A COMMON FIXED POINT THEOREM
FOR FOUR MAPS SATISFYING GENERALIZED
$\alpha$ - WEAKLY CONTRACTIVE CONDITION
IN ORDERED PARTIAL METRIC SPACES

K. P. R. Rao and A. Sombabu

Abstract. In this paper we obtain a common fixed point theorem for four maps satisfying generalized $\alpha$ - weakly contractive condition and we give an example to illustrate our main theorem. Our result generalize and improve the theorem of Seonghoon Cho [9].

1. Introduction and Preliminaries

There are many generalizations of the concept of metric spaces in the literature. One of them is a partial metric space introduced by Matthews [18] as a part of study of denotational semantics of data flow networks. After that fixed and common fixed point results in partial metric spaces were studied by many other authors, for example [26, 13, 14, 15, 6, 7, 22, 29, 12, 25, 3].

Throughout this paper, $\mathbb{R}^+$ and $\mathbb{N}$ denote the set of all non-negative real numbers and the set of all positive integers respectively.

First we recall some basic definitions and lemmas which play crucial role in the theory of partial metric spaces.

Definition 1.1. ([18]) A partial metric on a non empty set $X$ is a function $p : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$,

$(p_1) \; x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y),$

$(p_2) \; p(x, x) \leq p(x, y), p(y, y) \leq p(x, y),$

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The pair \((X, p)\) is called a partial metric space (PMS).

If \(p\) is a partial metric on \(X\), then the function \(d_p : X \times X \to \mathbb{R}^+\) given by

\[
d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y)
\]

is a metric on \(X\). It is clear that

(i) \(p(x, y) = 0 \Rightarrow x = y\),
(ii) \(x \neq y \Rightarrow p(x, y) > 0\) and
(iii) \(p(x, x)\) may not be 0.

**Example 1.1.** (See e.g. [14, 18, 3]) Consider \(X = \mathbb{R}^+\) with \(p(x, y) = \max\{x, y\}\). Then \((X, p)\) is a partial metric space. It is clear that \(p\) is not a (usual) metric. Note that in this case \(d_p(x, y) = |x - y|\).

We now state some basic topological notations (such as convergence, completeness, continuity) on partial metric spaces (See e.g. [14, 15, 6, 18, 3]).

**Definition 1.2.** Let \((X, p)\) be a partial metric space.

(i) A sequence \(\{x_n\}\) in \((X, p)\) is said to be convergent to \(x \in X\) if and only if \(p(x, x) = \lim_{n \to \infty} p(x, x_n)\).
(ii) A sequence \(\{x_n\}\) in \((X, p)\) is said to be Cauchy sequence if \(\lim_{n, m \to \infty} p(x_n, x_m)\) exists and is finite.
(iii) \((X, p)\) is said to be complete if every Cauchy sequence \(\{x_n\}\) in \(X\) converges with respect to \(\tau_p\), to a point \(x \in X\) such that \(p(x, x) = \lim_{n, m \to \infty} p(x_n, x_m)\).

The following lemma is one of the basic results in PMS ([14, 15, 6, 18, 3]).

**Lemma 1.1.** Let \((X, p)\) be a partial metric space.

(i) A sequence \(\{x_n\}\) in \((X, p)\) is said to be Cauchy sequence if and only if it is a Cauchy sequence in the metric space \((X, d_p)\).
(ii) \((X, p)\) is complete if and only if the metric space \((X, d_p)\) is complete.

Moreover, \(\lim_{n \to \infty} d_p(x_n, x_{n+1}) = 0\) if and only if \(\lim_{n \to \infty} p(x_n, x_{n+1}) = \lim_{n, m \to \infty} p(x_n, x_m)\).

Next we give a simple lemma which will be used in the proof of our main result. For the proof we refer to [3].

**Lemma 1.2 ([3]).** If \(\{x_n\}\) converges to \(z\) in a partial metric space \((X, p)\) and \(p(z, z) = 0\) then \(\lim_{n \to \infty} p(x_n, y) = p(z, y)\) for all \(y \in X\).

Samet et al. [27] introduced the notion of \(\alpha\)-admissible mappings associated with a single map. Later Karapinar et al. [16], Shahi et al. [28], Abdeljawad [4] and Rao et al. [23] extended \(\alpha\)-admissible mappings associated with two and four mappings and proved fixed and common fixed point theorems for mappings on various spaces.
**Definition 1.3.** Let $X$ be a non empty set and $\alpha : X \times X \to \mathbb{R}^+$

(i) ([27]): A mapping of $T : X \to X$ is called $\alpha$ - admissible if $\alpha(x, y) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$ for all $x, y \in X$.

(ii) ([16]): A mapping of $T : X \to X$ is called triangular $\alpha$ - admissible if $\alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1$ for all $x, y \in X$ and $\alpha(x, z) \geq 1$ and $\alpha(z, y) \geq 1 \Rightarrow \alpha(x, y) \geq 1$ for all $x, y, z \in X$.

