}, y_{2n(k)-2}) + d_p(y_{2n(k)-2}, y_{2n(k)-1}) + d_p(y_{2n(k)-1}, y_{2n(k)})\\ &< \epsilon + d_p(y_{2n(k)-2}, y_{2n(k)-1}) + d_p(y_{2n(k)-1}, y_{2n(k)}),\ \ from\ (2.11)\end{aligned}$$

Letting $k \to \infty$ and using (2.8), we have

$$\lim_{k\to\infty} d_p(y_{2m(k)}, y_{2n(k)}) = \epsilon$$

From the definition of d_p and (2.8), we have

$$\lim_{k\to\infty} p(y_{2m(k)}, y_{2n(k)}) = \frac{\epsilon}{2} \tag{2.12}$$

Letting $k \to \infty$ and using (2.12) and (2.8) in
$|d_p(y_{2n(k)-1}, y_{2m(k)}) - d_p(y_{2m(k)}, y_{2n(k)})| \leqslant d_p(y_{2n(k)-1}, y_{2n(k)})$,
we get

$$\lim_{k\to\infty} d_p(y_{2n(k)-1}, y_{2m(k)}) = \epsilon$$

From the definition of d_p, we have

$$\lim_{k\to\infty} d(y_{2n(k)-1}, y_{2m(k)}) = \frac{\epsilon}{2} \tag{2.13}$$

Letting $k \to \infty$ and using (2.12) and (2.8) in
$|d_p(y_{2n(k)}, y_{2m(k)+1}) - d_p(y_{2n(k)}, y_{2m(k)})| \leqslant d_p(y_{2m(k)+1}, y_{2m(k)})$,
we get

$$\lim_{k\to\infty} d_p(y_{2n(k)}, y_{2m(k)+1}) = \epsilon$$

From the definition of d_p, we have

$$\lim_{k\to\infty} p(y_{2n(k)}, y_{2m(k)+1}) = \frac{\epsilon}{2} \tag{2.14}$$

Letting $k \to \infty$ and using (2.12) and (2.8) in

$$\left|d_p(y_{2m(k)+1}, y_{2n(k)-1}) - d_p(y_{2m(k)}, y_{2n(k)})\right| \leqslant \begin{pmatrix} d_p(y_{2m(k)+1}, y_{2m(k)}) \\ +d_p(y_{2n(k)-1}, y_{2n(k)}) \end{pmatrix}$$

we get

$$\lim_{k\to\infty} d_p(y_{2m(k)+1}, y_{2n(k)-1}) = \epsilon$$

From the definition of d_p, we have

$$\lim_{k\to\infty} p(y_{2m(k)+1}, y_{2n(k)-1}) = \frac{\epsilon}{2} \tag{2.15}$$

$\alpha(Sx_{2m(k)+1}, Tx_{2n(k)}) = \alpha(y_{2m(k)}, y_{2n(k)-1}) \geqslant 1$ from (2.3). Also from (2.4), $x_{2n(k)} \preceq x_{2m(k)+1}$. From (2.1.3), we have

(2.16)

$$\begin{aligned}\psi\begin{pmatrix} (p(y_{2m(k)+1}, y_{2n(k)}) + \\ \varphi(y_{2m(k)+1}) + \varphi(y_{2n(k)})) \end{pmatrix} &= \psi\begin{pmatrix} p(fx_{2m(k)+1}, gx_{2n(k)}) + \\ \varphi(fx_{2m(k)+1}) + \varphi(gx_{2n(k)}) \end{pmatrix}\\ &\leqslant \alpha\left(Sx_{2m(k)+1}, Tx_{2n(k)}\right)\psi\begin{pmatrix} p(fx_{2m(k)+1}, gx_{2n(k)}) + \\ \varphi(fx_{2m(k)+1}) + \varphi(gx_{2n(k)}) \end{pmatrix}\\ &\leqslant \psi\left(M(x_{2m(k)+1}, x_{2n(k)})\right) - \phi\left(M(x_{2m(k)+1}, x_{2n(k)})\right)\end{aligned}$$

