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UNIQUE COMMON FIXED POINT THEOREM FOR FOUR MAPS IN COMPLEX VALUED $S-$ METRIC SPACES

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ABSTRACT. In this paper we obtain a common fixed point theorem for the two weakly compatible pairs of mappings satisfying a contractive condition in complex valued S-metric spaces.

1. Introduction

It is a well-known fact that the mathematical results regarding fixed points of contraction type mappings are very useful for determining the existence and uniqueness of solutions to various mathematical models. Over the last 40 years, the theory of fixed points has been developed regarding the results that are related to finding the fixed points of self and nonself nonlinear mappings in a metric space.

Several authors proved fixed point results in different types of generalized metric spaces.

Azam et al. [**2**] introduced the concept of a complex valued metric space and obtained sufficient conditions for the existence of common fixed points of a pair of mappings satisfying contractive type conditions. Later several authors proved fixed and common fixed point theorems in complex valued metric spaces, for example, refer [**1, 2, 3, 5, 13, 9, 11, 12, 14, 15**].

Throughout this paper, let $\mathbf{C}$ denote the set of all complex numbers.

A Complex number $z \in \mathbf{C}$ is an ordered pair of real numbers, whose first co-ordinate is called $Re(z)$ and second co-ordinate is called $Im(z)$. Let $z_1, z_2 \in \mathbf{C}$.

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Define a partial order $\precsim$ on $\mathbf{C}$ follows:

$$z_1 \precsim z_2 \text{ if and only if } Re(z_1) \leqslant Re(z_2),\ Im(z_1) \leqslant Im(z_2).$$

Thus $z_1 \precsim z_2$ if one of the following holds:
(1) $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$,
(2) $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$,
(3) $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$,
(4) $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.

Azam [**2**] defined the complex metric as follows:

Definition 1.1. ([**2**]) Let X be a non-empty set. A function $d : X \times X \to \mathbf{C}$ is called a complex valued metric on X if for all $x, y, z \in X$ the following conditions are satisfied:
(i) $0 \precsim d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, y) \precsim d(x, z) + d(z, y)$.
The pair (X, d) is called a complex valued metric space.

Sedghi et al. [**16**] introduced the concept of $S-$ metric space as follows.

Definition 1.2. ([**16**]) Let X be a non-empty set. A $S-$metric on X is a function $S : X^3 \longrightarrow [0, \infty)$ that satisfies the following conditions for all $x, y, z, a \in X$.
(S1) $S(x, y, z) = o$ if and only if $x = y = z$,
(S2) $S(x, y, z) \leqslant S(x, x, a) + S(y, y, a) + S(z, z, a)$.
The pair (X, S) is called an $S-$metric space.

Following examples of $S-$metric space are due to[**16**].

Example 1.1. 1) Let $X = \mathbf{R}^n$ and $\|.\|$ a norm on X. Then

$$S(x, y, z) = \|yz - 2x\| + \|x + y\|$$

is an $S-$metric space.
2) Let $X = \mathbf{R}^n$ and $\|.\|$ a norm on X. Then

$$S(x, y, z) = \|x - z\| + \|y - z\|$$

is an $S-$metric space.

Later some authors proved fixed point results in $S-$metric spaces, for example [**4, 6, 8, 10, 16**].

Lemma 1.1 ([**16**]). *Let (X, S) be a $S-$metric space. If there exist $\{x_n\}$ and $\{y_n\}$ such that*

$$\lim_{n\to\infty} x_n = x \text{ and } \lim_{n\to\infty} y_n = y,$$

then

$$\lim_{n\to\infty} S(x_n, x_n, y_n) = S(x, x, y).$$

For $\{y_n\} = y$ the above lemma becomes as follows.

LEMMA 1.2. *Let (X, S) be a $S-$metric space. If there exists $\{x_n\}$ such that $\lim_{n\to\infty} x_n = x$ then $\lim_{n\to\infty} S(x_n, x_n, y) = S(x, x, y)$.*

Nabil et al. [**7**] introduced the concept of complex valued $S-$ metric space as follows.

DEFINITION 1.3. ([**7**]) Let X be a non-empty set. A complex valued S-metric on X is a function $S : X^3 \to \mathbf{C}$ that satisfies the following conditions, for all $x, y, z, a \in X$:

(i) $0 \precsim S(x, y, z)$,

(ii) $S(x, y, z) = 0$ if and only if $x = y = z$,

(iii) $S(x, y, z) \precsim S(x, x, a) + S(y, y, a) + S(z, z, a)$.

The pair (X, S) is called a complex valued S-metric space.

EXAMPLE 1.2. Let X = $\mathbf{C}$. Define $S : \mathbf{C}^3 \to \mathbf{C}$ by:

$S(z_1, z_2, z_3) =$

$[|Re(z_1) - Re(z_3)| + |Re(z_2) - Re(z_3)|] + i[|Im(z_1) - Im(z_3)| + |Im(z_2) - Im(z_3)|]$.

Then (X, S) is a complex valued S-metric space.

DEFINITION 1.4. ([**7**]) If (X, S) is called a complex valued S-metric space, then

(1) A sequence $\{x_n\}$ in X converges to x if and only if for all ϵ such that $0 \prec \epsilon \in \mathbf{C}$, there exists a natural number n_0 such that for all $n \geqslant n_0$, we have $S(x_n, x_n, x) \prec \epsilon$ and we denote this by $\lim_{n\to\infty} x_n = x$.

(2) A sequence $\{x_n\}$ in X is called a Cauchy sequence if for all ϵ such that $0 \prec \epsilon \in \mathbf{C}$, there exists a natural number n_0 such that for all $n, m \geqslant n_0$, we have $S(x_n, x_n, x_m) \prec \epsilon$.

