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EXISTENCE OF THREE SYMMETRIC POSITIVE SOLUTIONS OF FOURTH ORDER BOUNDARY VALUE PROBLEMS ON SYMMETRIC TIME-SCALES

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Abstract. Let $\mathbb{T} \subset \mathbb{R}$ be a symmetric bounded time-scale, with $a = \min \mathbb{T}$ and $b = \max \mathbb{T}$. We consider the following fourth order boundary value problem

$$\begin{aligned} &\phi(-px^{\triangle\nabla})^{\triangle\nabla}(t) + f(t, x(t)) = 0, \qquad t \in \mathbb{T}^{\kappa^2}_{\kappa^2}, \\ &x(a) = x(b) = 0, \quad x^{\triangle\nabla}(\sigma(a)) = x^{\triangle\nabla}(\rho(b)) = 0 \end{aligned}$$

for a suitable function p and an increasing homeomorphism and homomorphism ϕ. By using the five-functionals fixed-point theorem, we present sufficient conditions for the existence of three symmetric positive solutions of the above problem on time-scales. As applications, an example is given to illustrate the main results.

1. Introductin

In recent years, the conditions for the existence and multiplicity of symmetric positive solutions to boundary value problems have been considered in many papers [**9, 5, 6**] and the references therein. In 2011, Sun [**21**] obtained at least three symmetric positive solutions to the second order nonlocal boundary value problem

$$\begin{aligned} &u''(t) + g(t)f(t, u(t)) = 0, \qquad t \in (0, 1), \\ &u(0) = u(1) = \int_0^1 m(s)u(s)ds. \end{aligned}$$

In 2013, Lin and Zhao [**13**] are concerned with the existence of symmetric positive solutions for the 2n-order boundary value problems

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$$(-1)^{2n}u^{(2n)}(t) + f(t,u(t)) = 0, \qquad t \in (0,1),$$
$$u^{(2k)}(0) = u^{(2n)}(1) = 0, \qquad k = 0,1,2,...,n-1.$$

The theory of dynamic equations on times-scales was introduced by Stefan Hilger in this Ph.D thesis in 1988 [**10**]. The time-scales approach, not only unifies differential and difference equations, but also provides accurate information of phenomena that manifest themselves partly in continuous time and partly in discrete time. By using the theory of time-scales we can also study biological, heat transfer, economic, stock market and epidemic models [**3, 11, 19, 22**]. Hence, the study of dynamic equations on time-scales is worthwhile and has theoretical and practical values. In the past few years, it is found that a considerable amount of interest and research in this area is rapidly growing.

In this paper, we are concerned with the existence of symmetric positive solutions of the following fourth order boundary value problem (FBVP)

$$\phi(-px^{\triangle\nabla})^{\triangle\nabla}(t) + f(t,x(t)) = 0, \qquad t \in \mathbb{T}^{\kappa^2}_{\kappa^2}, \tag{1.1}$$

$$x(a) = x(b) = 0, \quad x^{\triangle\nabla}(\sigma(a)) = x^{\triangle\nabla}(\rho(b)) = 0 \tag{1.2}$$

where $\mathbb{T}$ is a symmetric time scale, i.e., $b - t + a \in \mathbb{T}$ for any given $t \in \mathbb{T}$ and $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism and homomorphism and $\phi(0) = 0$. A projection $\phi : \mathbb{R} \to \mathbb{R}$, which generates the p-Laplacian operator $\phi_p(u) = |u|^{p-2}u$ for $p > 1$, is called an increasing homeomorphism and homomorphism if the following conditions are satisfied.

(i) If $x \leqslant y$, then $\phi(x) \leqslant \phi(y)$, for all $x, y \in \mathbb{R}$.

(ii) ϕ is a continuous bijection and its inverse mapping is also continuous.

(iii) $\phi(xy) = \phi(x)\phi(y)$, for all $x, y \in \mathbb{R}$.

Recently, for the existence problems of positive solutions of boundary value problems on time-scales, some authors have obtained many results; for details, see [**1, 2, 16, 17, 20**] and the references therein. However, they did not further provide characteristic of positive solutions, such as symmetry that not only has its theoretical value, such as in studying chemical structures [**9, 13, 14, 21**].

There is only a few work in the literature which discussed the fourth order boundary value problem for an increasing homeomorphism and homomorphism on symmetric time scales but in these studies, problems were given on symmetric time scales which is subset of $(0,1)$. However, there are so many symmetric time scales that such as $\mathbb{Z}$, $[-1,0] \cup [1,2]$, $\{3 - \frac{1}{n}\}_{n\in\mathbb{N}} \cup \{3\} \cup \{3 + \frac{1}{n}\}_{n\in\mathbb{N}}$, etc., the domain of the unknown function for these problems is empty set. Therefore, we can not say that these results yield a result for discrete equations and so on. So, our results in this paper are new for the special cases of continuous and discrete equations, as in the symmetric time-scale.

