UNIQUE COMMON FIXED POINT THEOREM
FOR FOUR MAPS IN
COMPLEX VALUED $S$– METRIC SPACES

K. P. R. Rao and Md. Mustaq Ali

Abstract. In this paper we obtain a common fixed point theorem for the two weakly compatible pairs of mappings satisfying a contractive condition in complex valued $S$-metric spaces.

1. Introduction

It is a well-known fact that the mathematical results regarding fixed points of contraction type mappings are very useful for determining the existence and uniqueness of solutions to various mathematical models. Over the last 40 years, the theory of fixed points has been developed regarding the results that are related to finding the fixed points of self and nonself nonlinear mappings in a metric space.

Several authors proved fixed point results in different types of generalized metric spaces.

Azam et al. [2] introduced the concept of a complex valued metric space and obtained sufficient conditions for the existence of common fixed points of a pair of mappings satisfying contractive type conditions. Later several authors proved fixed and common fixed point theorems in complex valued metric spaces, for example, refer [1, 2, 3, 5, 13, 9, 11, 12, 14, 15].

Throughout this paper, let $\mathbb{C}$ denote the set of all complex numbers.

A Complex number $z \in \mathbb{C}$ is an ordered pair of real numbers, whose first co-ordinate is called $Re(z)$ and second co-ordinate is called $Im(z)$. Let $z_1, z_2 \in \mathbb{C}$.
Define a partial order $\preceq$ on $\mathbb{C}$ follows:

$$z_1 \preceq z_2 \, \text{if and only if} \, \text{Re}(z_1) \leq \text{Re}(z_2), \, \text{Im}(z_1) \leq \text{Im}(z_2).$$

Thus $z_1 \preceq z_2$ if one of the following holds:

1. $\text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$,
2. $\text{Re}(z_1) < \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$,
3. $\text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) < \text{Im}(z_2)$,
4. $\text{Re}(z_1) < \text{Re}(z_2)$ and $\text{Im}(z_1) < \text{Im}(z_2)$.

Azam [2] defined the complex metric as follows:

**Definition 1.1.** Let $X$ be a non-empty set. A function $d : X \times X \to \mathbb{C}$ is called a complex valued metric on $X$ if for all $x, y, z \in X$ the following conditions are satisfied:

1. $0 \preceq d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$;
3. $d(x, y) \preceq d(x, z) + d(z, y)$.

The pair $(X, d)$ is called a complex valued metric space.

Sedghi et al. [16] introduced the concept of $S-$metric space as follows.

**Definition 1.2.** Let $X$ be a non-empty set. An $S-$metric on $X$ is a function $S : X^3 \to [0, \infty)$ that satisfies the following conditions for all $x, y, z, a \in X$.

1. $S(x, y, z) = 0$ if and only if $x = y = z$,
2. $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

The pair $(X, S)$ is called an $S-$metric space.

Following examples of $S-$metric space are due to[16].

**Example 1.1.** 1) Let $X = \mathbb{R}^n$ and $\|\cdot\|$ a norm on $X$. Then

$$S(x, y, z) = \|yz - 2x\| + \|x + y\|$$

is an $S-$metric space.

2) Let $X = \mathbb{R}^n$ and $\|\cdot\|$ a norm on $X$. Then

$$S(x, y, z) = \|x - z\| + \|y - z\|$$

is an $S-$metric space.

Later some authors proved fixed point results in $S-$metric spaces, for example [4, 6, 8, 10, 16].

**Lemma 1.1 ([16]).** Let $(X, S)$ be a $S-$metric space. If there exist $\{x_n\}$ and $\{y_n\}$ such that

$$\lim_{n \to \infty} x_n = x \quad \text{and} \quad \lim_{n \to \infty} y_n = y,$$

then

$$\lim_{n \to \infty} S(x_n, x, y_n) = S(x, x, y).$$

For $\{y_n\} = y$ the above lemma becomes as follows.
**Lemma 1.2.** Let \((X, S)\) be a \(S\)-metric space. If there exists \(\{x_n\}\) such that 
\[
\lim_{n \to \infty} x_n = x \quad \text{then} \quad \lim_{n \to \infty} S(x_n, x_n, y) = S(x, x, y).
\]

Nabil et al. [7] introduced the concept of complex valued \(S\)-metric space as follows.

