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RESTRAINED HUB NUMBER IN GRAPHS

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Abstract. In this paper, we initiate the study of a variation of standard hub, we called restrained hub. Let G be a graph, a hub set H_r of G is a restrained hub set of G if for any two vertices $u, v \in V(G) \setminus H_r$, there is a path between them with all intermediate vertices in $V(G) \setminus H_r$, the minimum cardinality of H_r in G is called a restrained hub number and denoted by $h_r(G)$. In this article we determine the restrained hub number of some standard graphs. Also, we obtain bounds for $h_r(G)$. In addition we characterize the class of all (p, q) graphs for which $h_r(G) = p - 2$.

1. Introduction

Let $G = (V, E)$ be a graph such that G is a finite and undirected graph without loops and multiple edges. A graph G with p vertices and q edges is called a (p, q) graph, the number p is referred to as the order of a graph G and q is referred to as the size of a graph G. In general, the degree of a vertex v in a graph G, denoted by $deg(v)$ is the number of edges of G incident with v. Also $\delta(G), \Delta(G)$ denote the minimum, maximum degree among the vertices of G, respectively [**5**]. The set of all end vertices in a graph G is denoted by E_n and it's cardinality is denoted by e_n. Given any vertex $v \in V(G)$, the graph obtained from G by removing the vertex v and all of its incident edges is denoted by $G - v$. In a tree, a leaf is a vertex of degree one. See [**5**] for terminology and notations not defined here.

Introduced by Walsh [**15**], a hub set in a graph G is a set H of vertices in G such that any two vertices outside H are connected by a path whose all internal vertices lie in H. The hub number of G, denoted by $h(G)$, is the minimum size of a hub set in G. A connected hub set in G is a vertex set F such that F is hub set and the subgraph of G induced by F (*denoted* $G[F]$) is connected. The connected hub number of G, denoted $h_c(G)$, is the minimum size of a connected hub set in G

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[15]. For more details on the hub studies we refer to [8, 10, 11, 12]. For graphs G_1 and G_2 having disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 respectively, their union, $G(V,E) = G_1 \cup G_2$ has, as expected, $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$. Their join is denoted by $G_1 + G_2$ and consists of $G_1 \cup G_2$ and all edges joining V_1 with V_2 [5].

A set D of vertices in a graph G is called dominating set of G if every vertex in $V(G) \setminus D$ is adjacent to some vertex in D, the minimum cardinality of a dominating set in G is called the domination number $\gamma(G)$ of a graph G, the dominating set D_c is connected if $G[D_c]$ is connected then the minimum cardinality of a connected dominating set is called connected domination number and denoted by $\gamma_c(G)$ [6]. A set $S \subseteq V(G)$ is a restrained dominating set if every vertex in $V(G) \setminus S$ is adjacent to a vertex in S and another vertex in $V(G) \setminus S$, the minimum cardinality of a restrained dominating set in G is called the restrained domination number $\gamma_r(G)$ of a graph G [3]. Let v be a vertex in G, define the contraction of v in G (denoted by G/v) as being the graph obtained by deleting v and putting a clique on the (open) neighbourhood of v. (Note that this operation does not create parallel edges, if two neighbours of v are already adjacent, then they remain simply adjacent). We call this operation vertex contraction by analogy with edge contraction [15].

A double star $S_{n,m}$ is the tree obtained from two disjoint stars $K_{1,n-1}$ and $K_{1,m-1}$ by connecting their centers [4]. The chromatic number $\chi(G)$ of a graph G is the minimum number of colors required to assign to the vertices of G in such a way that no two adjacent vertices of G receive the same color [5]. The following results will be useful in the proof of our results.

Theorem 1.1 ([15]). *Let S be a subset of $V(G)$. Then G/S is complete if and only if S is a hub set of G.*

Proposition 1.1 ([8]). *For any connected graph G,*

$$h(G) \leqslant h_c(G) \leqslant \gamma_c(G) \leqslant h(G) + 1.$$

Theorem 1.2 ([9]). *For any connected graph G,*

$$\gamma_c(G) \leqslant 2q - p.$$

Theorem 1.3 ([14]). *For any graph G,*

$$\lceil \frac{p}{\Delta(G) + 1} \rceil \leqslant \gamma(G),$$

where $\lceil x \rceil$ is a least integer not less than x.