Motivated by the references [**14, 18, 20**], we consider the FBVP for an increasing homeomorphism and homomorphism (1.1)-(1.2) on symmetric time scales. By using the symmetric technique and the five-functionals fixed-point theorem, we obtain the existence of three symmetric positive solutions of problem (1.1)-(1.2). As applications, an example is given to illustrate our main results.

The rest of the paper is organized as follows. In this section, we give some definitions. In section 2, we give the definition of cone and the five-functionals fixed-point theorem. We present some lemmas which needed to prove main result. In section 3, by using five-functionals fixed-point theorem, we obtain the existence of three symmetric positive solutions of the problem (1.1)-(1.2) and also we present an example to illustrate our main results.

We first briefly recall some basic definitions and results concerning time-scales. Further general details can be found in [**5, 6**]. Hereafter, we use the notation $[a,b]_{\mathbb{T}}$ to indicate the time scale interval $[a,b]\cap\mathbb{T}$.

Let $\mathbb{T}\subset\mathbb{R}$ be a bounded time-scale (a non-empty closed subset of $\mathbb{R}$), with $a=\inf\{s\in\mathbb{T}\},\ b=\sup\{s\in\mathbb{T}\}$. Define the jump operators $\sigma,\rho:\mathbb{T}\to\mathbb{T}$ by $\sigma(t):=\inf\{s\in\mathbb{T}:s>t\}$, $\rho(t):=\sup\{s\in\mathbb{T}:s<t\}$ where, in this definition, we write $\inf\emptyset=a,\ \sup\emptyset=b$ so that $\rho(a)=a,\ \ \sigma(b)=b$. A point $t\in\mathbb{T}$ is said to be left dense, left scattered, right dense, right scattered if $\rho(t)=t,\ \rho(t)<t,\sigma(t)=t,\ \sigma(t)>t$ respectively. We endow $\mathbb{T}$ with the subspace topology inherited from $\mathbb{R}$.

Now suppose that $x:\mathbb{T}\to\mathbb{R}$. Continuity of x is defined in the usual manner, while x is said to be ld-continuous on $\mathbb{T}$ if it is continuous at all left dense points and has finite right sided limits at all right dense points of $\mathbb{T}$. We let $\mathcal{C}_{ld}(\mathbb{T})$ denote the set of ld-continuous functions $x:\mathbb{T}\to\mathbb{R}$, and let

$$\|x\|:=\sup_{t\in\mathbb{T}}|x(t)|,\quad x\in\mathcal{C}_{ld}(\mathbb{T}).$$

With this norm $\mathcal{C}_{ld}$ is a Banach space.

We assume throughout that $\rho^2(b)>\sigma^2(a)$, where $\sigma^2(t):=\sigma(\sigma(t))$ and $\rho^2(t)=\rho(\rho(t))$ so that $\mathbb{T}$ must contained at least 6 points. Now define the sets $\mathbb{T}_\kappa:=\mathbb{T}-[a,\sigma(a))$, $\mathbb{T}^\kappa:=\mathbb{T}-(\rho(b),b]$, $\mathbb{T}^\kappa_\kappa:=\mathbb{T}-([a,\sigma(a))\cup(\rho(b),b])$ and $\mathbb{T}^{\kappa^2}_{\kappa^2}:=\mathbb{T}-([a,\sigma^2(a))\cup(\rho^2(b),b])$. These sets are closed, so they are time-scales and we can also define the above spaces and norms using $\mathbb{T}^\kappa_\kappa$ and $\mathbb{T}^{\kappa^2}_{\kappa^2}$ instead of $\mathbb{T}$.

A function $x:\mathbb{T}\to\mathbb{R}$ is delta differentiable at $t\in\mathbb{T}^\kappa$ if there exists a number $x^\Delta(t)$ with the following property: for any $\epsilon>0$ there exists a $\delta>0$ such that

$$s\in\mathbb{T}\text{ and }\ |t-s|<\delta\Rightarrow|x(\sigma(t))-x(s)-x^\Delta(t)(\sigma(t)-s)|\leqslant\epsilon|\sigma(t)-s|.$$

If x is delta differentiable at every $t\in\mathbb{T}^\kappa$ then x is said to be delta differentiable. Similarly, a function $x:\mathbb{T}\to\mathbb{R}$ is nabla differentiable at $t\in\mathbb{T}_\kappa$ if there exists a number $x^\nabla(t)$ with the following property: for any $\epsilon>0$ there exists a $\delta>0$ such that

$$s\in\mathbb{T}\text{ and }\ |t-s|<\delta\Rightarrow|x(\rho(t))-x(s)-x^\nabla(t)(\rho(t)-s)|\leqslant\epsilon|\rho(t)-s|.$$

If x is nabla differentiable at every $t\in\mathbb{T}_\kappa$ then x is said to be nabla differentiable.