**Definition 1.3.** ([7]) Let \(X\) be a non-empty set. A complex valued \(S\)-metric on \(X\) is a function \(S : X^3 \to \mathbb{C}\) that satisfies the following conditions, for all \(x, y, z, a \in X\):

(i) \(0 \preceq S(x, y, z)\),
(ii) \(S(x, y, z) = 0\) if and only if \(x = y = z\),
(iii) \(S(x, y, z) \preceq S(x, x, a) + S(y, y, a) + S(z, z, a)\).

The pair \((X, S)\) is called a complex valued \(S\)-metric space.

**Example 1.2.** Let \(X = \mathbb{C}\). Define \(S : \mathbb{C}^3 \to \mathbb{C}\) by:

\[
S(z_1, z_2, z_3) = [(\text{Re}(z_1) - \text{Re}(z_3) + |\text{Re}(z_2) - \text{Re}(z_3)|) + i|\text{Im}(z_1) - \text{Im}(z_3)| + |\text{Im}(z_2) - \text{Im}(z_3)|].
\]

Then \((X, S)\) is a complex valued \(S\)-metric space.

**Definition 1.4.** ([7]) If \((X, S)\) is called a complex valued \(S\)-metric space, then

(1) A sequence \(\{x_n\}\) in \(X\) converges to \(x\) if and only if for all \(\epsilon \in \mathbb{C}\), there exists a natural number \(n_0\) such that for all \(n \geq n_0\), we have \(S(x_n, x, x) < \epsilon\) and we denote this by \(\lim_{n \to \infty} x_n = x\).

(2) A sequence \(\{x_n\}\) in \(X\) is called a Cauchy sequence if for all \(\epsilon \in \mathbb{C}\), there exists a natural number \(n_0\) such that for all \(n, m \geq n_0\), we have \(S(x_n, x_n, x_m) < \epsilon\).

(3) An \(S\)-metric space \((X, S)\) is said to be complete if for every Cauchy sequence is convergent.

**Lemma 1.3 ([7]).** Let \((X, S)\) be a complex valued \(S\)-metric space and \(\{x_n\}\) be a sequence in \(X\). Then \(\{x_n\}\) converges to \(x\) if and only if \(|S(x_n, x_n, x)| \to 0\) as \(n \to \infty\).

**Lemma 1.4 ([7]).** Let \((X, S)\) be a complex valued \(S\)-metric space and \(\{x_n\}\) be a sequence in \(X\). Then \(\{x_n\}\) is a Cauchy sequence if and only if \(|S(x_n, x_n, x_{n+m})| \to 0\) as \(n \to \infty\) and \(m \to \infty\).

**Lemma 1.5 ([7]).** Let \((X, S)\) be a complex valued \(S\)-metric space. Then 
\[S(x, x, y) = S(y, y, x)\] for all \(x, y \in X\).

2. Main results

Recently Naval Singh et al. [13] proved the following theorem in complex valued metric spaces as follows.

**Theorem 2.1.** Let \((X, d)\) be a complete complex valued metric space and \(S, T : X \to X\). If there exist mappings \(\lambda, \mu, \gamma, \delta : X \times X \times X \to [0, 1)\) such that for all \(x, y \in X\), the following is valid

(a)
Assume that there exists a mapping for all a proposition which is needed to prove our main Theorem.

Thus

(2)

Thus

(2')

Proof. Proposition 2.1

In this paper we generalize the Theorem (2.1) in complex valued $S$–metric spaces for four maps satisfying more general contractive condition. First we prove a proposition which is needed to prove our main Theorem.

Proposition 2.1. Let $(X, S)$ be a complex valued $S$–metric space and $F, G, f, g : X \to X$. Let $y_0 \in X$ and define the sequence $(y_n)$ by

$$y_{2n+1} = g y_{2n+1} = F x_{2n}; \quad y_{2n+2} = f y_{2n+2} = G x_{2n+1}, \text{ for all } n = 0, 1, 2,...$$

Assume that there exists a mapping $\lambda_1 : X \times X \times X \to [0, 1)$ such that

(i) $\lambda_1(F x, y, a) \leq \lambda_1(f x, y, a)$ and $\lambda_1(x, G y, a) \leq \lambda_1(x, g y, a)$,

(ii) $\lambda_1(G x, y, a) \leq \lambda_1(g x, y, a)$ and $\lambda_1(x, F y, a) \leq \lambda_1(x, f y, a)$.