where

$$M(x_{2m(k)+1}, x_{2n(k)}) = \max \left\{ \begin{array}{l} p(y_{2m(k)}, y_{2n(k)-1}) + \varphi(y_{2m(k)}) + \varphi(y_{2n(k)-1}), \\ p(y_{2m(k)}, y_{2m(k)+1}) + \varphi(y_{2m(k)}) + \varphi(y_{2m(k)+1}), \\ p(y_{2n(k)-1}, y_{2n(k)}) + \varphi(y_{2n(k)-1}) + \varphi(y_{2n(k)}), \\ \frac{1}{2} \left[\begin{array}{l} p(y_{2m(k)}, y_{2n(k)}) + \varphi(y_{2m(k)}) + \varphi(y_{2n(k)}) \\ +p(y_{2n(k)-1}, y_{2m(k)+1}) \\ +\varphi(y_{2n(k)-1}) + \varphi(y_{2m(k)+1}) \end{array} \right] \end{array} \right\}$$

$$\to \max\{\tfrac{\epsilon}{2}, 0, 0, \tfrac{1}{2}(\tfrac{\epsilon}{2} + \tfrac{\epsilon}{2})\} = \tfrac{\epsilon}{2}$$

from (2.8),(2.9),(2.12),(2.13) and (2.15)

Letting $n \to \infty$ in (2.16) and using (2.14), we get $\psi(\frac{\epsilon}{2}) \leqslant \psi(\frac{\epsilon}{2}) - \phi(\frac{\epsilon}{2})$ which in turn yields that $\phi(\frac{\epsilon}{2}) = 0$. Hence $\epsilon = 0$. It is a contradiction. Hence $\{y_{2n}\}$ is Cauchy.

Letting $n, m \to \infty$ in

$$|d_p(y_{2n+1}, y_{2m+1}) - d_p(y_{2n}, y_{2m})| \leqslant d_p(y_{2n+1}, y_{2n}) + d_p(y_{2m}, y_{2m+1}),$$

we get

$$\lim_{n,m\to\infty} d_p(y_{2n+1}, y_{2m+1}) = 0.$$

Hence $\{y_{2n+1}\}$ is Cauchy. Thus $\{y_n\}$ is a Cauchy sequence in (X, d_p). Hence, we have $\lim_{n,m\to\infty} d_p(y_n, y_m) = 0$. Now from the definition of d_p, we have

$$\lim_{n,m\to\infty} p(y_n, y_m) = 0 \tag{2.17}$$

Suppose (2.1.7)(a) holds. Since $\{y_{2n}\} = \{Sx_{2n+1}\} \subseteq S(X)$ and $S(X)$ is a complete sub space of X, there exists $z \in S(X)$ such that $\{y_{2n}\}$ converges to z. There exists $u \in X$ such that $z = Su$. Since $\{y_n\}$ is a Cauchy and $\{y_{2n}\}$ converges to z, it follows that $\{y_{2n+1}\}$ also converges to z.

From Lemma 1.1(ii), we have

$$p(z, z) = \lim_{n\to\infty} p(y_{2n+1}, z) = \lim_{n\to\infty} p(y_{2n}, z) = \lim_{n,m\to\infty} p(y_n, y_m).$$

$$p(z, z) = \lim_{n\to\infty} p(y_{2n+1}, z) = \lim_{n\to\infty} p(y_{2n}, z) = 0, from (2.17)$$

Since φ is lower semi continuous, we have

$$\varphi(z) \leqslant \lim_{n\to\infty} inf\ \varphi(y_n) \leqslant \lim_{n\to\infty} \varphi(y_n) = 0,$$

from (2.9). Hence

$$\varphi(z) = 0. \tag{2.18}$$

and $\alpha(Su, Tx_{2n}) = \alpha(z, y_{2n-1}) \geqslant 1$, from (2.1.7)(a). Since S is dominating map, we have $u \preceq Su = z$. Since $gx_{2n} \preceq x_{2n}$ and $gx_{2n} \to z$, by (2.1.6), we have $z \preceq x_{2n}$. Thus $u \preceq x_{2n}$.