A function $F:\mathbb{T}\to\mathbb{R}$ is called a delta antiderivative of $f:\mathbb{T}\to\mathbb{R}$ provided $F^\Delta(t)=f(t)$ holds for all $t\in\mathbb{T}^\kappa$. We then define the delta integral of f by

$$\int_a^t f(s)\Delta s=F(t)-F(a)\ \text{ for all }\ a,t\in\mathbb{T}.$$

A function $G : \mathbb{T} \to \mathbb{R}$ is called a nabla antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided $G^{\nabla}(t) = f(t)$ holds for all $t \in \mathbb{T}_\kappa$. We then define the nabla integral of f by

$$\int_a^t f(s)\nabla s = G(t) - G(a) \quad \text{for all} \quad a, t \in \mathbb{T}.$$

For convenience, we now present some symmetric definitions which can be found in [**11**].

Definition 1.1. A time-scale $\mathbb{T}$ is said to be symmetric if for any given $t \in \mathbb{T}$, we have $b - t + a \in \mathbb{T}$.

We remark that since $\mathbb{T}$ is a symmetric time-scale, we have $b - \sigma(a) + a = \rho(b)$ and $b - \rho(b) + a = \sigma(a)$.

Definition 1.2. A function $x : \mathbb{T} \to \mathbb{R}$ is said to be symmetric on $\mathbb{T}$ if for any given $t \in \mathbb{T}$, $x(t) = x(b - t + a)$.

Definition 1.3. We say x is a symmetric solution of FBVP (1.1)-(1.2) on $\mathbb{T}$ provided x is a solution of FBVP (1.1)-(1.2) and is symmetric on $\mathbb{T}$.

Throughout this paper, $\mathbb{T}$ is a symmetric bounded time-scale with $a = \min \mathbb{T}$, $b = \max \mathbb{T}$ and we assume that

(H1) $p \in \mathcal{C}_{ld}(\mathbb{T})$ and p is positive and symmetric on $\mathbb{T}$, (H2) $f : \mathbb{T} \times [0, \infty) \to [0, \infty)$ is ld-continuous, and does not vanish identically, in addition $f(., x)$ is a symmetric function on $\mathbb{T}$, i.e., $f(b - t + a, x) = f(t, x)$ for all $(t, x) \in \mathbb{T} \times [0, \infty)$.

2. Preliminaries

In this section, we provide some background material from the theory of cones in Banach spaces, and we then state Five-functionals fixed-point theorem for a cone preserving operator. In the rest of this section, we present some lemmas and completely continuous operator, which will be needed in the proof of the main result. We provide some background material on the theory of cones in Banach spaces [**8**], which will be used in the rest of the paper.

Definition 2.1. Let B be a real Banach space. A nonempty, closed, convex set $P \subset B$ is called a cone if it satisfies the following two conditions:

(i) $x \in P$, $\lambda \geqslant 0$ implies $\lambda x \in P$;

(ii) $x \in P$, $-x \in P$ implies $x = 0$. Every cone $P \subset B$ induces an ordering in B given by $x \leqslant y$ if and only if $y - x \in P$.

Definition 2.2. An operator is called completely continuous, if it is continuous and maps bounded sets into precompact sets.

Definition 2.3. A map α is said to be a nonnegative, continuous, concave functional on a cone P of a real Banach space B, if $\alpha : P \to [0, \infty)$ is continuous and

$$\alpha(tx + (1-t)y) \geqslant t\alpha(x) + (1-t)\alpha(y) \text{ for all } x, y \in P \text{ and } t \in [0, 1].$$

Similarly, we say the map β is a nonnegative, continuous, convex functional on a cone P of a real Banach space B, if $\beta : P \to [0,\infty)$ is continuous and

$$\beta(tx+(1-t)y) \leqslant t\beta(x)+(1-t)\beta(y) \text{ for all } x,y \in P \text{ and } t \in [0,1].$$

Let $\gamma,\ \beta,\ \theta$ be nonnegative, continuous, convex functionals on P and $\alpha,\ \psi$ be nonnegative, continuous, concave functionals on P. Then, for nonnegative real numbers $h,\ a,\ b,\ d$ and c, we define the convex sets,

$$P(\gamma,c) = \{x \in P : \gamma(x) < c\},$$
$$P(\gamma,\alpha,a,c) = \{x \in P : a \leqslant \alpha(x),\ \gamma(x) \leqslant c\},$$
$$Q(\gamma,\beta,d,c) = \{x \in P : \beta(x) \leqslant d,\ \gamma(x) \leqslant c\},$$
$$P(\gamma,\theta,\alpha,a,b,c) = \{x \in P : a \leqslant \alpha(x),\ \theta(x) \leqslant b, \gamma(x) \leqslant c\},$$
$$P(\gamma,\beta,\psi,h,d,c) = \{x \in P : h \leqslant \psi(x),\ \beta(x) \leqslant d, \gamma(x) \leqslant c\}.$$

To prove our main results, we need the following five-functionals fixed-point theorem [**4**], which is a generalization of the Leggett-Williams fixed point theorem [**12**].