for all $x, y \in X$ and for a fixed element $a \in X$ and $n = 0, 1, 2,...$ Then

$$\lambda_1(y_{2n}, y, a) \leq \lambda_1(y_0, y, a) \text{ and } \lambda_1(x, y_{2n+1}, a) \leq \lambda_1(x, y_1, a), \text{ for all } x, y \in X.$$ 

Proof. Let $x, y \in X$ and $n = 0, 1, 2,...$ Then we have

$$\lambda_1(y_{2n}, y, a) = \lambda_1(G x_{2n-1}, y, a) \leq \lambda_1(g x_{2n-1}, y, a) = \lambda_1(x, y_{2n-1}, y, a) = \lambda_1(F x_{2n-2}, y, a) \leq \lambda_1(f x_{2n-2}, y, a) = \lambda_1(y_{2n-2}, y, a) = \lambda_1(G x_{2n-3}, y, a) \leq \lambda_1(x, y_{2n-3}, y, a) \cdots \leq \lambda_1(y_0, y, a).$$

Thus $\lambda_1(y_{2n}, y, a) \leq \lambda_1(y_0, y, a)$.

Similarly we have

$$\lambda_1(x, y_{2n+1}, a) = \lambda_1(x, F x_{2n}, a) \leq \lambda_1(x, f x_{2n}, a) = \lambda_1(x, G x_{2n-1}, a) \leq \lambda_1(x, g x_{2n-1}, a) = \lambda_1(x, F x_{2n-2}, a) \leq \lambda_1(x, f x_{2n-2}, a) = \lambda_1(x, y_{2n-2}, a) \cdots \leq \lambda_1(x, y_1, a).$$

Thus $\lambda_1(x, y_{2n+1}, a) \leq \lambda_1(x, y_1, a)$. \hfill $\Box$

Theorem 2.2. Let $(X, S)$ be a complex valued $S$–metric space and $F, G, f, g : X \to X$ satisfying the conditions.

(2.2.1) $G X \subseteq f X$ and $F X \subseteq g X$,

(2.2.2) The pairs $(F, f)$ and $(G, g)$ are weakly compatible,
(2.2.3) $fX$ or $gX$ is a complete subspace of $X$.

(2.2.4) If there exist mappings $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7 : X \times X \times X \to [0, 1]$ such that

$\lambda_n(fx, y, a) \leq \lambda_n(fx, y, a); \lambda_n(Gx, y, a) \leq \lambda_n(gx, y, a)$ and

$\lambda_n(x, Fy, a) \leq \lambda_n(x, fy, a); \lambda_n(x, Gy, a) \leq \lambda_n(x, gy, a)$, $\forall n = 1, 2, 3, ..., 7$,

for all $x, y \in X$ and for a fixed element $a \in X$.

(2.2.5)

\[
S(Fx, Fx, Gy) \leq \lambda_1(fx, fy, a)S(fx, fx, fy) + \lambda_2(fy, gy, a)S(fy, fy, fy) + \lambda_3(fx, fy, a)S(fx, fy, fy) + \lambda_4(fy, gy, a)[S(fy, gy, fy) + S(fx, fy, fy)]
\]

\[
+ \lambda_5(fx, fy, a) \left( \frac{S(fx, fx, fy)S(gy, gy, gy)}{1+S(fx, fy, fy)} \right) + \lambda_6(fy, gy, a) \left( \frac{S(fy, fy, fy)S(fx, fx, fy)}{1+S(fx, fy, fy)} \right)
\]

\[
+ \lambda_7(fx, fy, a) \left( \frac{S(fx, fx, fy)S(fx, fy, fy)S(gy, gy, fy)S(gy, gy, fy)}{1+S(fx, fy, fy) + S(gy, gy, fy)} \right)
\]

for all $x, y \in X$ and for a fixed element $a \in X$, where

(2.2.6) $(\lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_4 + \lambda_5 + \lambda_6 + \lambda_7)(x, y, a) < 1$.

Then $F, G, f$ and $g$ have a unique common fixed point.

**Proof.** Let $x_0 \in X$ be an arbitrary point. We define a sequence $\{y_n\}$ in $X$ such that $y_{n+1} = g y_{n+1} = Fx_{n+1}$ and $y_{n+1} = f y_{n+1} = Gy_{n+1}$, $n = 0, 1, 2, ...