(iii) ([28]): Let $f, g : X \to X$. Then $f$ is said to be $\alpha$ - admissible with respect to $g$ if $\alpha(gx, gy) \geq 1$ implies $\alpha(fx, fy) \geq 1$ for all $x, y \in X$.

(iv) ([4]): Let $f, g : X \to X$. Then the pair $(f, g)$ is said to be $\alpha$-admissible if $\alpha(x, y) \geq 1$ implies $\alpha(fx, fy) \geq 1$ and $\alpha(gx, fy) \geq 1$ for all $x, y \in X$.

(v) ([23]): Let $f, g, S, T : X \to X$. Then the pair $(f, g)$ is said to be $\alpha$ - admissible w.r.t to the pair $(S, T)$ if $\alpha(Sx, Ty) \geq 1$ implies $\alpha(fx, gx) \geq 1$ and $\alpha(Tx, Sy) \geq 1$ implies $\alpha(fx, fy) \geq 1$ for all $x, y \in X$. Furthermore, we say that the pair $(f, g)$ is triangular $\alpha$-admissible with respect to the pair $(S, T)$ if $(f, g)$ is $\alpha$-admissible w.r.t the pair $(S, T)$ and $\alpha(x, z) \geq 1, \alpha(z, y) \geq 1 \Rightarrow \alpha(x, y) \geq 1$ for all $x, y, z \in X$.

Recently Abbas et al. [1, 2] introduced the new concepts in a partially ordered set as follows.

**Definition 1.4.** ([1, 2]) Let $(X, \preceq)$ be a partially ordered set and $f : X \to X$.

(i) $f$ is said to be a dominating map if $x \preceq fx, \forall x \in X$.

(ii) $f$ is said to be dominated if $fx \preceq x, \forall x \in X$.

In 1977, Alber et al. [5] generalized the Banach contraction principle by introducing the concept weak contraction mappings in Hilbert space and proved that every weak contraction mapping on a Hilbert space has a unique fixed point.

Rhodes [24] extended weak contraction principle in Hilbert spaces to metric spaces. Later many authors, for example, [10, 11, 8, 17, 20, 21, 19] obtained generalizations and extensions of the weak contraction principle to obtain fixed, common fixed, coupled and common coupled fixed point theorems in various spaces.

**Definition 1.5.** Let $X$ be a non-empty set and $f : X \to \mathbb{R}^+$. Then $f$ is called lower semi continuous at $x \in X$ if $f(x) \leq \liminf_{n \to \infty} f(x_n)$ whenever $\{x_n\} \subset X$ with $\lim_{n \to \infty} x_n = x$. Let

\[
\Psi = \{ \psi : \mathbb{R}^+ \to \mathbb{R}^+ \text{ such that } \psi \text{ is continuous and } \psi(t) = 0 \Leftrightarrow t = 0 \},
\]

\[
\Phi = \{ \phi : \mathbb{R}^+ \to \mathbb{R}^+ \text{ such that } \phi \text{ is lower semi continuous and } \phi(t) = 0 \Leftrightarrow t = 0 \}.
\]

Recently Seonghoon Cho [9] proved the following theorem.

**Theorem 1.1** (Theorem 2.1 of [9]). Let $(X, d)$ be a complete metric space and $T : X \to X$ be a mapping satisfying

\[
\psi(d(Tx, Ty) + \varphi(Tx) + \varphi(Ty)) \leq \psi(m(x, y, d, T, \varphi)) - \phi(l(x, y, d, T, \varphi))
\]
for all \( x, y \in X \), where \( \psi \in \Psi \), \( \phi \in \Phi \), \( \varphi : X \to \mathbb{R}^+ \) is a lower semi continuous function and

\[
m(x, y, d, T, \varphi)) = \max \left\{ \begin{array}{l}
d(x, y) + \varphi(x) + \varphi(y), \\
d(x, Tx) + \varphi(x) + \varphi(Tx), \\
d(y, Ty) + \varphi(y) + \varphi(Ty), \\
\frac{1}{2} \left[ d(x, Ty) + \varphi(x) + \varphi(Ty) + \\
\frac{1}{2} d(y, Tx) + \varphi(y) + \varphi(Tx) \right] \end{array} \right\}
\]

and

\[
l(x, y, d, T, \varphi) = \max \{ d(x, y) + \varphi(x) + \varphi(y), d(y, Ty) + \varphi(y) + \varphi(Ty) \}.
\]

Then there exists a unique \( z \in X \) such that \( Tz = z \) and \( \varphi(z) = 0 \).

Using these concepts, we prove one common fixed point theorem for four maps in partially ordered partial metric spaces. Our theorem generalize and extend the Theorem 2.1 of Seonghoon Cho [9]. We also give an example to illustrate our theorem. We call the condition (2.1.3) as generalized \( \alpha \)-weakly contractive condition associated with four maps involved in it.