(3) An S-metric space (X, S) is said to be complete if for every Cauchy sequence is convergent.

LEMMA 1.3 ([**7**]). *Let (X, S) be a complex valued S-metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|S(x_n, x_n, x)| \to 0$ as $n \to \infty$.*

LEMMA 1.4 ([**7**]). *Let (X, S) be a complex valued S-metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|S(x_n, x_n, x_{n+m})| \to 0$ as $n \to \infty$ and $m \to \infty$.*

LEMMA 1.5 ([**7**]). *Let (X, S) be a complex valued S-metric space. Then $S(x, x, y) = S(y, y, x)$ for all $x, y \in X$.*

2. Main results

Recently Naval Singh et al. [**13**] proved the following theorem in complex valued metric spaces as follows.

THEOREM 2.1. *Let (X, d) be a complete complex valued metric space and $S, T : X \to X$. If there exist mappings $\lambda, \mu, \gamma, \delta : X \times X \times X \to [0, 1)$ such that for all $x, y \in X$, the following is valid*

(a)

$$\lambda(TSx,y,a) \leqslant \lambda(x,y,a) \text{ and } \lambda(x,STy,a) \leqslant \lambda(x,y,a),$$
$$\mu(TSx,y,a) \leqslant \mu(x,y,a) \text{ and } \mu(x,STy,a) \leqslant \mu(x,y,a),$$
$$\gamma(TSx,y,a) \leqslant \gamma(x,y,a) \text{ and } \gamma(x,STy,a) \leqslant \gamma(x,y,a),$$
$$\delta(TSx,y,a) \leqslant \delta(x,y,a) \text{ and } \delta(x,STy,a) \leqslant \delta(x,y,a);$$

(b)

$$d(Sx,Ty) \precsim \lambda(x,y,a)d(x,y) + \mu(x,y,a)\tfrac{d(x,Sx)d(y,Ty)}{1+d(x,y)} + \gamma(x,y,a)\tfrac{d(y,Sx)d(x,Ty)}{1+d(x,y)}$$
$$+\delta(x,y,a)\tfrac{d(x,Sx)d(x,Ty)+d(y,Ty)d(y,Sx)}{1+d(x,Ty)+d(y,Sx)};$$

(c) $\lambda(x,y,a)+\mu(x,y,a)+\gamma(x,y,a)+\delta(x,y,a) < 1$,

then S and T have a unique common fixed point.

In this paper we generalize the Theorem (2.1) in complex valued $S-$metric spaces for four maps satisfying more general contractive condition. First we prove a proposition which is needed to prove our main Theorem.

PROPOSITION 2.1. *Let (X,S) be a complex valued S-metric space and $F,G,f,g : X \to X$. Let $y_0 \in X$ and define the sequence $\{y_n\}$ by*

$$y_{2n+1} = gx_{2n+1} = Fx_{2n};\ y_{2n+2} = fx_{2n+2} = Gx_{2n+1}. \textit{for alln} = 0,1,2....$$

Assume that there exists a mapping $\lambda_1 : X \times X \times X \to [0,1)$ such that

(i) $\lambda_1(Fx,y,a) \leqslant \lambda_1(fx,y,a)$ *and* $\lambda_1(x,Gy,a) \leqslant \lambda_1(x,gy,a)$,

(ii) $\lambda_1(Gx,y,a) \leqslant \lambda_1(gx,y,a)$ *and* $\lambda_1(x,Fy,a) \leqslant \lambda_1(x,fy,a)$.

fora ll $x,y \in X$ and for a fixed element $a \in X$ and $n = 0,1,2,...$ Then

$$\lambda_1(y_{2n},y,a) \leqslant \lambda_1(y_0,y,a) \text{ and } \lambda_1(x,y_{2n+1},a) \leqslant \lambda_1(x,y_1,a), \textit{ for allx}, y \in X$$

.

PROOF. Let $x,y \in X$ and $n = 0,1,2....$ Then we have

$$\lambda_1(y_{2n},y,a) = \lambda_1(Gx_{2n-1},y,a) \leqslant \lambda_1(gx_{2n-1},y,a) = \lambda(y_{2n-1},y,a) =$$
$$\lambda_1(Fx_{2n-2},y,a) \leqslant \lambda_1(fx_{2n-2},y,a) = \lambda(y_{2n-2},y,a) = \lambda_1(Gx_{2n-3},y,a)$$
$$\leqslant \lambda_1(gx_{2n-3},y,a) = \lambda_1(y_{2n-3},y,a)\cdots = \lambda_1(y_0,y,a).$$

Thus $\lambda_1(y_{2n},y,a) \leqslant \lambda_1(y_0,y,a)$.