From (2.2.5) we have

\[
S(y_{n+1}, y_{n+1}, y_{n+1}) = S(Fx_{n+1}, Fx_{n+1}, Gy_{n+1})
\]

\[
\leq \lambda_1(y_{n+1}, y_{n+1}, y_{n+1})S(y_{n+1}, y_{n+1}, y_{n+1}) + \lambda_2(y_{n+1}, y_{n+1}, y_{n+1})S(y_{n+1}, y_{n+1}, y_{n+1})
\]

\[
+ \lambda_3(y_{n+1}, y_{n+1}, y_{n+1})S(y_{n+1}, y_{n+1}, y_{n+1}) + \lambda_4(y_{n+1}, y_{n+1}, y_{n+1})S(y_{n+1}, y_{n+1}, y_{n+1})
\]

\[
+ \lambda_5(y_{n+1}, y_{n+1}, y_{n+1}) \left( \frac{S(y_{n+1}, y_{n+1}, y_{n+1})S(y_{n+1}, y_{n+1}, y_{n+1})}{1+S(y_{n+1}, y_{n+1}, y_{n+1})} \right)
\]

\[
+ \lambda_6(y_{n+1}, y_{n+1}, y_{n+1}) \left( \frac{S(y_{n+1}, y_{n+1}, y_{n+1})S(y_{n+1}, y_{n+1}, y_{n+1})}{1+S(y_{n+1}, y_{n+1}, y_{n+1})} \right)
\]

\[
+ \lambda_7(y_{n+1}, y_{n+1}, y_{n+1}) \left( \frac{S(y_{n+1}, y_{n+1}, y_{n+1})S(y_{n+1}, y_{n+1}, y_{n+1})}{1+S(y_{n+1}, y_{n+1}, y_{n+1})} \right)
\]

Since $S(x, x, x) = 0$, we have

\[
\left| S(y_{n+1}, y_{n+1}, y_{n+1}) \right| \leq \lambda_1(y_{n+1}, y_{n+1}, y_{n+1}) \left| S(y_{n+1}, y_{n+1}, y_{n+1}) \right|
\]

\[
+ \lambda_2(y_{n+1}, y_{n+1}, y_{n+1}) \left| S(y_{n+1}, y_{n+1}, y_{n+1}) \right|
\]

\[
+ \lambda_3(y_{n+1}, y_{n+1}, y_{n+1}) \left| S(y_{n+1}, y_{n+1}, y_{n+1}) \right|
\]

\[
+ \lambda_4(y_{n+1}, y_{n+1}, y_{n+1}) \left| S(y_{n+1}, y_{n+1}, y_{n+1}) \right|
\]

\[
+ \lambda_5(y_{n+1}, y_{n+1}, y_{n+1}) \left| S(y_{n+1}, y_{n+1}, y_{n+1}) \right|
\]

\[
+ \lambda_6(y_{n+1}, y_{n+1}, y_{n+1}) \left| S(y_{n+1}, y_{n+1}, y_{n+1}) \right|
\]

\[
+ \lambda_7(y_{n+1}, y_{n+1}, y_{n+1}) \left| S(y_{n+1}, y_{n+1}, y_{n+1}) \right|
\]
and that where

Let $h_{26} K. P. R. RAO, MD. MUSTAQ ALI

which in turn implies that

$$S(y_{2n+1}, y_{2n+1}, y_{2n+2}) \leq \left( \frac{\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4}{1 - (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)(y_0, y_1, a)} \right) S(y_{2n}, y_{2n+1})$$

Using Proposition (2.1), we get

$$S(y_{2n+1}, y_{2n+1}, y_{2n+2}) \leq (\lambda_1 + \lambda_2 + \lambda_4 + \lambda_7)(y_0, y_1, a) |S(y_{2n}, y_{2n+1})|$$

Similarly using $S(x, y, z) = S(x, x, y)$ and proceeding as above we can show that

$$S(y_{2n+2}, y_{2n+2}, y_{2n+3}) \leq h_2 |S(y_{2n+1}, y_{2n+1}, y_{2n+2})|$$

Let $h = \max\{h_1, h_2\}$, then $0 \leq h < 1$, since $h_1, h_2 \in [0, 1]$. Hence from (1) and (2), we have $|S(y_n, y_{n+1})| \leq h|S(y_{n-1}, y_{n-1}, y_n)|, n = 1, 2, 3, \ldots$. Repeated use of above inequality gives

$$|S(y_k, y_{k+1})| \leq h^k |S(y_0, y_1)| \quad \ldots \ldots \quad (3)$$

$$\rightarrow 0 \quad \text{as} \quad k \rightarrow \infty \quad \ldots \ldots \quad (4)$$