Now from(2.1.3), we have

$$\begin{aligned}\psi(p(fu,y_{2n})+\varphi(fu)+\varphi(fy_{2n})) &= \psi(p(fu,gx_{2n})+\varphi(fu)+\varphi(gx_{2n})) \\ &\leqslant \alpha(Su,Tx_{2n})\psi\begin{pmatrix} p(fu,gx_{2n})+ \\ \varphi(fu)+\varphi(gx_{2n})\end{pmatrix} \\ &\leqslant \psi(M(u,x_{2n}))-\phi(M(u,x_{2n}))\end{aligned} \tag{2.19}$$

where

$$M(u,x_{2n}) = \max\left\{\begin{array}{l} p(z,y_{2n-1})+\varphi(z)+\varphi(y_{2n-1}), \\ p(z,fu)+\varphi(z)+\varphi(fu), \\ p(y_{2n-1},y_{2n})+\varphi(y_{2n-1})+\varphi(y_{2n}), \\ \frac{1}{2}\left[\begin{array}{l} p(z,y_{2n})+\varphi(z)+\varphi(y_{2n})+ \\ p(y_{2n-1},fu)+\varphi(y_{2n-1})+\varphi(fu)\end{array}\right]\end{array}\right\}$$

$$\to max\{0,p(z,fu)+\varphi(fu),0,\tfrac{1}{2}[p(z,fu)+\varphi(fu)]\}$$

from (2.8), (2.9), (2.18) and Lemma 1.2.

Letting $n\to\infty$ in (2.19), we get

$$\psi\left(p(fu,z)+\varphi(fu)\right)\leqslant\psi\left(p(z,fu)+\varphi(fu)\right)-\phi\left(p(z,fu)+\varphi(fu)\right)$$

which in turn yields that $\phi(p(z,fu)+\varphi(fu))=0$. Hence $p(z,fu)+\varphi(fu)=0$. Thus $fu=z$. Hence $Su=z=fu$. Since f is dominated and S is dominating maps, we have $z=fu\preceq u$ and $u\preceq Su=z$. Thus $u=z$. Hence

$$Sz=z=fz. \tag{2.20}$$

Since $f(X)\subseteq T(X)$, there exists $v\in X$ such that $z=fz=Tv$. Since T is dominating map, we have $v\preceq Tv=z$. From (2.1.7)(a) $\alpha(Sz,Tv)=\alpha(z,z)\geqslant 1$.

(2.21)

$$\begin{aligned}\psi(p(z,gv)+\varphi(z)+\varphi(gv)) &= \psi(p(fz,gv)+\varphi(fz)+\varphi(gv)) \\ &\leqslant \alpha(Sz,Tv)\ \psi(p(fz,gv)+\varphi(fz)+\varphi(gv)) \\ &\leqslant \psi(M(z,v))-\phi(M(z,v))\end{aligned}$$

where

$$M(z,v) = \max\left\{\begin{array}{l} p(z,z)+\varphi(z)+\varphi(z), \\ p(z,z)+\varphi(z)+\varphi(z), \\ p(z,gv)+\varphi(z)+\varphi(gv), \\ \frac{1}{2}\left[\begin{array}{l} p(z,gv)+\varphi(z)+\varphi(gv) \\ +p(z,z)+\varphi(z)+\varphi(z)\end{array}\right]\end{array}\right\}$$

$$=p(z,gv)+\varphi(gv), \text{ from (2.18).}$$

Now (2.21) becomes

$$\psi(p(z,gv)+\varphi(gv))\leqslant\psi(p(z,gv)+\varphi(gv))-\phi(p(z,gv)+\varphi(gv))$$

which in turn yields that $\phi(p(z,gv)+\varphi(gv))=0$. Thus $gv=z$ and $\varphi(gv)=0$. Hence $gv=z=Tv$. Since g is dominated and T is dominating maps, we have