Similarly we have

$$\lambda_1(x,y_{2n+1},a) = \lambda_1(x,Fx_{2n},a) \leqslant$$
$$\lambda_1(x,fx_{2n},a) = \lambda_1(x,y_{2n},a) = \lambda_1(x,Gx_{2n-1},a)$$
$$\leqslant \lambda_1(x,gx_{2n-1},a) = \lambda_1(x,y_{2n-1},a) = \lambda_1(x,Fx_{2n-2},a) \leqslant \lambda_1(x,fx_{2n-2},a) =$$
$$\lambda_1(x,y_{2n-2},a)\cdots = \lambda_1(x,y_1,a).$$

Thus $\lambda_1(x,y_{2n+1},a) \leqslant \lambda_1(x,y_1,a)$. □

THEOREM 2.2. *Let (X,S) be a complex valued $S-$metric space and $F,G,f,g : X \to X$ satisfying the conditions .*

(2.2.1) $GX \subseteq fX$ *and* $FX \subseteq gX$,

(2.2.2) *The pairs (F,f) and (G,g) are weakly compatible ,* (2.2.3) *fX or gX is a complete subspace of X,*

(2.2.4) *If there exist mappings $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7 : X\times X\times X \to [0,1)$ such that $\lambda_n(Fx, y, a) \leqslant \lambda_n(fx, y, a)$; $\lambda_n(Gx, y, a) \leqslant \lambda_n(gx, y, a)$ and $\lambda_n(x, Fy, a) \leqslant \lambda_n(x, fy, a)$; $\lambda_n(x, Gy, a) \leqslant \lambda_n(x, gy, a), \forall n = 1, 2, 3..., 7,$ for all $x, y \in X$ and for a fixed element $a \in X$,*

(2.2.5)

$$\begin{aligned}
S(Fx, Fx, Gy) \precsim\; & \lambda_1(fx, gy, a)S(fx, fx, gy) + \lambda_2(fx, gy, a)S(fx, fx, Fx) \\
& +\lambda_3(fx, gy, a)S(gy, gy, Gy) \\
& +\lambda_4(fx, gy, a)[S(gy, gy, Fx) + S(fx, fx, Gy)] \\
& +\lambda_5(fx, gy, a)\left(\tfrac{S(fx,fx,Fx)S(gy,gy,Gy)}{1+S(fx,fx,gy)}\right) \\
& +\lambda_6(fx, gy, a)\left(\tfrac{S(gy,gy,Fx)S(fx,fx,Gy)}{1+S(fx,fx,gy)}\right) \\
& +\lambda_7(fx, gy, a)\left(\tfrac{S(fx,fx,Fx)S(fx,fx,Gy)+S(gy,gy,Gy)S(gy,gy,Fx)}{1+S(fx,fx,Gy)+S(gy,gy,Fx)}\right)
\end{aligned}$$

for all $x, y \in X$ and for a fixed element $a \in X$, where

(2.2.6) $(\lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_4 + \lambda_5 + \lambda_6 + \lambda_7)(x, y, a) < 1.$

Then F, G, f and g have a unique common fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point. We define a sequence $\{y_n\}$ in X such that $y_{2n+1} = gx_{2n+1} = Fx_{2n}$ and $y_{2n+2} = fx_{2n+2} = Gx_{2n+1}, n = 0, 1, 2, ...$ From(2.2.5) we have

$$\begin{aligned}
& S(y_{2n+1}, y_{2n+1}, y_{2n+2}) = S(Fx_{2n}, Fx_{2n}, Gx_{2n+1}) \\
& \precsim \lambda_1(y_{2n}, y_{2n+1}, a)S(y_{2n}, y_{2n}, y_{2n+1}) + \lambda_2(y_{2n}, y_{2n+1}, a)S(y_{2n}, y_{2n}, y_{2n+1}) \\
& \quad +\lambda_3(y_{2n}, y_{2n+1}, a)S(y_{2n+1}, y_{2n+1}, y_{2n+2}) \\
& \quad +\lambda_4(y_{2n}, y_{2n+1}, a)[S(y_{2n+1}, y_{2n+1}, y_{2n+1}) + S(y_{2n}, y_{2n}, y_{2n+2})] \\
& \quad +\lambda_5(y_{2n}, y_{2n+1}, a)\left(\tfrac{S(y_{2n},y_{2n},y_{2n+1})S(y_{2n+1},y_{2n+1},y_{2n+2})}{1+S(y_{2n},y_{2n},y_{2n+1})}\right) \\
& \quad +\lambda_6(y_{2n}, y_{2n+1}, a)\left(\tfrac{S(y_{2n+1},y_{2n+1},y_{2n+1})S(y_{2n},y_{2n},y_{2n+1})}{1+S(y_{2n},y_{2n},y_{2n+1})}\right) \\
& \quad +\lambda_7(y_{2n}, y_{2n+1}, a)\left(\frac{\begin{array}{c} S(y_{2n}, y_{2n}, y_{2n+1})S(y_{2n}, y_{2n}, y_{2n+2}) \\ +S(y_{2n+1}, y_{2n+1}, y_{2n+2})S(y_{2n+1}, y_{2n+1}, y_{2n+1}) \end{array}}{1+S(y_{2n},y_{2n},y_{2n+1})+S(y_{2n+1},y_{2n+1},y_{2n+1})}\right)
\end{aligned}$$

Since $S(x, x, x) = 0$, we have

$$\begin{aligned}
& |S(y_{2n+1}, y_{2n+1}, y_{2n+2})| \\
& \quad \leqslant \lambda_1(y_{2n}, y_{2n+1}, a)\,|S(y_{2n}, y_{2n}, y_{2n+1})| \\
& \qquad +\lambda_2(y_{2n}, y_{2n+1}, a)\,|S(y_{2n}, y_{2n}, y_{2n+1})| \\
& \qquad +\lambda_3(y_{2n}, y_{2n+1}, a)\,|S(y_{2n+1}, y_{2n+1}, y_{2n+2})| \\
& \qquad +\lambda_4(y_{2n}, y_{2n+1}, a)\,|S(y_{2n}, y_{2n}, y_{2n+1})| \\
& \qquad +\lambda_4(y_{2n}, y_{2n+1}, a)\,|S(y_{2n+1}, y_{2n+1}, y_{2n+2})| \\
& \qquad +\lambda_5(y_{2n}, y_{2n+1}, a)\,|S(y_{2n+1}, y_{2n+1}, y_{2n+2})|\left|\tfrac{S(y_{2n},y_{2n},y_{2n+1})}{1+S(y_{2n},y_{2n},y_{2n+1})}\right| \\
& \qquad +\lambda_7(y_{2n}, y_{2n+1}, a)\,|S(y_{2n}, y_{2n}, y_{2n+1})|\left|\tfrac{S(y_{2n},y_{2n},y_{2n+2})}{1+S(y_{2n},y_{2n},y_{2n+2})}\right|.
\end{aligned}$$