Hence for any $m > n$, we have

$$S(y_n, y_n, y_m) = 2 \left[ |S(y_n, y_{n+1})| + |S(y_{n+1}, y_{n+1}, y_{n+2})| + \right]$$

$$\ldots + |S(y_{m-1}, y_{m-1}, y_m)|$$

$$\leq \frac{2h^n}{1 - h} |S(y_0, y_0, y_1)|$$

and

$$S(y_n, y_n, y_m) \leq \frac{2h^n}{1 - h} |S(y_0, y_0, y_1)| \rightarrow 0 \quad \text{as} \quad m, n \rightarrow \infty.$$ 

Hence $\{y_n\}$ is a Cauchy sequence in $X$.

Now suppose $fX$ is a complete subspace of $X$. Since $y_{2n+2} = f x_{2n+2} \in f(X)$ and $\{y_n\}$ is a Cauchy sequence, there exists $z \in f(X)$ such that $y_{2n+2} \rightarrow z$ as $n \rightarrow \infty$. Then there exists $u \in X$ such that $fu = z$. Thus

$$\lim_{n \rightarrow \infty} f x_{2n} = \lim_{n \rightarrow \infty} g x_{2n+1} = \lim_{n \rightarrow \infty} G x_{2n+1} = \lim_{n \rightarrow \infty} f x_{2n+2} = z.$$
Consider

\[ S(Fu, Fu, Gx_{2n+1}) \]
\[ \leq \lambda_1(fu, y_{2n+1}, a) S(fu, fu, y_{2n+1}) + \lambda_2(fu, y_{2n+1}, a) S(fu, Fu) + \lambda_3(fu, y_{2n+1}, a) S(y_{2n+1}, y_{2n+1}, y_{2n+2}) + \lambda_4(fu, y_{2n+1}, a) S(y_{2n+1}, y_{2n+1}, Fu) + S(fu, fu, y_{2n+2}) \]
\[ + \lambda_5(fu, y_{2n+1}, a) \left( \frac{S(fu, fu, Fu) S(y_{2n+1}, y_{2n+1}, y_{2n+2})}{1+S(fu, y_{2n+1})} \right) \]
\[ + \lambda_6(fu, y_{2n+1}, a) \left( \frac{S(y_{2n+1}, y_{2n+1}, Fu) S(fu, fu, y_{2n+2})}{1+S(fu, y_{2n+1})} \right) \]
\[ + \lambda_7(fu, y_{2n+1}, a) \left( \frac{S(fu, fu, Fu) S(fu, y_{2n+2}) + S(y_{2n+1}, y_{2n+1}, y_{2n+2}) S(y_{2n+1}, y_{2n+1}, Fu)}{1+S(fu, y_{2n+1}) + S(y_{2n+1}, y_{2n+1}, Fu)} \right) \]

Letting \( n \to \infty \) and using Lemma 1.2 and 1.5, we get

\[ |S(Fu, Fu, z)| \leq \lambda_3(z, z, a) |S(z, z, Fu)| + \lambda_4(z, z, a) |S(z, z, Fu)| \]

from (4), Lemma 1.3 \((1 - (\lambda_2 + \lambda_4)(z, z, a)) |S(z, z, Fu)| \leq 0 \) which in turn yields from (2.2.6) that \(|S(Fu, Fu, z)| \leq 0\). Therefore \(|S(Fu, Fu, z)| = 0\). Hence \( Fu = z \).

Thus \( fu = Fu = z \). Since \( Fx \subseteq gX \), there exists \( v \in X \) such that \( Fu = gv \). Thus \( fu = Fu = gv = z \). Again from (2.2.5), we have

\[ |S(z, z, Gv)| = |S(Fu, Fu, Gv)| \]
\[ \leq \lambda_1(fu, gv, a) |S(fu, fu, gv)| + \lambda_2(fu, gv, a) |S(fu, Fu)| + \lambda_3(fu, gv, a) |S(gv, gv, Gv)| + \lambda_4(fu, gv, a) |S(gv, gv, Fu)| + \lambda_5(fu, gv, a) |S(fu, fu, Gv)| + \lambda_6(fu, gv, a) |S(fu, fu, Gv)| + \lambda_7(fu, gv, a) |S(fu, fu, Gv)| + |S(y_{2n+1}, y_{2n+1}, Gv)+S(y_{2n+1}, y_{2n+1}, Fu)| \]

so that

\[ |S(z, z, Gv)| \leq \lambda_3(z, z, a) |S(z, z, Gv)| + \lambda_4(z, z, a) |S(z, z, Gv)| \]

\[(1 - (\lambda_3 + \lambda_4)(z, z, a)) |S(z, z, Gv)| \leq 0\]
which in turn yields from (2.2.6) that \( |S(z, z, Gv)| \leq 0 \). Therefore \( |S(z, z, Gv)| = 0 \). Hence \( Gv = z \). Thus

\[
Gv = z = f u = F u = g v. \quad \text{(5)}
\]