Now we give our main result.

2. The Main Result

**Theorem 2.1.** Let \( (X, p, \preceq) \) be a partially ordered partial metric space, \( \alpha : X \times X \to \mathbb{R}^+ \) be an admissible function and \( f, g, S, T : X \to X \) be mappings satisfying

1. \( f \) and \( g \) are dominated and \( S, T \) are dominating mappings respectively,
2. \( f(X) \subseteq T(X) \) and \( g(X) \subseteq S(X) \),
3. \( \alpha(Sx, Ty)\psi(p(fx, gy) + \varphi(fx) + \varphi(gy)) \leq \psi(M(x, y)) - \phi(M(x, y)) \) for all comparable elements \( x, y \in X \), where \( \psi \in \Psi \), \( \phi \in \Phi \),
4. \( \varphi : X \to \mathbb{R}^+ \) is a lower semi continuous function and

\[
M(x, y) = \max \left\{ \begin{array}{l}
p(Sx, Ty) + \varphi(Sx) + \varphi(Ty), \\
p(Sx, fx) + \varphi(Sx) + \varphi(fx), \\
p(Ty, gy) + \varphi(Ty) + \varphi(gy), \\
\frac{1}{2} \left[ p(Sx, gy) + \varphi(Sx) + \varphi(gy) + \\
\frac{1}{2} p(Ty, fx) + \varphi(Ty) + \varphi(fx) \right] \end{array} \right\}
\]

4.1. the pair \((f, g)\) is triangular \( \alpha \)-admissible with respect to the pair \((S, T)\),
4.2. \( \alpha(Sx_1, fx_1) \geq 1 \) and \( \alpha(fx_1, Sx_1) \geq 1 \) for some \( x_1 \in X \),
4.3. If for a non-increasing sequence \( \{x_n\} \) in \( X \) with \( y_n \preceq x_n \) for all \( n \in \mathbb{N} \) and \( y_n \to u \) for some \( u \in X \) implies \( u \preceq x_n \) for all \( n \in \mathbb{N} \),
4.4. Suppose \( S(X) \) is a complete sub space of \( X \). Further assume that \( \alpha(\theta, y_{2n-1}) \geq 1 \), for all \( n \in \mathbb{N} \) and \( \alpha(\theta, y_0) \geq 1 \) whenever there exists a sequence \( \{y_n\} \) in \( X \) such that \( \alpha(y_n, y_{n+1}) \geq 1 \), \( \alpha(y_{n+1}, y_n) \geq 1 \) for all \( n \in \mathbb{N} \) and \( y_n \to \theta \) for some \( \theta \in X \).
From (2.1.3) and (2.4), we have
\[ y \]
Then from (2.1.1), we have
\[ x \]
Continuing in this way, we have
\[ y \]
Proof. From (2.1.5), there exists \( x_1 \in X \) such that \( \alpha(Sx_1, fx_1) \geq 1 \) and \( \alpha(fx_1, Sx_1) \geq 1 \). From (2.1.2), there exist sequences \( \{x_n\} \) and \( \{y_n\} \) in \( X \) such that
\[ y_{2n+1} = fx_{2n+1} = Tx_{2n+2}, \quad n = 0, 1, 2, \ldots \]
and
\[ y_{2n} = gx_{2n} = Sx_{2n+1}, \quad n = 1, 2, \ldots \]
Now we have
\[ \alpha(Sx_1, fx_1) \geq 1 \Rightarrow \alpha(Sx_1, Tx_2) \geq 1, \text{ from the definition of } \{y_n\} \]
\[ \Rightarrow \alpha(fx_1, gx_2) \geq 1, \text{ from (2.1.4), i.e., } \alpha(y_1, y_2) \geq 1 \]
\[ \Rightarrow \alpha(Tx_2, Sx_3) \geq 1, \text{ from the definition of } \{y_n\} \]
\[ \Rightarrow \alpha(gx_2, fx_3) \geq 1, \text{ from (2.1.4), i.e., } \alpha(y_2, y_3) \geq 1 \]
\[ \Rightarrow \alpha(Sx_3, Tx_4) \geq 1, \text{ from the definition of } \{y_n\} \]
\[ \Rightarrow \alpha(fx_3, gx_4) \geq 1, \text{ from (2.1.4), i.e., } \alpha(y_3, y_4) \geq 1. \]
Continuing in this way, we have
\[ (2.1) \quad \alpha(y_n, y_{n+1}) \geq 1, \forall n \in \mathbb{N}. \]
Similarly by using \( \alpha(fx_1, Sx_1) \geq 1 \) we can show that
\[ (2.2) \quad \alpha(y_{n+1}, y_n) \geq 1, \forall n \in \mathbb{N}. \]
From triangular \( \alpha \) - admissible condition (2.1.4), we have
\[ (2.3) \quad \alpha(y_m, y_n) \geq 1, \forall m, n \in \mathbb{N}, m \geq n \]
From (2.1.1), we have
\[ x_{2n+1} \preceq Sx_{2n+1} = gx_{2n} \preceq x_{2n} \preceq Tx_{2n} = fx_{2n-1} \preceq x_{2n-1} \]
Thus
\[ (2.4) \quad x_{n+1} \preceq x_n, \forall n \in \mathbb{N}. \]
Case(i): Suppose \( p(y_n, y_{n+1}) + \varphi(y_n) + \varphi(y_{n+1}) = 0 \) for some \( n \). Let \( n = 2m \). Then \( y_{2m} = y_{2m+1} \) and \( \varphi(y_{2m}) = \varphi(y_{2m+1}) = 0 \). From (2.1.3) and (2.4) and (2.1), we have
\[ \alpha(Sx_{2m+1}, Tx_{2m+2}) = \alpha(y_{2m}, y_{2m+1}) \geq 1. \]
From (2.1.3) and (2.4), we have
\[ (2.5) \quad \psi(p(y_{2m+1}, y_{2m+2}) + \varphi(y_{2m+1}) + \varphi(y_{2m+2})) \]
\[ = \psi(p(fx_{2m+1}, gx_{2m+2}) + \varphi(fx_{2m+1}) + \varphi(gx_{2m+2})) \]
\[ \leq \alpha(Sx_{2m+1}, Tx_{2m+2}) \psi(p(fx_{2m+1}, gx_{2m+2}) + \varphi(fx_{2m+1}) + \varphi(gx_{2m+2})) \]
\[ \leq \psi(M(x_{2m+1}, x_{2m+2})) - \phi(M(x_{2m+1}, x_{2m+2})) \]
\[ M(x_{2m+1}, x_{2m+2}) = \max \begin{cases} 
\varphi(y_{2m}) + \varphi(y_{2m+1}) + \varphi(y_{2m+2}), \\
\varphi(y_{2m}) + \varphi(y_{2m+1}), \\
p(y_{2m+1}, y_{2m+2}) + \varphi(y_{2m+1}) + \varphi(y_{2m+2}), \\
p(y_{2m+1}, y_{2m+2}) + \varphi(y_{2m+1}) + \varphi(y_{2m+2}) + \frac{1}{2} \varphi(y_{2m+2}). 
\end{cases} \]