$$\begin{aligned}&|S(y_{2n+1}, y_{2n+1}, y_{2n+2})| \\ &\quad \leqslant (\lambda_1 + \lambda_2 + \lambda_4 + \lambda_7)(y_{2n}, y_{2n+1}, a)\,|S(y_{2n}, y_{2n}, y_{2n+1})| \\ &\qquad +(\lambda_3 + \lambda_4 + \lambda_5)(y_{2n}, y_{2n+1}, a)\,|S(y_{2n+1}, y_{2n+1}, y_{2n+2})|\,.\end{aligned}$$

Using Proposition (2.1), we get

$$\begin{aligned}|S(y_{2n+1}, y_{2n+1}, y_{2n+2})| \leqslant{}& (\lambda_1 + \lambda_2 + \lambda_4 + \lambda_7)(y_0, y_1, a)\,|S(y_{2n}, y_{2n}, y_{2n+1})| \\ &+(\lambda_3 + \lambda_4 + \lambda_5)(y_0, y_1, a)\,|S(y_{2n+1}, y_{2n+1}, y_{2n+2})|\end{aligned}$$

which in turn implies that

$$|S(y_{2n+1}, y_{2n+1}, y_{2n+2})| \leqslant \left(\tfrac{(\lambda_1+\lambda_2+\lambda_4+\lambda_7)(y_0,y_1,a)}{1-(\lambda_3+\lambda_4+\lambda_5)(y_0,y_1,a)}\right)|S(y_{2n}, y_{2n}, y_{2n+1})|\,.$$

Let $h_1 = \left(\tfrac{(\lambda_1+\lambda_2+\lambda_4+\lambda_7)(y_0,y_1,a)}{1-(\lambda_3+\lambda_4+\lambda_5)(y_0,y_1,a)}\right)$. Thus

$$|S(y_{2n+1}, y_{2n+1}, y_{2n+2})| \leqslant h_1\,|S(y_{2n}, y_{2n}, y_{2n+1})|\,. \;........(1)$$

Similarly using $S(x, y, y) = S(x, x, y)$ and proceeding as above we can show that

$$|S(y_{2n+2}, y_{2n+2}, y_{2n+3})| \leqslant h_2\,|S(y_{2n+1}, y_{2n+1}, y_{2n+2})| ....(2)$$

where $h_2 = \left(\tfrac{(\lambda_1+\lambda_3+\lambda_4+\lambda_7)(y_0,y_1,a)}{1-(\lambda_2+\lambda_4+\lambda_5)(y_0,y_1,a)}\right)$.

Let $h = max\{h_1, h_2\}$, then $0 \leqslant h < 1$, since $h_1, h_2 \in [0, 1)$. Hence from (1) and (2), we have $|S(y_n, y_n, y_{n+1})| \leqslant h\,|S(y_{n-1}, y_{n-1}, y_n)|\,, n = 1, 2, 3, ...$ Repeated use of above inequality gives

$$\begin{aligned}|S(y_k, y_k, y_{k+1})| &\leqslant h^k\,|S(y_0, y_0, y_1)| \;..........(3) \\ &\to 0 \quad as \quad k \to \infty \;...........(4)\end{aligned}$$

Hence for any $m > n$, we have

$$\begin{aligned}|S(y_n, y_n, y_m)| &= 2\left[\begin{array}{l}|S(y_n, y_n, y_{n+1})| + |S(y_{n+1}, y_{n+1}, y_{n+2})| + \\ ...+ |S(y_{m-1}, y_{m-1}, y_m)|\end{array}\right] \\ &= 2(h^n + h^{n+1} + ... + h^{m-1})\,|S(y_0, y_0, y_1)| \text{ from (3)} \\ &\leqslant \tfrac{2h^n}{1-h}\,|S(y_0, y_0, y_1)|\end{aligned}$$

and

$$|S(y_n, y_n, y_m)| \leqslant \frac{2h^n}{1-h}\,|S(y_0, y_0, y_1)| \to 0 \text{ as } m, n \to \infty.$$

Hence $\{y_n\}$ is a Cauchy sequence in X.

Now suppose fX is a complete subspace of X. Since $y_{2n+2} = fx_{2n+2} \in f(X)$ and $\{y_n\}$ is a Cauchy sequence, there exists $z \in f(X)$ such that $y_{2n+2} \to z$ as $n \to \infty$. Then there exists $u \in X$ such that $fu = z$. Thus

$$\lim_{n\to\infty} Fx_{2n} = \lim_{n\to\infty} gx_{2n+1} = \lim_{n\to\infty} Gx_{2n+1} = \lim_{n\to\infty} fx_{2n+2} = z.$$