$z = gv \preceq v$ and $v \preceq Tv = z$. Thus $v = z$. Hence

$$gz = z = Tz. \tag{2.22}$$

From (2.20) and (2.22), it follows that z is a common fixed point of f, g, S and T.
Similarly, we can prove this theorem when (2.1.7)(b) holds. □

Now we give an example to illustrate our main Theorem 2.1.

EXAMPLE 2.1. Let $X = [0, \infty)$ and $p(x, y) = max\{x, y\}$, $\forall\ x, y \in X$. Let $\preceq$ be the ordinary $\leqslant$. Let $f, g, S, T : X \to X$ be defined by $fx = \frac{x}{2}$, $gx = \frac{x}{3}$, $Sx = 6x$ and $Tx = 4x$. Let $\psi, \phi, \varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by $\psi(t) = t$, $\phi(t) = \frac{5t}{6}$, $\varphi(t) = t$, for all $t \in \mathbb{R}^+$. Define

$$\alpha : X \times X \to \mathbb{R}^+ \text{ by } \alpha(x, y) = \begin{cases} 1, & \text{if } x, y \in [0, 1], \\ 2, & \text{otherwise.} \end{cases}$$

We have $fx = \frac{x}{2} \leqslant x$, $gx = \frac{x}{3} \leqslant x$. Also $x \leqslant 6x = Sx$, $x \leqslant 4x = Tx$.

Now we will verify the condition (2.1.3). If $x > \frac{1}{6}$ and $y \in X$ or $x \in X$ and $y > \frac{1}{4}$, then $\alpha(Sx, Ty) = \alpha(6x, 4y) = 2$.

$$\begin{aligned} \alpha(Sx, Ty)[d(fx, gy) + \varphi(fx) + \varphi(gy)] &= 2[max\{\tfrac{x}{2}, \tfrac{y}{3}\} + \tfrac{x}{2} + \tfrac{y}{3}] \\ &= \tfrac{1}{6}[max\{6x, 4y\} + 6x + 4y] \\ &= \tfrac{1}{6}[p(Sx, Ty) + \varphi(Sx) + \varphi(Ty)]. \end{aligned}$$

If $x \leqslant \frac{1}{6}$ and $y \leqslant \frac{1}{4}$, then $\alpha(Sx, Ty) = 1$.

$$\begin{aligned} \alpha(Sx, Ty)[d(fx, gy) + \varphi(fx) + \varphi(gy)] &= max\{\tfrac{x}{2}, \tfrac{y}{3}\} + \tfrac{x}{2} + \tfrac{y}{3} \\ &= \tfrac{1}{12}[max\{6x, 4y\} + 6x + 4y] \\ &< \tfrac{1}{6}[p(Sx, Ty) + \varphi(Sx) + \varphi(Ty)]. \end{aligned}$$

Thus the condition (2.1.3)

$$\begin{aligned} &\alpha(Sx, Ty)[p(fx, gy) + \varphi(fx) + \varphi(gy)] \\ &\leqslant \tfrac{1}{6} \max \left\{ \begin{array}{c} p(Sx, Ty) + \varphi(Sx) + \varphi(Ty), p(Sx, fx) + \varphi(Sx) + \varphi(fx), \\ p(Ty, gy) + \varphi(Ty) + \varphi(gy), \\ \frac{1}{2}\left[p(Sx, gy) + \varphi(Sx) + \varphi(gy) + p(Ty, fx) + \varphi(Ty) + \varphi(fx)\right] \end{array} \right\} \end{aligned}$$

for all $x, y \in X$ is satisfied. One can easily verify the remaining conditions of Theorem 2.1. Clearly 0 is a common fixed point of f, g, S and T.

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