Theorem 2.1. Let P be a cone in a real Banach space B. Suppose there exist positive numbers c and M, nonnegative, continuous, concave functions α and ψ on P, and nonnegative continuous, convex functionals $\gamma, \beta,$ and θ and P, with

$$\alpha(x) \leqslant \beta(x) \text{ and } \|x\| \leqslant M\gamma(x) \text{ for all } x \in \overline{P(\gamma,c)}.$$

Suppose $T : \overline{P(\gamma,c)} \to \overline{P(\gamma,c)}$ is completely continuous and there exist nonnegative numbers h, a, k, b, with $0 < a < b$ such that:

(i) $\{x \in P(\gamma,\theta,\alpha,b,k,c) : \alpha(x) > b\} \neq \emptyset$ and $\alpha(Tx) > b$ for $x \in P(\gamma,\theta,\alpha,b,k,c)$;

(ii) $\{x \in Q(\gamma,\beta,\psi,h,a,c) : \alpha(x) > b\} \neq \emptyset$ and $\beta(Tx) < a$ for $x \in Q(\gamma,\beta,\psi,h,a,c)$;

(iii) $\alpha(Tx) > b$ for $x \in P(\gamma,\alpha,b,c)$ with $\theta(Tx) > k$;

(iv) $\beta(Tx) < a$ for $x \in Q(\gamma,\beta,a,c)$ with $\psi(Tx) < h$.

Then T has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\gamma,c)}$ such that

$$\beta(x_1) < a, b < \alpha(x_2), \text{ and } a < \beta(x_3) \text{ with } \alpha(x_3) < b.$$

To obtain our main result, we will make use of the following lemmas. For their proofs we refer the reader to [**7**].

Lemma 2.1. Assume that (H1) holds. Let $y \in \mathcal{C}_{ld}(\mathbb{T})$ and $y(t) \not\equiv 0$. Then the BVP

$$\phi(-px^{\triangle\nabla})(t) - y(t) = 0, \qquad t \in \mathbb{T}^{\kappa}_{\kappa}, \tag{2.1}$$

$$x(a) = x(b) = 0, \tag{2.2}$$

has a unique solution

$$x(t) = \int_a^b G(t,s)\frac{1}{p(s)}\phi^{-1}(y(s))\nabla s \tag{2.3}$$

where

$$G(t,s) = \frac{1}{b-a}\begin{cases} (t-a)(b-s), & t \leqslant s; \\ (s-a)(b-t), & s \leqslant t. \end{cases} \tag{2.4}$$

LEMMA 2.2. Assume that (H2) holds. Then for $x \in \mathcal{C}_{ld}(\mathbb{T})$, the BVP

$$-y^{\triangle\nabla}(t) = f(t, x(t)), \qquad t \in \mathbb{T}_{\kappa^2}^{\kappa^2}, \tag{2.5}$$

$$y(\sigma(a)) = y(\rho(b)) = 0, \tag{2.6}$$

has a unique solution

$$y(t) = \int_{\sigma(a)}^{\rho(b)} H(t,s) f(t, x(t)) \nabla s \tag{2.7}$$

where

$$H(t,s) = \frac{1}{\rho(b) - \sigma(a)} \begin{cases} (t - \sigma(a))(\rho(b) - s), & t \leqslant s; \\ (s - \sigma(a))(\rho(b) - t), & s \leqslant t. \end{cases} \tag{2.8}$$

Assume that x is a solution of problem (1.1)-(1.2). From Lemma 2.1, we have

$$x(t) = \int_a^b G(t,s) \frac{1}{p(s)} \phi^{-1}(y(s)) \nabla s$$

and then from Lemma 2.2, we have

$$x(t) = \int_a^b G(t,s) \frac{1}{p(s)} \phi^{-1} \left(\int_{\sigma(a)}^{\rho(b)} H(s,\tau) f(\tau, x(\tau)) \nabla \tau \right) \nabla s.$$

LEMMA 2.3. The following is valid
(i) For $t, s \in \mathbb{T}$, we have $G(t,s) \geqslant 0$ and $G(t,s) \leqslant G(s,s)$,
(ii) For $t, s \in T_\kappa^\kappa$, we have $H(t,s) \geqslant 0$ and $H(t,s) \leqslant H(s,s)$.

LEMMA 2.4. Let $\delta \in \left(0, \dfrac{1}{2}\right)$ be a given constant, then we have

$$(i)\ G(t,s) \geqslant \frac{\delta}{b-a} G(s,s), \quad t \in [q_1, q_2]_{\mathbb{T}}, \ \ s \in \mathbb{T} \tag{2.9}$$

where $q_1 := \min\{t \in \mathbb{T} : a + \delta \leqslant t\}$ and $q_2 := \max\{t \in \mathbb{T} : t \leqslant b - \delta\}$,

$$(ii)\ H(t,s) \geqslant \frac{\delta}{\rho(b) - \sigma(a)} H(s,s), \quad t \in [q_1^*, q_2^*]_{\mathbb{T}}, \ \ s \in \mathbb{T}_\kappa^\kappa \tag{2.10}$$

where $q_1^* := \min\{t \in \mathbb{T} : \sigma(a) + \delta \leqslant t\}$ and $q_2^* := \max\{t \in \mathbb{T} : t \leqslant \rho(b) - \delta\}$.

LEMMA 2.5. The following is valid
(i) $G(b - t + a, b - s + a) = G(t,s)$ for all $t, s \in \mathbb{T}$,
(ii) $H(b - t + a, b - s + a) = H(t,s)$ for all $t, s \in \mathbb{T}_\kappa^\kappa$.