Consider

$$\begin{aligned}
&S(Fu,Fu,Gx_{2n+1})\\
&\precsim \lambda_1(fu,y_{2n+1},a)S(fu,fu,y_{2n+1})\\
&\quad+\lambda_2(fu,y_{2n+1},a)S(fu,fu,Fu)\\
&\quad+\lambda_3(fu,y_{2n+1},a)S(y_{2n+1},y_{2n+1},y_{2n+2})\\
&\quad+\lambda_4(fu,y_{2n+1},a)[S(y_{2n+1},y_{2n+1},Fu)+S(fu,fu,y_{2n+2})]\\
&\quad+\lambda_5(fu,y_{2n+1},a)\left(\tfrac{S(fu,fu,Fu)S(y_{2n+1},y_{2n+1},y_{2n+2})}{1+S(fu,fu,y_{2n+1})}\right)\\
&\quad+\lambda_6(fu,y_{2n+1},a)\left(\tfrac{S(y_{2n+1},y_{2n+1},Fu)S(fu,fu,y_{2n+2})}{1+S(fu,fu,y_{2n+1})}\right)\\
&+\lambda_7(fu,y_{2n+1},a)\left(\tfrac{S(fu,fu,Fu)S(fu,fu,y_{2n+2})+S(y_{2n+1},y_{2n+1},y_{2n+2})S(y_{2n+1},y_{2n+1},Fu)}{1+S(fu,fu,y_{2n+2})+S(y_{2n+1},y_{2n+1},Fu)}\right)
\end{aligned}$$

$$\begin{aligned}
&|S(Fu,Fu,Gx_{2n+1})|\\
&\leqslant \lambda_1(fu,y_{2n+1},a)|S(fu,fu,y_{2n+1})|\\
&\quad+\lambda_2(fu,y_{2n+1},a)|S(fu,fu,Fu)|\\
&\quad+\lambda_3(fu,y_{2n+1},a)|S(y_{2n+1},y_{2n+1},y_{2n+2})|\\
&\quad+\lambda_4(fu,y_{2n+1},a)|S(y_{2n+1},y_{2n+1},Fu)+S(fu,fu,y_{2n+2})|\\
&\quad+\lambda_5(fu,y_{2n+1},a)\left(\tfrac{|S(fu,fu,Fu)||S(y_{2n+1},y_{2n+1},y_{2n+2})|}{|1+S(fu,fu,y_{2n+1})|}\right)\\
&\quad+\lambda_6(fu,y_{2n+1},a)\left(\tfrac{|S(y_{2n+1},y_{2n+1},Fu)||S(fu,fu,y_{2n+2})|}{|1+S(fu,fu,y_{2n+1})|}\right)\\
&\quad+\lambda_7(fu,y_{2n+1},a)\left(\frac{\begin{array}{c}|S(fu,fu,Fu)||S(fu,fu,y_{2n+2})|\\+|S(y_{2n+1},y_{2n+1},y_{2n+2})||S(y_{2n+1},y_{2n+1},Fu)\end{array}}{|1+S(fu,fu,y_{2n+1})+S(y_{2n+1},y_{2n+1},Fu)|}\right)
\end{aligned}$$

Letting $n\longrightarrow\infty$ and using Lemma 1.2 and 1.5, we get

$$|S(Fu,Fu,z)|\leqslant\lambda_2(z,z,a)\,|S(z,z,Fu)|+\lambda_4(z,z,a)\,|S(z,z,Fu)|$$

from(4), Lemma 1.3 $(1-(\lambda_2+\lambda_4)(z,z,a))\,|S(z,z,Fu)|\leqslant 0$ which in turn yields from(2.2.6) that $|S(Fu,Fu,z)|\leqslant 0$. Therefore $|S(Fu,Fu,z)|=0$. Hence $Fu=z$. Thus $fu=Fu=z$. Since $FX\subseteq gX$, there exists $v\in X$ such that $Fu=gv$. Thus $fu=Fu=gv=z$. Again from (2.2.5), we have

$$\begin{aligned}
|S(z,z,Gv)|&=|S(Fu,Fu,Gv)|\\
&\leqslant\lambda_1(fu,gv,a)|S(fu,fu,gv)|+\lambda_2(fu,gv,a)|S(fu,fu,Fu)|\\
&\quad+\lambda_3(fu,gv,a)|S(gv,gv,Gv)|\\
&\quad+\lambda_4(fu,gv,a)|S(gv,gv,Fu)+S(fu,fu,Gv)|\\
&\quad+\lambda_5(fu,gv,a)\left(\tfrac{|S(fu,fu,Fu)||S(gv,gv,Gv)|}{|1+S(fu,fu,gv)|}\right)\\
&\quad+\lambda_6(fu,gv,a)\left(\tfrac{|S(gv,gv,Fu)||S(fu,fu,Gv)|}{|1+S(fu,fu,gv)|}\right)\\
&\quad+\lambda_7(fu,gv,a)\left(\tfrac{|S(fu,fu,Fu)||S(fu,fu,Gv)|+|S(gv,gv,Gv)||S(gv,gv,Fu)|}{|1+S(fu,fu,Gv)+S(gv,gv,Fu)|}\right)
\end{aligned}$$

so that

$$|S(z,z,Gv)|\leqslant\lambda_3(z,z,a)\,|S(z,z,Gv)|+\lambda_4(z,z,a)\,|S(z,z,Gv)|.$$
$$(1-(\lambda_3+\lambda_4)(z,z,a))\,|S(z,z,Gv)|\leqslant 0$$

which in turn yields from(2.2.6) that $|S(z,z,Gv)| \leqslant 0$. Therefore $|S(z,z,Gv)| = 0$. Hence $Gv = z$. Thus