But
\[
\frac{1}{2} \left[ p(y_{2m}, y_{2m+2}) + \varphi(y_{2m}) + \varphi(y_{2m+2}) - \frac{1}{2} \left[ p(y_{2m+1}, y_{2m+2}) + \varphi(y_{2m+1}) + \varphi(y_{2m+2}) \right] \right] \leq \max \left\{ \frac{1}{2} \left[ p(y_{2m}, y_{2m+1}) + \varphi(y_{2m}) + \varphi(y_{2m+1}) \right] \right\}
\]

Thus
\[ M(x_{2m+1}, x_{2m+2}) = \max \left\{ \frac{1}{2} \left[ p(y_{2m}, y_{2m+1}) + \varphi(y_{2m}) + \varphi(y_{2m+1}) \right] \right\}.
\]

Now (2.5) becomes
\[
\psi \left( \frac{1}{2} \left[ p(y_{2m+1}, y_{2m+2}) + \varphi(y_{2m+1}) + \varphi(y_{2m+2}) \right] \right) \leq \psi \left( \frac{1}{2} \left[ p(y_{2m}, y_{2m+1}) + \varphi(y_{2m}) + \varphi(y_{2m+1}) \right] \right)
\]
which in turn yields that
\[
\psi(p(y_{2m+1}, y_{2m+2}) + \varphi(y_{2m+1}) + \varphi(y_{2m+2})) = 0.
\]
Hence
\[
p(y_{2m+1}, y_{2m+2}) + \varphi(y_{2m+1}) + \varphi(y_{2m+2}) = 0.
\]
Thus
\[
y_{2m+1} = y_{2m+2}, \quad \varphi(y_{2m+1}) = \varphi(y_{2m+2}) = 0.
\]
Continuing in this way, we get \( y_{2m} = y_{2m+1} = y_{2m+2} = \ldots \). Thus \( \{y_n\} \) is a constant Cauchy sequence.