2. Definition and values of some standard graphs

Definition 2.1. Let G be a graph, a restrained hub set H_r of G is a subset $H_r \subseteq V(G)$ with the property for any two vertices $u, v \in V(G) \setminus H_r$, there is a path between them with all intermediate vertices in H_r as well as another path with all intermediate vertices in $V(G) \setminus H_r$, the minimum cardinality of H_r in G is called a restrained hub number of G and denoted by $h_r(G)$.

It is obvious that $h_r(G) \geqslant h(G)$, since every restrained hub set is a hub set. If G is a disconnected graph then any restrained hub set must contain all of the vertices in all but one of the components, as well as a restrained hub set in the remaining component.

It is clear that $h_r(G)$ is well-defined for any graph G, since $V(G)$ is a restrained hub set.

A restrained hub set $H_r \subseteq V(G)$ is called a connected restrained hub set, if the subgraph $G[H_r]$ is connected. The minimum cardinality of a connected restrained hub set of G is called a connected restrained hub number and is denoted by $h_{rc}(G)$. The connected restrained hub number is well-defined for any connected graph G, since $V(G)$ is a connected restrained hub set. It is obvious that $h_r(G) \leqslant h_{rc}(G)$, since every connected restrained hub set is a restrained hub set.

In the following proposition we determine the restrained hub number and connected restrained hub number of some standard graphs.

PROPOSITION 2.1. *The following hold*

(1) *For any complete graph* K_p,

$$h_r(K_p) = h_{rc}(K_p) = 0.$$

(2) *For any cycle* C_p,

$$h_r(C_p) = h_{rc}(C_p) = p - 3.$$

(3) *For any path* P_p *with* $p \geqslant 3$,

$$h_r(P_p) = h_{rc}(P_p) = p - 2.$$

(4) *For the wheel graph* $W_{1,p-1}$,

$$h_r(W_{1,p-1}) = h_{rc}(W_{1,p-1}) = 1.$$

(5) *For the star* $K_{1,p-1}$, $p \geqslant 4$,

$$h_r(K_{1,p-1}) = p - 2, \text{ and } h_{rc}(K_{1,p-1}) = p - 1.$$

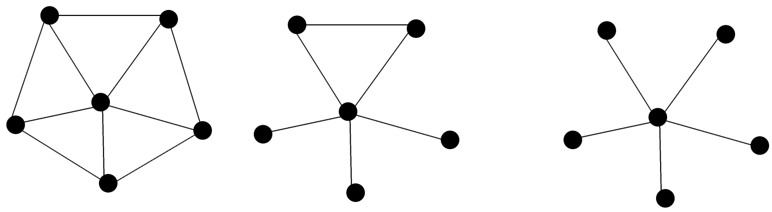
(6) *For the double star* $S_{n,m}$,

$$h_r(S_{n,m}) = p - 2, \text{ and } h_{rc}(S_{n,m}) = p - 1.$$

(7) *For the complete bipartite graph* $K_{n,m}$, $m, n \geqslant 3$,

$$h_r(K_{n,m}) = h_{rc}(K_{n,m}) = 2.$$

Now we will check if the restrained hub parameters are suitable measures of stability?. We ask, do the restrained hub parameters discriminate between graphs?. There are many examples of graphs which prove that $h_r(G)$ and $h_{rc}(G)$ are suitable measures of stability which are able to discriminate between graphs. For example, consider the graphs G_1, G_2 and G_3 in Figure 1.



G_1 G_2 G_3

Figure 1

From Figure 1, it is clear that $h(G_1) = h(G_2) = h(G_3) = 1$, also $\gamma(G_1) = \gamma(G_2) = \gamma(G_3) = 1$, then the hub number and the domination number do not discriminate between graphs G_1, G_2 and G_3, but $h_r(G_1) = 1$, $h_r(G_2) = 3$, $h_r(G_3) = 4$, so $h_r(G_1) \neq h_r(G_2) \neq h_r(G_3)$, and $h_{rc}(G_1) = 1$, $h_{rc}(G_2) = 4$, $h_{rc}(G_3) = 5$, then $h_{rc}(G_1) \neq h_{rc}(G_2) \neq h_{rc}(G_3)$, hence the restrained hub parameters discriminate between graphs G_1, G_2 and G_3.