Since \((F, f)\) is weakly compatible, we have

\[
fz = f F u = F f u = F z. \quad \text{(6)}
\]

\[
S(Fz, Fz, z) = S(Fz, Fz, Gv) \leq \lambda_1(fz, gv, a)S(fz, fz, gv) + \lambda_2(fz, gv, a)S(fz, fz, Fz) + \lambda_3(fz, gv, a)S(gv, gv, Gv) + \lambda_4(fz, gv, a)[S(gv, gv, Fz) + S(fz, fz, Gv)] + \lambda_5(fz, gv, a)\left( \frac{S(fz, fz, Fz)}{1 + S(fz, fz, Gv)} \right) + \lambda_6(fz, gv, a)\left( \frac{S(gv, gv, Fz)}{1 + S(fz, fz, Gv)} \right) + \lambda_7(fz, gv, a)\left( \frac{S(gv, gv, Fz)S(fz, fz, Gv)}{1 + S(fz, fz, Gv) + S(gv, gv, Fz)} \right)
\]

\[
= \lambda_1(Fz, z, a)S(Fz, Fz, z) + \lambda_4(Fz, z, a)[S(z, z, Fz) + S(Fz, Fz, z)] + \lambda_6(Fz, z, a)\left( \frac{S(z, z, Fz)}{1 + S(Fz, Fz, z)} \right) \quad \text{from (5) and (6)}
\]

\[
|S(Fz, Fz, z)| \leq \lambda_1(Fz, z, a)|S(Fz, Fz, z)| + \lambda_4(Fz, z, a)|S(z, z, Fz) + S(Fz, Fz, z)| + \lambda_6(Fz, z, a)|S(z, z, Fz)| \left| \frac{S(Fz, Fz, z)}{1 + S(Fz, Fz, z)} \right|
\]

\[
(1 - (\lambda_1 + 2\lambda_4 + \lambda_6)(Fz, z, a))|S(Fz, Fz, z)| \leq 0
\]

which in turn yields from (2.2.6) that \( |S(Fz, Fz, z)| \leq 0 \). Therefore \( |S(Fz, Fz, z)| = 0 \). Hence \( Fz = z \). Thus

\[
z = Fz = fz. \quad \text{(7)}
\]

Since the pair \((G, g)\) is weakly compatible, we have

\[
gz = g Gv = Gv = Gz. \quad \text{(8)}
\]

From (2.2.5)

\[
S(z, z, Gz) = S(Fz, Fz, Gz) \leq \lambda_1(fz, gz, a)S(fz, fz, gz) + \lambda_2(fz, gz, a)S(fz, fz, Fz) + \lambda_3(fz, gz, a)S(gz, gz, Gz) + \lambda_4(fz, gz, a)[S(gz, gz, Fz) + S(fz, fz, Gz)] + \lambda_5(fz, gz, a)\left( \frac{S(fz, fz, Fz)}{1 + S(fz, fz, Gz)} \right) + \lambda_6(fz, gz, a)\left( \frac{S(gz, gz, Fz)}{1 + S(fz, fz, Gz)} \right) + \lambda_7(fz, gz, a)\left( \frac{S(gz, gz, Fz)S(fz, fz, Gz)}{1 + S(fz, fz, Gz) + S(gz, gz, Fz)} \right)
\]

\[
|S(z, z, Gz)| \leq \lambda_1(z, Gz, a)|S(z, z, Gz)| + \lambda_4(z, Gz, a)|S(Gz, Gz, z) + S(z, z, Gz)| + \lambda_6(z, Gz, a)|S(Gz, Gz, z)| \left| \frac{S(z, z, Gz)}{1 + S(z, z, Gz)} \right| \quad \text{from (7), (8)}
\]
(1 - (λ_1 + 2λ_3 + λ_6)(z,Gz,a)) |S(z,z,Gz)| ≤ 0 which in turn yields from (2.2.6) that |S(z,z,Gz)| ≤ 0. Therefore |S(z,z,Gz)| = 0. Hence Gz = z, so that

\[ Gz = gz = z. \quad \ldots \ldots \ldots (9) \]

Thus from (7) and (9), z is a common fixed point of \( F, G, f \) and \( g \). For uniqueness, let \( z^* \in X \) be such that \( fz^* = Fz^* = z^* = gz^* = Gz^* \).