**Case (ii):** Assume that \( p(y_n, y_{n+1}) + \varphi(y_n) + \varphi(y_{n+1}) \neq 0 \) for all \( n \). Then as in Case (i) and (2.5) we have
\[
(2.6) \quad \psi \left( p(y_{2n+1}, y_{2n+2}) + \varphi(y_{2n+1}) + \varphi(y_{2n+2}) \right) \leq \psi \left( M(x_{2n+1}, x_{2n+2}) \right) - \phi \left( M(x_{2n+1}, x_{2n+2}) \right)
\]
where
\[ M(x_{2n+1}, x_{2n+2}) = \max \left\{ p(y_{2n}, y_{2n+1}) + \varphi(y_{2n}) + \varphi(y_{2n+1}), \right. \\
p(y_{2n+1}, y_{2n+2}) + \varphi(y_{2n+1}) + \varphi(y_{2n+2}) \left. \right\} \]
If
\[
p(y_{2n}, y_{2n+1}) + \varphi(y_{2n}) + \varphi(y_{2n+1}) < p(y_{2n+1}, y_{2n+2}) + \varphi(y_{2n+1}) + \varphi(y_{2n+2}),
\]
then

\[ M(x_{2n+1}, x_{2n+2}) = p(y_{2n+1}, y_{2n+2}) + \varphi(y_{2n+1}) + \varphi(y_{2n+2}) \]

Now (2.6) becomes

\[ \psi \left( p(y_{2n+1}, y_{2n+2}) + \varphi(y_{2n+1}) + \varphi(y_{2n+2}) \right) \leq \psi \left( p(y_{2n+1}, y_{2n+2}) + \varphi(y_{2n+1}) + \varphi(y_{2n+2}) \right) - \phi \]

which in turn yields that

\[ \phi(p(y_{2n+1}, y_{2n+2}) + \varphi(y_{2n+1}) + \varphi(y_{2n+2})) = 0. \]

Thus

\[ p(y_{2n+1}, y_{2n+2}) + \varphi(y_{2n+1}) + \varphi(y_{2n+2}) = 0 \]

which is a contradiction to Case (ii). Hence

\[ p(y_{2n+1}, y_{2n+2}) + \varphi(y_{2n+1}) + \varphi(y_{2n+2}) \leq p(y_{2n}, y_{2n+1}) + \varphi(y_{2n}) + \varphi(y_{2n+1}). \]

Similarly we can show that

\[ p(y_{2n}, y_{2n+1}) + \varphi(y_{2n}) + \varphi(y_{2n+1}) \leq p(y_{2n-1}, y_{2n}) + \varphi(y_{2n-1}) + \varphi(y_{2n}). \]

Thus \( \{p(y_n, y_{n+1}) + \varphi(y_n) + \varphi(y_{n+1})\} \) is non-increasing sequence of non-negative real numbers and hence converges to \( r \geq 0 \).

Now from (2.6), we have

\[ \psi \left( p(y_{2n+1}, y_{2n+2}) + \varphi(y_{2n+1}) + \varphi(y_{2n+2}) \right) \leq \psi \left( p(y_{2n}, y_{2n+1}) + \varphi(y_{2n}) + \varphi(y_{2n+1}) \right) - \phi \]

Assume \( r > 0 \).

Letting \( n \to \infty \) in (2.7) and using continuity of \( \psi \) and lower semi continuity of \( \phi \), we get \( \psi(r) \leq \psi(r) - \phi(r) \) which in turn yields that \( \phi(r) = 0 \) so that \( r = 0 \).

Thus \( \lim_{n \to \infty} [p(y_n, y_{n+1}) + \varphi(y_n) + \varphi(y_{n+1})] = 0. \) Hence

\[ \lim_{n \to \infty} p(y_n, y_{n+1}) = 0 \]

and

\[ \lim_{n \to \infty} \varphi(y_n) = 0 \]

Now we prove that \( \{y_{2n}\} \) is Cauchy. On contrary, suppose that \( \{y_{2n}\} \) is not Cauchy. Then there exist \( \epsilon > 0 \) and monotone increasing sequences of natural numbers \( \{2m(k)\} \) and \( \{2n(k)\} \) such that \( n(k) > m(k) \),

\[ d_p(y_{2m(k)}, y_{2n(k)}) \geq \epsilon \]

and

\[ d_p(y_{2n(k)}, y_{2n(k)-2}) < \epsilon \]
From (2.10)
\[ \epsilon \leq d_p(y_{2n(k)}, y_{2n(k)}) \]
\[ \leq d_p(y_{2n(k)}, y_{2n(k)} - 2) + d_p(y_{2n(k)} - 2, y_{2n(k)} - 1) + d_p(y_{2n(k)} - 1, y_{2n(k)}) \]
\[ < \epsilon + d_p(y_{2n(k)} - 2, y_{2n(k)} - 1) + d_p(y_{2n(k)} - 1, y_{2n(k)}), \quad \text{from (2.11)} \]

Letting \( k \to \infty \) and using (2.8), we have
\[ \lim_{k \to \infty} d_p(y_{2n(k)}, y_{2n(k)}) = \epsilon \]

From the definition of \( d_p \) and (2.8), we have
\[ \lim_{k \to \infty} p(y_{2n(k)}, y_{2n(k)}) = \frac{\epsilon}{2} \]