$$Gv = z = fu = Fu = gv. ......(5)$$

Since (F,f) is weakly compatible, we have

$$fz = fFu = Ffu = Fz. ....(6)$$

$$\begin{aligned}
S(Fz,Fz,z) &= S(Fz,Fz,Gv) \\
&\precsim \lambda_1(fz,gv,a)S(fz,fz,gv) + \lambda_2(fz,gv,a)S(fz,fz,Fz) \\
&\quad +\lambda_3(fz,gv,a)S(gv,gv,Gv) \\
&\quad +\lambda_4(fz,gv,a)[S(gv,gv,Fz) + S(fz,fz,Gv)] \\
&\quad +\lambda_5(fz,gv,a)\left(\tfrac{S(fz,fz,Fz)S(gv,gv,Gv)}{1+S(fz,fz,Gv)}\right) \\
&\quad +\lambda_6(fz,gv,a)\left(\tfrac{S(gv,gv,Fz)S(fz,fz,Gv)}{1+S(fz,fz,Gv)}\right) \\
&\quad +\lambda_7(fz,gv,a)\left(\tfrac{S(fz,fz,Fz)S(fz,fz,Gv)+S(gv,gv,Gv)S(gv,gv,Fz)}{1+S(fz,fz,Gv)+S(gv,gv,Fz)}\right) \\
&= \lambda_1(Fz,z,a)S(Fz,Fz,z) + \lambda_4(Fz,z,a)[S(z,z,Fz) + S(Fz,Fz,z)] \\
&\quad +\lambda_6(Fz,z,a)\left(\tfrac{S(z,z,Fz)S(Fz,Fz,z)}{1+S(Fz,Fz,z)}\right) from(5)\, and\, (6)
\end{aligned}$$

$$\begin{aligned}
|S(Fz,Fz,z)| &\leqslant \lambda_1(Fz,z,a)\,|S(Fz,Fz,z)| \\
&\quad +\lambda_4(Fz,z,a)\,|S(z,z,Fz) + S(Fz,Fz,z)| \\
&\quad +\lambda_6(Fz,z,a)\,|S(z,z,Fz)| \left|\tfrac{S(Fz,Fz,z)}{1+S(Fz,Fz,z)}\right|.
\end{aligned}$$

$(1-(\lambda_1+2\lambda_4+\lambda_6)(Fz,z,a))\,|S(Fz,Fz,z)| \leqslant 0$
which in turn yields from (2.2.6) that $|S(Fz,Fz,z)| \leqslant 0$. Therefore $|S(Fz,Fz,z)| = 0$. Hence $Fz = z$. Thus

$$z = Fz = fz. ....(7)$$

Since the pair (G,g) is weakly compatible, we have

$$gz = gGv = Ggv = Gz. ......(8)$$

From (2.2.5)

$$\begin{aligned}
S(z,z,Gz) &= S(Fz,Fz,Gz) \\
&\precsim \lambda_1(fz,gz,a)S(fz,fz,gz) + \lambda_2(fz,gz,a)S(fz,fz,Fz) \\
&\quad +\lambda_3(fz,gz,a)S(gz,gz,Gz) \\
&\quad +\lambda_4(fz,gz,a)[S(gz,gz,Fz) + S(fz,fz,Gz)] \\
&\quad +\lambda_5(fz,gz,a)\left(\tfrac{S(fz,fz,FZ)S(gz,gz,Gz)}{1+S(fz,fz,gz)}\right) \\
&\quad +\lambda_6(fz,gz,a)\left(\tfrac{S(gz,gz.Fz)S(fz,fz,Gz)}{1+S(fz,fz,gz)}\right) \\
&\quad +\lambda_7(fz,gz,a)\left(\tfrac{S(fz,fz,Fz)S(fz,fz,Gz)+S(gz,gz,Gz)S(gz,gz,Fz)}{1+S(fz,fz,Gz)+S(gz,gz,Fz)}\right)
\end{aligned}$$

$$\begin{aligned}
|S(z,z,Gz)| &\leqslant \lambda_1(z,Gz,a)\,|S(z,z,Gz)| \\
&\quad +\lambda_4(z,Gz,a)\,|S(Gz,Gz,z) + S(z,z,Gz)| \\
&\quad +\lambda_6(z,Gz,a)\,|S(Gz,Gz,z)| \left|\tfrac{S(z,z,Gz)}{1+S(z,z,Gz)}\right| \text{ from (7), (8)}
\end{aligned}$$

$(1-(\lambda_1+2\lambda_4+\lambda_6)(z,Gz,a))\,|S(z,z,Gz)| \leqslant 0$ which in turn yields from (2.2.6) that $|S(z,z,Gz)| \leqslant 0$. Therefore $|S(z,z,Gz)| = 0$. Hence $Gz = z$, so that

$$Gz = gz = z. \text{ ....}(9)$$

Thus from (7) and (9), z is a common fixed point of F, G, f and g. For uniqueness, let $z^* \in X$ be such that $fz^* = Fz^* = z^* = gz^* = Gz^*$.

Now from (2.2.5)

$$\begin{aligned} S(z,z,z^*) &= S(Fz,Fz,Gz^*) \\ &\precsim \lambda_1(fz,gz^*,a)S(fz,fz,gz^*) + \lambda_2(fz,gz^*,a)S(fz,fz,Fz) \\ &\quad +\lambda_3(fz,gz^*,a)S(gz^*,gz^*,Gz^*) \\ &\quad +\lambda_4(fz,gz^*,a)[S(gz^*,gz^*,Fz)+S(fz,fz,Gz^*)] \\ &\quad +\lambda_5(fz,gz^*,a)\left(\tfrac{S(fz,fz,Fz)S(gz^*,gz^*,Gz^*)}{1+S(fz,fz,gz^*)}\right) \\ &\quad +\lambda_6(fz,gz^*,a)\left(\tfrac{S(gz^*,gz^*,Fz)S(fz,fz,Gz^*)}{1+S(fz,fz,gz^*)}\right) \\ &\quad +\lambda_7(fz,gz^*,a)\left(\tfrac{S(fz,fz,Fz)S(fz,fz,Gz^*)+S(gz^*,gz^*,Gz^*)S(gz^*,gz^*,Fz)}{1+S(fz,fz,Gz^*)+S(gz^*,gz^*,Fz)}\right). \end{aligned}$$