Now, we let $B = \mathcal{C}_{ld}(\mathbb{T})$ then B is a Banach space with the $\|x\| = \max_{t \in \mathbb{T}} |x(t)|$, and define a cone $P \subset B$ by $P = \{x \in B : x(t) \geqslant 0, \text{ for } t \in \mathbb{T}, \ x^{\triangle\nabla}(t) \leqslant 0, \text{ for } t \in \mathbb{T}_\kappa^\kappa, \ x(t) \text{ is symmetric on } \mathbb{T} \text{ and } x(t) \geqslant \Lambda\|x\|\}$ where $\Lambda := \dfrac{\delta}{b-a} \phi^{-1} \left(\dfrac{\delta}{\rho(b) - \sigma(a)} \right)$.

REMARK 2.1. For $x \in P$, since $x^{\triangle\nabla}(t) \leqslant 0$ for $t \in \mathbb{T}_\kappa^\kappa$ and x is symmetric then x has a maximum such that $\max_{t \in \mathbb{T}} |x(t)| = x(\xi)$ where $\xi = \max\{t \in \mathbb{T} : t \leqslant \dfrac{b-a}{2}\}$.

Secondly, we define the integral operator $T : P \to B$ by

$$Tx(t) = \int_a^b G(t,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{\sigma(a)}^{\rho(b)} H(s,\tau)f(\tau,x(\tau))\nabla\tau\right)\nabla s. \tag{2.11}$$

So, we have

$$\begin{aligned} T^{\Delta}x(t) &= \frac{1}{b-a}\int_a^t (a-s)\frac{1}{p(s)}\phi^{-1}\left(\int_{\sigma(a)}^{\rho(b)} H(s,\tau)f(\tau,x(\tau))\nabla\tau\right)\nabla s \\ &\quad +\frac{1}{b-a}\int_t^b (b-s)\frac{1}{p(s)}\phi^{-1}\left(\int_{\sigma(a)}^{\rho(b)} H(s,\tau)f(\tau,x(\tau))\nabla\tau\right)\nabla s, \\ T^{\Delta\nabla}x(t) &= -\frac{1}{p(t)}\phi^{-1}\left(\int_{\sigma(a)}^{\rho(b)} H(t,s)f(s,x(s))\nabla s\right). \end{aligned}$$

Hence, for $x \in P$, $Tx(t) \geqslant 0$ on $\mathbb{T}$ and $T^{\Delta\nabla}x(t) \leqslant 0$ on $\mathbb{T}^{\kappa}_{\kappa}$.

Using that $p(t)$, $x(t)$ and $f(t,x(t))$ are symmetric on $\mathbb{T}$, we have

$$\begin{aligned} &Tx(b-t+a) = \\ &\int_a^b G(b-t+a,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{\sigma(a)}^{\rho(b)} H(b-s+a,\tau)f(\tau,x(\tau))\nabla\tau\right)\nabla s = \\ &\int_b^a G(b-t+a,b-s+a)\frac{1}{p(b-s+a)}\phi^{-1}\left(\int_{\sigma(a)}^{\rho(b)} H(b-s+a,\tau)f(\tau,x(\tau))\nabla\tau\right)\nabla(b-s+a) \\ &= \\ &\int_a^b G(t,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{\sigma(a)}^{\rho(b)} H(b-s+a,\tau)f(\tau,x(\tau))\nabla\tau\right)\nabla s = \\ &\int_a^b G(t,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{\rho(b)}^{\sigma(a)} H(b-s+a,b-\tau+a)f(b-\tau+a,x(b-\tau+a))\nabla(b-\tau+a)\right)\nabla s \\ &= \\ &\int_a^b G(t,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{\sigma(a)}^{\rho(b)} H(s,\tau)f(\tau,x(\tau))\nabla\tau\right)\nabla s = \\ &Tx(t) \end{aligned}$$

for every $t \in \mathbb{T}$. This implies that $Tx(t)$ is symmetric on $\mathbb{T}$. It is easy to verify that $Tx(t) \geqslant \Lambda\|Tx\|$. So, $T : P \to P$.

Lemma 2.6. Assume (H1) and (H2) hold. Then $x \in B$ is a solution of FBVP (1.1)- (1.2) if and only if x is a fixed point of the operator T.

Lemma 2.7. Assume (H1) and (H2) hold. Then, the operator $T : P \to P$ is completely continuous.