Letting \( k \to \infty \) and using (2.12) and (2.8) in
\[ |d_p(y_{2n(k)} - 1, y_{2n(k)}) - d_p(y_{2n(k)}, y_{2n(k)})| \leq d_p(y_{2n(k)} - 1, y_{2n(k)}), \]
we get
\[ \lim_{k \to \infty} d_p(y_{2n(k)} - 1, y_{2n(k)}) = \epsilon \]

From the definition of \( d_p \), we have
\[ \lim_{k \to \infty} d_p(y_{2n(k)} + 1, y_{2n(k)}) = \frac{\epsilon}{2} \]

Letting \( k \to \infty \) and using (2.12) and (2.8) in
\[ |d_p(y_{2n(k)} + 1, y_{2n(k)}) - d_p(y_{2n(k)} + 1, y_{2n(k)})| \leq d_p(y_{2n(k)} + 1, y_{2n(k)}), \]
we get
\[ \lim_{k \to \infty} d_p(y_{2n(k)} + 1, y_{2n(k)}) = \epsilon \]

From the definition of \( d_p \), we have
\[ \lim_{k \to \infty} p(y_{2n(k)}, y_{2n(k)} + 1) = \frac{\epsilon}{2} \]

Letting \( k \to \infty \) and using (2.12) and (2.8) in
\[ |d_p(y_{2n(k) + 1}, y_{2n(k)}) - d_p(y_{2n(k)}, y_{2n(k)})| \leq \left( d_p(y_{2n(k) + 1}, y_{2n(k)}) + d_p(y_{2n(k)} + 1, y_{2n(k)}) \right) \]
we get
\[ \lim_{k \to \infty} d_p(y_{2n(k) + 1}, y_{2n(k)}) = \epsilon \]

From the definition of \( d_p \), we have
\[ \lim_{k \to \infty} p(y_{2n(k) + 1}, y_{2n(k) - 1}) = \frac{\epsilon}{2} \]

\( \alpha(Sx_{2m(k) + 1}, Tx_{2n(k)}) = \alpha(y_{2m(k)}, y_{2n(k) - 1}) \geq 1 \) from (2.3). Also from (2.4),
\( x_{2n(k)} \geq x_{2m(k) + 1} \). From (2.1.3), we have
\[ (2.16) \]
\[ \psi \left( p(y_{2m(k) + 1}, y_{2n(k)}) + \varphi(y_{2m(k) + 1}) \right) = \psi \left( p(f x_{2m(k) + 1}, g x_{2n(k)}) + \varphi(f x_{2m(k) + 1}) \right) \]
\[ \leq \alpha(Sx_{2m(k) + 1}, Tx_{2n(k)}) \psi \left( p(f x_{2m(k) + 1}, g x_{2n(k)}) + \varphi(f x_{2m(k) + 1}) \right) \]
\[ \leq \psi \left( M(x_{2m(k) + 1}, x_{2n(k)}) \right) \]
Thus \( u \) and \( (2.18) \)

\[
\phi
\]

sub space of \( X \)

\[
\text{where}
\]

\[
M(x_{2m(k)+1},x_{2n(k)}) = \max \begin{cases}
  p(y_{2m(k)},y_{2n(k)} - 1) + \phi(y_{2m(k)} - 1) + \phi(y_{2n(k)} - 1), \\
  p(y_{2m(k)},y_{2m(k)} + 1) + \phi(y_{2m(k)}), \\
  p(y_{2n(k)} - 1,y_{2n(k)} + 1) + \phi(y_{2n(k)} - 1) + \phi(y_{2n(k)}), \\
  \frac{1}{2} \left[ p(y_{2m(k)},y_{2n(k)}) + \phi(y_{2m(k)}) + \phi(y_{2n(k)}) \right] + \phi(y_{2m(k)} - 1) + \phi(y_{2n(k)} - 1)
\end{cases}
\]

\[
\to \max \{ \xi, 0, 0, \frac{1}{2} (\xi + \frac{\varepsilon}{2}) \} = \frac{\varepsilon}{2}
\]

from (2.8), (2.9), (2.12), (2.13) and (2.15)

Letting \( n \to \infty \) in (2.16) and using (2.14), we get \( \psi(\frac{\varepsilon}{2}) < \psi(\frac{\varepsilon}{2}) - \phi(\frac{\varepsilon}{2}) \) which in turn yields that \( \phi(\frac{\varepsilon}{2}) = 0 \). Hence \( \epsilon = 0 \). It is a contradiction. Hence \( \{y_{2n}\} \) is Cauchy.