3. Some properties of restrained hub number

Proposition 3.1. *For any connected graph G, $\gamma_c(G) \leqslant h_r(G) + 1$.*

Proof. It is obvious that $h_r(G) \geqslant h(G)$, and by Proposition 1.1, the proof is done. □

Corollary 3.1. *Let G be a (p, q) graph, then $h_r(G) \geqslant \lceil \frac{p}{\Delta(G)+1} \rceil - 1$.*

Proof. By Theorem 1.3, and the previous Proposition we get the result. □

If we take a graph $G \cong K_{1,p-1}$, then $h_r(G) - h(G) = 1$ when $p = 4$, and $h_r(G) - h(G) = k$, if $p = k + 3$ for any $k \geqslant 1$. Then we have

Proposition 3.2. *There exists a graph G for which $h_r(G) - h(G)$ can be made arbitrarily large.*

Remark 3.1. Let F be a subgraph of G, then need not hold true $h_r(F) \leqslant h_r(G)$. For example, $G \cong W_{1,p-1}$, and $F \cong K_{1,p-1}$, $h_r(G) = 1$ and $h_r(F) = p - 2$, then $h_r(F) \nleqslant h_r(G)$.

Lemma 3.1. *For any nontrivial graph G, $h_r(G) \leqslant p - 2$.*

Proof. Let G be any nontrivial graph, so we can fined two vertices u, v such that v is adjacent to u, then clearly that $V(G) \smallsetminus \{u, v\}$ is a restrained hub set of a graph G. □

Theorem 3.1. *Let G be a graph, the subset A of $V(G)$ is a restrained hub set of G if and only if G/A is a complete and $G - A$ is connected.*

Proof. Suppose that A is a restrained hub set of G, then A is a hub set of G, so G/A is complete by Theorem1.1. Now let u, v be any two vertices in $G - A$, since A is a restrained hub set, then there exist a path between u and v with all internal vertices in $V(G) \setminus A$. Therefore by the definition of a connected graph we get that $[G - A]$ is connected. Conversely, since G/A is complete, then by Theorem1.1, A is a hub set of G, now let $u, v \in G - A$, since $G - A$ is connected then there is $u - v$ path with all internal vertices in $G - A$. Hence A is a restrained hub set of G. □

THEOREM 3.2. *For any graph G, $\gamma_r(G) \leqslant h_r(G) + 1$.*

PROOF. Let H_r be a restrained hub set in G, and let Q be the set of vertices which are not adjacent to anything in H_r (nor in H_r themselves). By the definition of H_r the only H_r-paths between the vertices in Q must be trivial, then $G[Q]$ is complete and every vertex in Q is adjacent to every vertex in $V(G) \setminus (H_r \cup Q)$. Let $B = V(G) \setminus (H_r \cup Q)$, by the previous Theorem $G[B \cup Q]$ is connected. Now we discuss the following cases:

Case 1: B and Q are nonempty. Let b be any vertex in B, since b is adjacent to all vertices in Q, and by definition of B its a neighborhood of H_r, so $H_r \cup \{b\}$ is a dominating set of G, and by definition of Q, any vertex $q \in Q$ is adjacent to all vertices in $V(G) \setminus (V(H_r) \cup \{b\})$. Therefore $H_r \cup \{b\}$ is a restrained dominating set of G.

Case 2: B is nonempty and Q is empty. Since B is connected and B its a neighborhood of H_r, then H_r is a restrained dominating set of G.

Case 3: B is empty. If Q is empty, then $H_r = V(G)$. If Q is not empty, then $H_r \cup \{q\}$ is a restrained dominating set of G for any $q \in Q$, since $G[Q]$ is complete and $G[Q \setminus \{q\}]$ is connected. From all of the above cases we get that $\gamma_r(G) \leqslant h_r(G) + 1$. □

PROPOSITION 3.3. *For any graph G, if v is an end vertex and H_r is a restrained hub set, then either $v \in H_r$ or $|H_r| = p - 2$.*

PROOF. Suppose that v is an end vertex in a graph G and H_r is a restrained hub set. Let $w \in V(G)$ such that $vw \in E$, clearly w is a cut vertex, if $w \in H_r$ then there is a vertex u such that w is any $uv-$path, so $v \in H_r$ or $A = V(G) \setminus \{v, w\} \subseteq H_r$, therefore either $v \in H_r$ or $|H_r| \geqslant p - 2$, since $|H_r| \leqslant p - 2$ we get $|H_r| = p - 2$. □

COROLLARY 3.2. *Let G be a graph, H_r is a restrained hub set, then any H_r contains all end vertices or all of end vertices except one.*

REMARK 3.2. $h_r(G) \geqslant e_n - 1$, and it is sharp on the star.