Now from (2.2.5)

\[
S(z,z,z^*) = S(Fz,Fz,Gz^*)
\]

\[
\leq \lambda_1(fz,gz^*,a)S(fz,fz,gz^*) + \lambda_2(fz,gz^*,a)S(fz,fz,Fz) + \lambda_3(fz,gz^*,a)S(gz^*,gz^*,Gz^*)
+ \lambda_4(fz,gz^*,a)[S(gz^*,gz^*,Fz) + S(fz,fz,Gz^*)]
+ \lambda_5(fz,gz^*,a)\left[\frac{S(fz,fz,Fz)S(gz^*,gz^*,Gz^*)}{1+S(fz,fz,Gz^*)}\right]
+ \lambda_6(fz,gz^*,a)\left[\frac{S(gz^*,Gz^*)S(fz,Gz^*)}{1+S(fz,fz,Gz^*)}\right]
+ \lambda_7(fz,gz^*,a)\left[\frac{S(gz^*,Gz^*)S(gz^*,Fz)}{1+S(fz,Gz^*)}\right].
\]

For uniqueness, let \( S(z,z,z^*) \leq \lambda_1(z,z^*,a) |S(z,z,z^*)| + \lambda_4(z,z^*,a) |S(z^*,z^*,z^*)| + \lambda_6(z,z^*,a) |S(z^*,z^*,z^*)| \left[\frac{S(z,z,z^*)}{1+S(z,z,z^*)}\right].
\]

(1 - (λ_1 + 2λ_3 + λ_6)(z,z^*,a)) |S(z,z,z^*)| ≤ 0 which in turn yields from (2.2.6) that |S(z,z,z^*)| = 0. Hence z is the unique common fixed point of \( F,G,f \) and \( g \). Similarly we can prove the theorem if \( gX \) is a complete subspace of \( X \).

Now we give an example to illustrate our main Theorem 2.2.

**Example 2.1.** Let \( X = [0,1] \) and \( S : X \times X \times X \to C \) be defined by \( S(x,y,z) = |x - z| + |y - z| \). Then \( (X,S) \) is a complex valued \( S- \) metric space. Define \( F,G,f \) and \( g : X \to X \) by \( Fx = \frac{x}{10} \), \( Gx = \frac{x}{2} \), \( fx = \frac{x}{3} \) and \( gx = \frac{2x}{3} \) for all \( x \in X \). For fixed element \( a = \frac{1}{3} \), define \( \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7 : X \times X \to [0,1] \) by

\[
\lambda_1(x,y,a) = \left( \frac{\pi}{10} + \frac{\pi}{10} + a \right), \lambda_2(x,y,a) = \frac{\pi a}{10}, \lambda_3(x,y,a) = \frac{\pi a^2}{10}, \lambda_4(x,y,a) = \frac{\pi a^3}{10}, \lambda_5(x,y,a) = \frac{\pi a^3}{10}, \lambda_6(x,y,a) = \frac{\pi a^3}{10}, \lambda_7(x,y,a) = \frac{\pi a^4}{10},
\]

for all \( x, y \in X \). Then

\[
\lambda_1(x,y) + \lambda_2(x,y) + \lambda_3(x,y) + 2\lambda_4(x,y) + \lambda_5(x,y) + \lambda_6(x,y) + \lambda_7(x,y) = \left( \frac{\pi}{10} + \frac{\pi}{10} + a \right) + \frac{\pi a}{10} + \frac{\pi a^2}{10} + 2 ( \frac{\pi a^3}{10} ) + \frac{\pi a^3}{10} + \frac{\pi a^3}{10} + \frac{\pi a^4}{40}
\]

\[
\leq \frac{3\pi}{5} + \frac{\pi}{10} \leq \frac{3\pi}{10} < 1.
\]