Proof. Suppose that $K \subset P$ is a bounded set. Let $M > 0$ be such that $\|x\| \leqslant M$ for $x \in K$, we have

$$|Tx(t)| = \left| \int_a^b G(t,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{\sigma(a)}^{\rho(b)} H(s,\tau)f(\tau,x(\tau))\nabla\tau\right)\nabla s\right|$$

$$\leqslant \int_a^b G(s,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{\sigma(a)}^{\rho(b)} H(\tau,\tau)f(\tau,x(\tau))\nabla\tau\right)\nabla s$$

$$\leqslant \left\{\int_a^b G(s,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{\sigma(a)}^{\rho(b)} H(\tau,\tau)f(\tau,x(\tau))\nabla\tau\right)\nabla s\right\}\phi^{-1}(\sup_{x\in K,\ t\in\mathbb{T}} f(t,x(t)))$$

for every $t \in \mathbb{T}$. This implies that $T(K)$ is bounded. By the Arzela-Ascoli theorem and the Lebesgue dominated convergent theorem on time-scales, we can easily seen that T is a completely continuous operator. □

3. Existence of three symmetric positive solutions

In this section, we consider the existence of three positive symmetric solutions of the FBVP (1.1)- (1.2). Let us define the nonnegative, continuous, concave functionals α, ψ, and the nonnegative, continuous, convex functionals β, θ, γ on the cone P by:

$$\gamma(x) = \theta(x) := \max_{t\in\mathbb{T}} x(t) = x(\xi),$$
$$\beta(x) := \max_{t\in[q_1^*,q_2^*]_{\mathbb{T}}} x(t) = x(\xi),$$
$$\alpha(x) = \psi(x) := \min_{t\in[q_1^*,q_2^*]_{\mathbb{T}}} x(t) = x(q_1^*).$$

We see that, for each $x \in P$, $\alpha(x) = x(q_1^*) \leqslant x(\xi) = \beta(x)$. In addition, for each $x \in P$, $\|x\| \leqslant \frac{1}{\Lambda}x(t)$ for $t \in [q_1^*, q_2^*]_{\mathbb{T}}$ so $\|x\| \leqslant \frac{1}{\Lambda}x(q_1^*) = \gamma(x)$. That is $\|x\| \leqslant \frac{1}{\Lambda}\gamma(x)$ for all $x \in P$. For convenience, we denote

$$m := \int_a^b G(s,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{\sigma(a)}^{\rho(b)} H(\tau,\tau)\nabla\tau\right)\nabla s,$$

$$M := \int_{q_1^*}^{q_2^*} G(s,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{q_1^*}^{q_2^*} H(\tau,\tau)\nabla\tau\right)\nabla s,$$

$$M_* := \int_{q_1^*}^{q_2^*} G(q_1^*,s)\frac{1}{p(s)}\phi^{-1}\left(\frac{1}{\rho(b)-\sigma(a)}\int_{q_1^*}^{q_2^*} H(\tau,\tau)\nabla\tau\right)\nabla s,$$

$$M^* := \int_a^b G(\xi,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{\sigma(a)}^{\rho(b)} H(\tau,\tau)\nabla\tau\right)\nabla s,$$

satisfy the inequality $M_* \geqslant \Lambda M^*$.

Theorem 3.1. Assume that (H1) and (H2) are satisfied. Suppose that there exist positive number $u < y < \frac{y}{\Lambda} < z$ with $\frac{z}{M^*} > \frac{y}{\Lambda M_*}$ such that the function f satisfies the following conditions:

(i)$f(t,x) < \phi\left(\frac{z}{M^*}\right)$ for all $(t,x) \in \mathbb{T} \times [0,z]$;

(ii)$f(t,x) > \phi\left(\frac{b}{\Lambda M_*}\right)$ for all $(t,x) \in [q_1^*, q_2^*]_{\mathbb{T}} \times [y, \frac{y}{\Lambda}]$;

(iii) $f(t,x) < \phi\left(\frac{a}{M^*}\right)$ for all $(t,x) \in [q_1^*, q_2^*]_{\mathbb{T}} \times [\Lambda u, u]$.

Then there exist at least three positive symmetric solutions x_1, x_2, x_3 of the FBVP (1.1)-(1.2) such that

$$\max_{t\in[q_1^*,q_2^*]_{\mathbb{T}}} x_1(t) < a < \max_{t\in[q_1^*,q_2^*]_{\mathbb{T}}} x_3(t) \text{ and } \min_{t\in[q_1^*,q_2^*]_{\mathbb{T}}} x_3(t) < b < \min_{t\in[q_1^*,q_2^*]_{\mathbb{T}}} x_2(t)$$

Proof. By the definition of completely continuous operator T and its properties, it suffices to show that all the conditions of the Theorem 2.1 hold with respect to T. First we show that T maps $\overline{P(\gamma,z)}$ into itself. In fact, for each $x \in \overline{P(\gamma,z)}$, from $\gamma(x) = \|x\| = \max_{t\in\mathbb{T}} \leqslant z$ and the condition (i), it follows that $f(t,x) < \phi\left(\frac{z}{M^*}\right)$ for all $(t,x) \in \mathbb{T} \times [0,z]$.

Applying this together with $Tx \in P$, we have the following estimate

$$\begin{aligned}
\|Tx\| &= \sup_{t\in\mathbb{T}} |Tx(t)| \\
&= \int_a^b G(\xi,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{\sigma(a)}^{\rho(b)} H(s,\tau)f(\tau,x(\tau))\nabla\tau\right)\nabla s \\
&< \frac{z}{M^*}\int_a^b G(\xi,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{\sigma(a)}^{\rho(b)} H(\tau,\tau)\nabla\tau\right)\nabla s = z.
\end{aligned}$$

Hence, $T : \overline{P(\gamma,z)} \longrightarrow \overline{P(\gamma,z)}$.