THEOREM 3.3. *Let G be a graph with at least one end vertex, $h_r(G) = p - 2$ if and only if there exists minimum restrained hub set not containing an end vertex.*

PROOF. Suppose that there exists minimum restrained hub set not containing an end vertex, then by Proposition 2.4, $h_r(G) = p-2$. Conversely let $H_r(G) = p-2$, and let u be an end vertex, take w such that $uw \in E$, then clearly $V(G) \setminus \{u, w\}$ is a restrained hub set, and it has $p - 2$ vertices. Since $h_r(G) = p - 2$, $V(G) \setminus \{u, w\}$ is a minimum restrained hub set not containing end vertex. □

OBSERVATION 3.1. *If G has at least two internal vertices, then there exists minimum H_r such that it contains all end vertices.*

THEOREM 3.4. *Let G be a graph with at least two internal vertices, then $h_r(G) = e_n$ if and only if $G - E_n$ is complete graph.*

PROOF. Suppose that $G - E_n$ is complete graph, clearly E_n is a restrained hub set by Theorem 2.2, then clearly by Observation 2.1, E_n is minimum restrained hub set. Conversely let $h_r(G) = e_n$, since $h_r(G) = e_n$, H_r does not contain any vertex from $G - E_n$, suppose that $G - E_n$ is not complete graph, let $u, v \in G - E_n$ such that $uv \notin E$, since H_r is a set of all end vertices, then there is no $uv-$path with all internal vertices in H_r , which is contradiction, therefore $G - E_n$ is complete graph. □

PROPOSITION 3.4. *If a graph G has two vertices with degree $p-1$ and $G \neq K_p$, then $h(G) = h_c(G) = h_r(G) = \gamma(G) = \gamma_c(G) = \gamma_r(G) = 1$.*

PROOF. Suppose that u, v are the two vertices with degree $p - 1$, let $A = \{v\}$, clearly G/A is complete, and as $deg(u) = p - 1$, then clearly $G - A$ is connected, hence A is a restrained hub set, so $h(G) = h_c(G) = h_r(G) = 1$. Since v is adjacent to every vertex in G we get $\gamma(G) = \gamma_c(G) = 1$, and since u is also adjacent to every vertex in G and $u \notin A$, then A is a restrained dominating set, so $\gamma_r(G) = 1$. □

THEOREM 3.5. *Let G, F be any two non trivial graphs, with orders p_1, p_2, respectively. Then*

$$h_r(G+F) = \begin{cases} 0, & \text{if } G, F \text{ are complete ;} \\ 1, & \text{if } \{\Delta(G) = p_1 - 1, \text{ or } \Delta(F) = p_2 - 1\} \text{ and} \\ & \qquad G \neq K_{p_1} \text{ or } F \neq K_{p_2}; \\ 2, & \text{otherwise.} \end{cases}$$

PROOF. Let G, F be non trivial graphs, with orders p_1, p_2, respectively. We have the following cases:

Case 1: G, F are complete graphs, then $G + F = K_{p_1+p_2}$, so $h_r(G + F) = 0$.

Case 2: Without loss of generality let $v \in G$, and $deg(v) = p_1 - 1$. In $G + F$ $deg(v) = p_1 + p_2 - 1$, therefore $(G + F)/v$ is complete and $(G + F) - v$ is connected, so $\{v\}$ is a restrained hub set of $(G + F)$ and it is minimum since at least G or F is not complete, then $h_r(G + F) = 1$.

Case 3: Let v be any vertex in G or F with $deg(v) \leqslant p_1 - 2$ or $deg(v) \leqslant p_2 - 2$. Then in $G + F$ $deg(v) \leqslant p_1 + p_2 - 2$, hence $(G + F)/v$ is not complete, then $h_r(G + F) \geqslant 2$. By Proposition 2.1 we get $h_r(G + F) = 2$. □

COROLLARY 3.3. *Let G, F be any two non trivial graphs, then $h_r(G + F) = h(G + F) = h_c(G + F)$.*

PROOF. Let H be any minimum hub set of $G + F$, then $(G + F) - H$ is connected, therefore H is a restrained hub set, thus we get the result. □

THEOREM 3.6. *Let G ba a (p, q) graph, then $h_r(G) \leqslant 2q - \gamma_c(G) - 2$.*

PROOF. By Theorem 1.4, we get $p - 2 \leqslant 2q - \gamma_c(G) - 2$, and by Lemma 2.1, we get the result. □

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