Hence \((\lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_4 + \lambda_5 + \lambda_6 + \lambda_7)(x,y,a) < 1 \). We have

\[
\lambda_1(Fx,y,a) = \lambda_1 \left( \frac{x}{16}, y, a \right) = \left( \frac{x}{640} + \frac{y}{50} + a \right)
\]

\[
\lambda_1(fx,y,a) = \lambda_1 \left( \frac{x}{4}, y, a \right) = \left( \frac{x}{160} + \frac{y}{50} + a \right).
\]
Clearly $\lambda_1(Fx, y, a) \leq \lambda_1(fx, y, a)$. We have
\[
\lambda_1(x, Fy, a) = \lambda_1(x, \frac{y}{16}, a) = \left(\frac{x}{40} + \frac{y}{800} + a\right)
\]
\[
\lambda_1(x, fy, a) = \lambda_1(x, \frac{y}{4}, a) = \left(\frac{x}{40} + \frac{y}{200} + a\right).
\]
Clearly $\lambda_1(x, Fy, a) \leq \lambda_1(x, fy, a)$. We have
\[
\lambda_1(Gx, y, a) = \lambda_1(x, \frac{y}{12}, y, a) = \left(\frac{x}{480} + \frac{y}{50} + a\right)
\]
\[
\lambda_1(gx, y, a) = \lambda_1(x, \frac{y}{3}, y, a) = \left(\frac{x}{120} + \frac{y}{50} + a\right).
\]
Clearly $\lambda_1(Gx, y, a) \leq \lambda_1(gx, y, a)$. We have
\[
\lambda_1(x, Gy, a) = \lambda_1(x, \frac{y}{12}, a) = \left(\frac{x}{40} + \frac{y}{600} + a\right)
\]
\[
\lambda_1(x, gy, a) = \lambda_1(x, \frac{y}{3}, a) = \left(\frac{x}{40} + \frac{y}{150} + a\right).
\]
Clearly $\lambda_1(x, Gy, a) \leq \lambda_1(x, gy, a)$.

Similarly we can prove that
\[
\lambda_n(Fx, y, a) \leq \lambda_n(fx, y, a), \lambda_n(x, Fy, a) \leq \lambda_n(x, fy, a)
\]
\[
\lambda_n(Gx, y, a) \leq \lambda_n(gx, y, a), \lambda_n(x, Gy, a) \leq \lambda_n(x, gy, a) \forall n = 2, 3, 4, \ldots 7.
\]

Consider
\[
|S(Fx, Fx, Gy)| = |S(\frac{x}{16}, \frac{x}{16}, \frac{y}{12})|
\]
\[
= |\frac{x}{16} - \frac{y}{12}| + \frac{y}{16} - \frac{y}{12} = \frac{1}{4}(|\frac{x}{3} - \frac{y}{3}| + |\frac{x}{3} - \frac{y}{3}|)
\]
\[
< \frac{1}{3}||x - y|| + |i\frac{x}{3} - \frac{y}{3}||
\]
\[
\leq (\frac{1}{160} + \frac{1}{150})||\frac{x}{3} - \frac{y}{3}|| = \lambda_1(fx, gy, a)S(fx, fx, gy)
\]
\[
\leq \lambda_1(fx, gy, a)S(fx, fx, gy) + \lambda_2(fx, gy, a)S(fx, fx, Fx)
\]
\[
+ \lambda_3(fx, gy, a)S(gy, gy, Gy) + \lambda_4(fx, gy, a)S(\frac{Fx, Fx, Gy}{S}\frac{S}{1+S(Fx, Fx, Gy)} S(\frac{Fx, Fx, Gy}{S}\frac{S}{1+S(Fx, Fx, Gy)})
\]
\[
+ \lambda_3(fx, gy, a)\left(\frac{S(fx, Fx, Gy)}{1+S(fx, Fx, Gy)}\frac{S(fx, Fx, Gy)}{1+S(fx, Fx, Gy)}\right).
\]
Thus (2.2.5) is satisfied.

One can easily verify the remaining conditions of Theorem 2.2. Clearly $x = 0$ is the unique common fixed point of $F, G, f$ and $g$.

References


K. P. R. Rao: Department of Mathematics, Acharya Nagarjuna University, Nagarjuna Nagar - 522 510, A.P., INDIA.

E-mail address: kprrao2004@yahoo.com

Md. Mustaq Ali: Department of Mathematics, USHA RAMA College of Engineering and Technology, Talaprolu-521109, A.P., INDIA.

E-mail address: alimustaq9@gmail.com