Next, let $N := \frac{M_* + \Lambda M^*}{2}$. Thus $M_* > N > \Lambda M^*$, and if we define

$$x_P(t) = \frac{y}{N}\int_a^b G(t,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{\sigma(a)}^{\rho(b)} H(s,\tau)\nabla\tau\right)\nabla s \text{ and}$$
$$x_Q(t) = \frac{\Lambda u}{N}\int_a^b G(t,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{\sigma(a)}^{\rho(b)} H(s,\tau)\nabla\tau\right)\nabla s$$

then, clearly $x_p, x_Q \in P$. Furthermore,

$$\begin{aligned}
\alpha(x_P) &= x_P(q_1^*) = \frac{y}{N}\int_a^b G(q_1^*,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{\sigma(a)}^{\rho(b)} H(s,\tau)\nabla\tau\right)\nabla s \\
&\geqslant \frac{y}{N}\int_{q_1^*}^{q_2^*} G(q_1^*,s)\frac{1}{p(s)}\phi^{-1}\left(\frac{1}{\rho(b)-\sigma(a)}\int_{q_1^*}^{q_2^*} H(\tau,\tau)\nabla\tau\right)\nabla s \\
&= \frac{y}{N}M_* > y
\end{aligned}$$

and

$$\begin{aligned}\theta(x_P) &= x_P(\xi)=\frac{y}{N}\int_a^b G(\xi,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{\sigma(a)}^{\rho(b)} H(s,\tau)\nabla\tau\right)\nabla s\\ &\leqslant \frac{y}{N}M^* < \frac{y}{\Lambda}\end{aligned}$$

as well as

$$\begin{aligned}\psi(x_Q) &= x_Q(q_1^*)=\frac{\Lambda u}{N}\int_a^b G(q_1^*,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{\sigma(a)}^{\rho(b)} H(s,\tau)\nabla\tau\right)\nabla s\\ &\geqslant \frac{\Lambda u}{N}\int_{q_1^*}^{q_2^*} G(q_1^*,s)\frac{1}{p(s)}\phi^{-1}\left(\frac{1}{\rho(b)-\sigma(a)}\int_{q_1^*}^{q_2^*} H(\tau,\tau)\nabla\tau\right)\nabla s\\ &= \frac{\Lambda u}{N}M_* > \Lambda u\end{aligned}$$

and

$$\begin{aligned}\beta(x_Q) &= x_Q(\xi)=\frac{\Lambda u}{N}\int_a^b G(q_1^*,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{\sigma(a)}^{\rho(b)} H(s,\tau)\nabla\tau\right)\nabla s\\ &\leqslant \frac{\Lambda u}{N}M^* < u.\end{aligned}$$

Therefore, $x_p \in \{x\in P(\gamma,\theta,\alpha,y,\frac{y}{\Lambda},z) : \alpha(x) > y\}$ and

$$x_Q \in \{x\in Q(\gamma,\beta,\psi,\Lambda u,u,z) : \beta(x) < u\}.$$

Hence, these sets are nonempty.

If $x\in P(\gamma,\theta,\alpha,y,\frac{y}{\Lambda},z)$, then $y\leqslant x(t)\leqslant \frac{y}{\Lambda}$ for all $t\in[q_1^*,q_2^*]_{\mathbb{T}}$ and by condition (ii) of this theorem

$$\begin{aligned}\alpha(Tx) &= Tx(q_1^*)=\int_a^b G(q_1^*,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{\sigma(a)}^{\rho(b)} H(s,\tau)f(\tau,x(\tau))\nabla\tau\right)\nabla s\\ &\geqslant \int_{q_1^*}^{q_2^*} G(q_1^*,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{q_1^*}^{q_2^*} H(s,\tau)f(\tau,x(\tau))\nabla\tau\right)\nabla s\\ &\geqslant \int_{q_1^*}^{q_2^*} G(q_1^*,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{q_1^*}^{q_2^*}\frac{\delta}{\rho(b)-\sigma(a)} H(\tau,\tau)f(\tau,x(\tau))\nabla\tau\right)\nabla s\\ &\geqslant \frac{y}{M_*}\int_{q_1^*}^{q_2^*} G(q_1^*,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{q_1^*}^{q_2^*}\frac{\delta}{\rho(b)-\sigma(a)} H(\tau,\tau)\nabla\tau\right)\nabla s\\ &= \frac{y}{M_*}M_* = y.\end{aligned}$$

Hence, condition (i) of the five-functionals fixed-point theorem is satisfied.

If $x\in P(\gamma,\alpha,y,z)$ with $\theta(Tx) > \frac{y}{\Lambda}$, then we have

$$\alpha(Tx)=Tx(q_1^*)\geqslant \Lambda\theta(Tx) > y.$$

Thus, condition (ii) of the five-functionals fixed-point theorem is satisfied.