Letting \( n,m \to \infty \) in

\[
|d_p(y_{2n+1},y_{2n+1}) - d_p(y_{2n},y_{2m})| \leq d_p(y_{2n+1},y_{2n}) + d_p(y_{2m},y_{2m+1}).
\]

we get

\[
\lim_{n,m \to \infty} d_p(y_{2n+1},y_{2m+1}) = 0.
\]

Hence \( \{y_{2n+1}\} \) is Cauchy. Thus \( \{y_n\} \) is a Cauchy sequence in \((X,d_p)\). Hence, we have \( \lim_{n,m \to \infty} d_p(y_n,y_m) = 0 \). Now from the definition of \( d_p \), we have

\[
(2.17) \quad \lim_{n,m \to \infty} p(y_n,y_m) = 0
\]

Suppose (2.1.7)(a) holds. Since \( \{y_{2n}\} = \{Sx_{2n+1}\} \subseteq S(X) \) and \( S(X) \) is a complete sub space of \( X \), there exists \( z \in S(X) \) such that \( \{y_{2n}\} \) converges to \( z \). There exists \( u \in X \) such that \( z = Su \). Since \( \{y_n\} \) is a Cauchy and \( \{y_{2n}\} \) converges to \( z \), it follows that \( \{y_{2n+1}\} \) also converges to \( z \).

From Lemma 1.1(ii), we have

\[
p(z,z) = \lim_{n \to \infty} p(y_{2n+1},z) = \lim_{n \to \infty} p(y_{2n},z) = \lim_{n,m \to \infty} p(y_n,y_m).
\]

\[
p(z,z) = \lim_{n \to \infty} p(y_{2n+1},z) = \lim_{n \to \infty} p(y_{2n},z) = 0, \text{ from (2.17)}
\]

Since \( \phi \) is lower semi continuous, we have

\[
\phi(z) \leq \lim_{n \to \infty} \inf \phi(y_n) \leq \lim_{n \to \infty} \phi(y_n) = 0,
\]

from (2.9). Hence

\[
(2.18) \quad \phi(z) = 0.
\]

and \( \alpha(Su,Tx_{2n}) = \alpha(z,y_{2n-1}) \geq 1 \), from (2.1.7)(a). Since \( S \) is dominating map, we have \( u \preceq Su = z \). Since \( gx_{2n} \preceq x_{2n} \) and \( gx_{2n} \to z \), by (2.1.6), we have \( z \preceq x_{2n} \).

Thus \( u \preceq x_{2n} \).
Now from (2.1.3), we have
\[ \psi(p(fu, y_{2n}) + \varphi(fu) + \varphi(fy_{2n})) = \psi(p(fu, gx_{2n}) + \varphi(fu) + \varphi(gx_{2n})) \]
(2.19)
\[ \leq \alpha(Su, Tx_{2n}) \psi \left( p(fu, gx_{2n}) + \varphi(fu) + \varphi(gx_{2n}) \right) \]
\[ \leq \psi(M(u, x_{2n})) - \phi(M(u, x_{2n})) \]
where
\[ M(u, x_{2n}) = \max \left\{ \begin{array}{l}
p(z, y_{2n-1}) + \varphi(z) + \varphi(y_{2n-1}), \\
p(z, fu) + \varphi(z) + \varphi(fu), \\
p(y_{2n-1}, y_{2n}) + \varphi(y_{2n-1}) + \varphi(y_{2n}), \\
\frac{1}{2} \left[ p(z, y_{2n}) + \varphi(z) + \varphi(y_{2n}) + p(y_{2n-1}, fu) + \varphi(y_{2n-1}) + \varphi(fu) \right] \end{array} \right\} \]
\[ \rightarrow \max\{0, p(z, fu) + \varphi(fu), 0, \frac{1}{2}[p(z, fu) + \varphi(fu)]\} \]
from (2.8), (2.9), (2.18) and Lemma 1.2.
Letting \( n \to \infty \) in (2.19), we get
\[ \psi(p(fu, z) + \varphi(fu)) \leq \psi(p(z, fu) + \varphi(fu)) - \phi(p(z, fu) + \varphi(fu)) \]
which in turn yields that \( \phi(p(z, fu) + \varphi(fu)) = 0 \). Hence \( p(z, fu) + \varphi(fu) = 0 \).
Thus \( fu = z \). Hence \( Su = z = fu \). Since \( f \) is dominated and \( S \) is dominating maps, we have \( z = fu \preceq u \) and \( u \preceq Su = z \). Thus \( u = z \). Hence
(2.20)
\[ Sz = z = fz. \]
Since \( f(X) \subseteq T(X) \), there exists \( v \in X \) such that \( z = fz = Tv \). Since \( T \) is dominating map, we have \( v \preceq Tv = z \). From (2.1.7)(a) \( \alpha(Sz, Tv) = \alpha(z, z) \geq 1 \).
(2.21)
\[ \psi(p(z, gv) + \varphi(z) + \varphi(gv)) = \psi(p(fz, gv) + \varphi(fz) + \varphi(gv)) \]
\[ \leq \alpha(Sz, Tv) \psi(p(fz, gv) + \varphi(fz) + \varphi(gv)) \]
\[ \leq \psi(M(z, v)) - \phi(M(z, v)) \]
where
\[ M(z, v) = \max \left\{ \begin{array}{l}
p(z, z) + \varphi(z) + \varphi(z), \\
p(z, z) + \varphi(z) + \varphi(z), \\
p(z, gv) + \varphi(z) + \varphi(gv), \\
\frac{1}{2} \left[ p(z, gv) + \varphi(z) + \varphi(gv) \right] + p(z, z) + \varphi(z) + \varphi(z) \end{array} \right\} \]
\[ = p(z, gv) + \varphi(gv), \text{ from (2.18).} \]
Now (2.21) becomes
\[ \psi(p(z, gv) + \varphi(gv)) \leq \psi(p(z, gv) + \varphi(gv)) - \phi(p(z, gv) + \varphi(gv)) \]
which in turn yields that \( \phi(p(z, gv) + \varphi(gv)) = 0 \). Thus \( gv = z \) and \( \varphi(gv) = 0 \).
Hence \( gv = z = Tv \). Since \( g \) is dominated and \( T \) is dominating maps, we have
Thus the condition (2.22) holds.