$$\begin{aligned} |S(z,z,z^*)| \leqslant{}& \lambda_1(z,z^*,a)\,|S(z,z,z^*)| + \lambda_4(z,z^*,a)\,|S(z^*,z^*,z)+S(z,z,z^*)| \\ &+\lambda_6(z,z^*,a)\,|S(z^*,z^*,z)|\left|\tfrac{S(z,z,z^*)}{1+S(z,z,z^*)}\right|. \end{aligned}$$

$|S(z,z,z^*)| \leqslant (\lambda_1+2\lambda_4+\lambda_6)(z,z^*,a)\,|S(z,z,z^*)|$.
$(1-(\lambda_1+2\lambda_4+\lambda_6)(z,z^*,a))\,|S(z,z,z^*)| \leqslant 0$ which in turn yields from (2.2.6) that $|S(z,z,z^*)| \leqslant 0$. Therfore $|S(z,z,z^*)| = 0$. Thus $z = z^*$.
Hence z is the unique common fixed point of F, G, f and g. Similarly we can prove the theorem if gX is a complete subspace of X. □

Now wegive an example to illustrate our main Theorem 2.2.

EXAMPLE 2.1. Let $X = [0,1]$ and $S : X \times X \times X \to C$ be defined by $S(x,y,z) = |x-z| + i|y-z|$. Then (X,S) is a complex valued $S-$ metric space. Define F, G, f and $g : X \to X$ by $Fx = \frac{x}{16}, Gx = \frac{x}{12}, fx = \frac{x}{4}$ and $gx = \frac{x}{3}$, for all $x \in X$. For fixed element $a = \frac{1}{3}$, define $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7 : X \times X \times X \to [0,1]$ by

$$\lambda_1(x,y,a) = (\tfrac{x}{40} + \tfrac{y}{50} + a), \lambda_2(x,y,a) = \tfrac{xya}{10}, \lambda_3(x,y,a) = \tfrac{x^2y^2a^2}{10}, \lambda_4(x,y,a) = \tfrac{x^3y^3a^3}{10}, \lambda_5(x,y,a) = \tfrac{x^3+y^3+a^3}{10}, \lambda_6(x,y,a) = \tfrac{x^2ya^3}{50}, \lambda_7(x,y,a) = \tfrac{xy^3a^2}{40},$$

for all $x, y \in X$. Then
$\lambda_1(x,y,a)+\lambda_2(x,y,a)+\lambda_3(x,y,a)+2\lambda_4(x,y,a)+\lambda_5(x,y,a)+\lambda_6(x,y,a)+\lambda_7(x,y,a)$
$= (\frac{x}{40} + \frac{y}{50} + a) + \frac{xya}{10} + \frac{x^2y^2a^2}{10} + 2(\frac{x^3y^3a^3}{10}) + \frac{x^3+y^3+a^3}{10} + \frac{x^2ya^3}{50} + \frac{xy^3a^2}{40}$
$\leqslant (\frac{1}{40} + \frac{1}{50} + \frac{1}{3}) + \frac{1}{30} + \frac{1}{90} + \frac{2}{270} + \frac{55}{270} + \frac{1}{1350} + \frac{1}{360}$
$= \frac{3442}{5400} < 1$.
Hence $(\lambda_1+\lambda_2+\lambda_3+2\lambda_4+\lambda_5+\lambda_6+\lambda_7)(x,y,a) < 1$. We have

$$\lambda_1(Fx,y,a) = \lambda_1(\frac{x}{16},y,a) = (\frac{x}{640} + \frac{y}{50} + a)$$

$$\lambda_1(fx,y,a) = \lambda_1(\frac{x}{4},y,a) = (\frac{x}{160} + \frac{y}{50} + a).$$

Clearly $\lambda_1(Fx,y,a) \leqslant \lambda_1(fx,y,a)$. We have

$$\lambda_1(x,Fy,a) = \lambda_1(x,\frac{y}{16},a) = (\frac{x}{40}+\frac{y}{800}+a)$$

$$\lambda_1(x,fy,a) = \lambda_1(x,\frac{y}{4},a) = (\frac{x}{40}+\frac{y}{200}+a).$$

Clearly $\lambda_1(x,Fy,a) \leqslant \lambda_1(x,fy,a)$. We have

$$\lambda_1(Gx,y,a) = \lambda_1(\frac{x}{12},y,a) = (\frac{x}{480}+\frac{y}{50}+a)$$

$$\lambda_1(gx,y,a) = \lambda_1(\frac{x}{3},y,a) = (\frac{x}{120}+\frac{y}{50}+a).$$

Clearly $\lambda_1(Gx,y,a) \leqslant \lambda_1(gx,y,a)$. We have

$$\lambda_1(x,Gy,a) = \lambda_1(x,\frac{y}{12},a) = (\frac{x}{40}+\frac{y}{600}+a)$$

$$\lambda_1(x,gy,a) = \lambda_1(x,\frac{y}{3},a) = (\frac{x}{40}+\frac{y}{150}+a).$$

Clearly $\lambda_1(x,Gy,a) \leqslant \lambda_1(x,gy,a)$.