If $x \in Q(\gamma, \beta, \psi, \Lambda u, u, z)$ then $\Lambda u \leqslant \psi(x) \leqslant a$ for all $t \in [q_1^*, q_2^*]$. Thus by condition (iii) of this theorem

$$\begin{aligned}
\beta(Tx) &= Tx(\xi) = \int_a^b G(\xi, s)\frac{1}{p(s)}\phi^{-1}\left(\int_{\sigma(a)}^{\rho(b)} H(s,\tau)f(\tau, x(\tau))\nabla\tau\right)\nabla s \\
&\leqslant \frac{u}{M^*}\int_a^b G(\xi, s)\frac{1}{p(s)}\phi^{-1}\left(\int_{\sigma(a)}^{\rho(b)} H(\tau,\tau)\nabla\tau\right)\nabla s = u.
\end{aligned}$$

Hence, condition (iii) of the five-functionals fixed-point theorem is satisfied.

If $x \in Q(\gamma, \beta, u, z)$ with $\psi(Tx) < \Lambda u$, then we have

$$\beta(Tx) = Tx(\xi) \leqslant \frac{1}{\Lambda}Tx(q_1^*) = \frac{1}{\Lambda}\psi(Tx) < u.$$

Consequently, condition (iv) of the five-functionals fixed-point theorem is also satisfied. Therefore, the hypotheses of the five-functionals fixed-point Theorem 2.1 are satisfied, and there exist at least three positive symmetric solutions x_1, x_2, x_3 of the FBVP (1.1)-(1.2) such that

$$\max_{t\in[q_1^*,q_2^*]_{\mathbb{T}}} x_1(t) < a < \max_{t\in[q_1^*,q_2^*]_{\mathbb{T}}} x_3(t) \quad \text{and} \quad \min_{t\in[q_1^*,q_2^*]_{\mathbb{T}}} x_3(t) < b < \min_{t\in[q_1^*,q_2^*]_{\mathbb{T}}} x_2(t).$$

□

Now, we present a simple example to illustrate our result.

Example 3.1. Let $\mathbb{T} = \{-5, -4, -3, -2, -1, 0, 0, 1, 2, 3, 4, 5\}$ be a bounded symmetric time-scale. We consider the following problem:

$$\phi((t^2+5)x^{\triangle\nabla})^{\triangle\nabla}(t) + f(t, x(t)) = 0, \qquad t \in \mathbb{T}_{\kappa^2}^{\kappa^2}, \tag{3.1}$$

$$x(-5) = x(5) = 0, \quad x^{\triangle\nabla}(-4) = x^{\triangle\nabla}(4) = 0 \tag{3.2}$$

where $\phi(x) = x^2$ and

$$f(t, x(t)) = \begin{cases} 2 + cost + x(t), & (t, x) \in \mathbb{T}] \times [0, 100]; \\ 102 + cost + (x(t) - 100)^4, & (t, x) \in \mathbb{T} \times [100, 64 \times 10^4]; \\ 102 + cost + (64 \times 10^4 - 100)^4, & (t, x) \in \mathbb{T} \times [64 \times 10^4, \infty). \end{cases}$$

We notice that $a = -5,\ b = 5,\ \sigma(a) = -4,\ \rho(b) = 4$ and $p(t) = t^2 + 5$ is symmetric on $[-5, 5]$. It is obvious $f : \mathbb{T} \times [0, +\infty] \to [0, +\infty]$ is ld-continuous and symmetric on $\mathbb{T}$. Let $\delta = \dfrac{1}{4} \in (0, \dfrac{1}{2})$, then $q_1^* = \min\{t \in \mathbb{T} : -4 + \delta \leqslant t\} = -3$ and $q_2^* = \max\{t \in \mathbb{T} : t \leqslant 4 - \delta\} = 3$. Then by calculations, we can obtain that $\Lambda = \frac{1}{640}, M_* \cong 0{,}4$ and $M^* \cong 5{,}4$. If we choose $u = 100$, $y = 10^6$ and $z = 10^{16}$, then we have $f(t, x(t)) < \phi\left(\frac{z}{M^*}\right) = \phi\left(\frac{10^{16}}{5{,}4}\right) = 12 \times 10^{29}$ for $(t, x) \in \mathbb{T} \times [0, 10^{16}]$, $f(t, x(t)) > \phi\left(\frac{y}{\epsilon M_*}\right) = \phi\left(\frac{640\times10^6}{0{,}4}\right) = 256 \times 10^{16}$ for $(t, x) \in [-4, 4]_{\mathbb{T}} \times [10^6, 640 \times 10^6]$, and $f(t, x(t)) < \phi\left(\frac{u}{M^*}\right) = \phi\left(\frac{100}{5{,}4}\right) = 120$ for $(t, x) \in [-4, 4]_{\mathbb{T}} \times [\frac{10}{64}, 100]$. By

Theorem 3.1 the FBVP (3.1)-(3.2) has at least three positive symmetric solutions x_1, x_2 and x_3 such that

$$x_1(0) < 100 < x_3(0) \quad \text{and}\, x_3(0) < 10^6 < x_2(-3).$$

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