\[ gz = z = Tz. \]

From (2.20) and (2.22), it follows that \( z \) is a common fixed point of \( f, g, S \) and \( T \).

Similarly, we can prove this theorem when (2.17)(b) holds. □

Now we give an example to illustrate our main Theorem 2.1.

**Example 2.1.** Let \( X = [0, \infty) \) and \( p(x, y) = \max\{x, y\}, \forall x, y \in X \). Let \( \leq \) be the ordinary \( \leq \). Let \( f, g, S, T : X \to X \) be defined by \( fx = \frac{x}{2}, gx = \frac{x}{3}, Sx = 6x \) and \( Tx = 4x \).

Let \( \psi, \phi, \varphi : \mathbb{R}^+ \to \mathbb{R}^+ \) be defined by \( \psi(t) = t, \phi(t) = \frac{2t}{3}, \varphi(t) = t, \) for all \( t \in \mathbb{R}^+ \). Define

\[
\alpha : X \times X \to \mathbb{R}^+ \text{ by } \alpha(x, y) = \begin{cases} 
1, & \text{if } x, y \in [0, 1], \\
2, & \text{otherwise.}
\end{cases}
\]

We have \( fx = \frac{x}{2} \leq x, gx = \frac{x}{3} \leq x \). Also \( x \leq 6x = Sx, x \leq 4x = Tx \).

Now we will verify the condition (2.1.3). If \( x > \frac{1}{2} \) and \( y \in X \) or \( x \in X \) and \( y > \frac{1}{2} \), then \( \alpha(Sx, Ty) = \alpha(6x, 4y) = 2 \).

\[
\alpha(Sx, Ty)[d(fx, gy) + \varphi(fx) + \varphi(gy)] = 2[\max\{\frac{x}{2}, \frac{y}{3}\} + \frac{x}{2} + \frac{y}{3}]
\]

\[
= \frac{1}{2}[\max\{6x, 4y\} + 6x + 4y]
\]

\[
= \frac{1}{2}[p(Sx, Ty) + \varphi(Sx) + \varphi(Ty)].
\]

If \( x \leq \frac{1}{2} \) and \( y \leq \frac{1}{2} \), then \( \alpha(Sx, Ty) = 1 \).

\[
\alpha(Sx, Ty)[d(fx, gy) + \varphi(fx) + \varphi(gy)] = \max\{\frac{x}{2}, \frac{y}{3}\} + \frac{x}{2} + \frac{y}{3}
\]

\[
= \frac{1}{6}[\max\{6x, 4y\} + 6x + 4y]
\]

\[
< \frac{1}{2}[p(Sx, Ty) + \varphi(Sx) + \varphi(Ty)].
\]

Thus the condition (2.1.3)

\[
\alpha(Sx, Ty)[p(fx, gy) + \varphi(fx) + \varphi(gy)] \\
\leq \frac{1}{6} \max \left\{ \begin{array}{c}
p(Sx, Ty) + \varphi(Sx) + \varphi(Ty), p(Sx, fx) + \varphi(Sx) + \varphi(fx), \\
p(Ty, gy) + \varphi(Ty) + \varphi(gy), \\
\frac{1}{2}[p(Sx, gy) + \varphi(Sx) + \varphi(gy) + p(Ty, fx) + \varphi(Ty) + \varphi(fx)] \end{array} \right\}
\]

for all \( x, y \in X \) is satisfied. One can easily verify the remaining conditions of Theorem 2.1. Clearly 0 is a common fixed point of \( f, g, S \) and \( T \).

**References**


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K.P.R.Rao: Department of Mathematics, Acharya Nagarjuna University, Nagarjuna Nagar - 522 510, A.P., INDIA.
E-mail address: kprao2004@yahoo.com

A.Sombabu: Department of Mathematics, NRI Institute of Technology, Agiripalli-521211, A.P., INDIA.
E-mail address: somu.mphil@gmail.com