Similarly we can prove that

$$\lambda_n(Fx,y,a) \leqslant \lambda_n(fx,y,a), \lambda_n(x,Fy,a) \leqslant \lambda_n(x,fy,a)$$

$$\lambda_n(Gx,y,a) \leqslant \lambda_n(gx,y,a), \lambda_n(x,Gy,a) \leqslant \lambda_n(x,gy,a) \forall n = 2,3,4,...7.$$

Consider

$$\begin{aligned}
&|S(Fx,Fx,Gy)| = \left|S(\tfrac{x}{16},\tfrac{x}{16},\tfrac{y}{12})\right| \\
&= |\tfrac{x}{16}-\tfrac{y}{12}| + i|\tfrac{x}{16}-\tfrac{y}{12}| = \tfrac{1}{4}[|\tfrac{x}{4}-\tfrac{y}{3}| + i|\tfrac{x}{4}-\tfrac{y}{3}|] \\
&< \tfrac{1}{3}[|\tfrac{x}{4}-\tfrac{y}{3}| + i|\tfrac{x}{4}-\tfrac{y}{3}|] \\
&\leqslant (\tfrac{x}{160}+\tfrac{y}{150}+\tfrac{1}{3})[|\tfrac{x}{4}-\tfrac{y}{3}| + i|\tfrac{x}{4}-\tfrac{y}{3}|] = \lambda_1(fx,gy,a)S(fx,fx,gy) \\
&\leqslant \lambda_1(fx,gy,a)S(fx,fx,gy) + \lambda_2(fx,gy,a)S(fx,fx,Fx) \\
&\quad +\lambda_3(fx,gy,a)S(gy,gy,Gy) + \lambda_4(fx,gy,a)[S(gy,gy,Fx)+S(fx,fx,Gy)] \\
&\quad +\lambda_5(fx,gy,a)\left(\tfrac{S(fx,fx,Fx)S(gy,gy,Gy)}{1+S(fx,fx,gy)}\right) \\
&\quad +\lambda_6(fx,gy,a)\left(\tfrac{S(gy,gy,Fx)S(fx,fx,Gy)}{1+S(fx,fx,gy)}\right) \\
&\quad +\lambda_7(fx,gy,a)\left(\tfrac{S(fx,fx,Fx)S(fx,fx,Gy)+S(gy,gy,Gy)S(gy,gy,Fx)}{1+S(fx,fx,Gy)+S(gy,gy,Fx)}\right).
\end{aligned}$$

Thus (2.2.5) is satisfied.

One can easily verify the remaining conditions of Theorem 2.2. Clearly $x=0$ is the unique common fixed point of F,G,f and g.

References

[1] M. Abbas, M. Arshad and A. Azam. Fixed points of asymptotically regular mappings in complex valued metric spaces. *Georgian Math. J.*, **20**(2)(2013), 213–221.

[2] A. Azam, B. Fisher and M. Khan. Common fixed point theorems in complex valued metric spaces. *Numerical Functional Analysis and Optimization*, **32**(3)(2011), 243–253.

[3] M. Abbas, B. Fisher and T. Nazir. Well-posedness and peroidic point property of mappings satisfying a rational inequality in an ordered complex valued metric spaces. *Scientific Studies and Research Series Mathematics and Informatics*, **22**(1)(2012), 5–24.

[4] L. Liu and F. Gu. Some common fixed point results in two $S-$metric spaces and applications to integral equations. *J. Nonlinear Sci. Appl.*, **9**(6((2016), 3527–3544.

[5] S. Manro. Some common fixed point theorems in complex valued metric space using implicit function. *Int. J. Anal. Appl.*, **2**(1)(2013), 62–70.

[6] J. Mojardi Afra. Double contraction in $S-$metric spaces. *Int. J. Math. Anal.*, **9**(3)(2015), 117–125.

[7] N. M. Mlaiki. Common fixed points in complex S-metric space. *Adv. Fixed Point Theory*, **4**(4)(2014), 509–524.

[8] N. M. Mlaiki. $\alpha-\psi-$contractive mappings on $S-$metric spaces. *Math. Sci. Lett.*, **4**(1)(2015), 9-12.

[9] H. K. Nashine, M. Imdad and M. Hassan. Common fixed point theorems under rational contractions in complex valued metric spaces. *J. Nonlinear Sci. Appl.*, **7**(2014), 42–50.

[10] K. Prudhvi. Some fixed point results in $S-$metric spaces. *J. Math. Sci. Appl.*, **4**(1)(2016), 1–3.

[11] K. P. R. Rao, P. Ranga Swamy and A. Azam. Common coupled fixed point theorems for a pair of hybrid maps in complex valued metric spaces. *Advances in Fixed Point Theory*, **6**(3)(2016), 277–286.

[12] F. Rouzkard and M. Imdad. Some common fixed point theorems on complex valued metric spaces. *Comput. Math. Appl.*, **64**(6)(2012), 1866–1874.

[13] N. Singh, D. Singh, A. Badal and V. Joshi. Fixed point theorems in complex valued metric space. *J. Egyptian. Math. Soc.*, **24**(2016), 402–409.

[14] W. Sintunavarant and P. Kumam. Generalized common fixed point theorem in complex valued metric spaces and applications. *J. Inequal. Appl.*, **2012**(2012): 84

[15] K. Sitthikul, S. Saejung. Some fixed point theorems in complex valud metric spaces. *Fixed Point Theory Appl.*, **2012**(2012):189.

[16] S. Sedghi, N. Shobe and A. Aliouche. A generalization of fixed point theorem in $S-$metric spaces. *Mat. Vesnik*, **64**(3)(2012), 258–266.

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