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# ON A SPECTRAL IDENTITY AND COEFFICIENTS ASSOCIATED WITH GEGENBAUER POLYNOMIALS $C_k^\nu$ $(k \geqslant 0,\ \nu > -1/2)$

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Abstract. The Gegenbauer coefficients $c_j^\ell(\nu)$ $(1 \leqslant j \leqslant \ell,\ \nu > -1/2)$ describe the Maclaurin expansion of the heat kernels on the $n$-dimensional unit sphere $\mathbb{S}^n$ $(n \geqslant 1)$ and the $n$-dimensional real projective space $\mathbf{P}^n(\mathbb{R})$ $(n \geqslant 1)$. These coefficients are associated with an interesting spectral identity on Gegenbauer polynomials $C_k^\nu$ $(k \geqslant 0, \nu > -1/2)$. In this note we give another proof of this identity and compute the higher coefficients $c_j^\ell(\nu)$ $(5 \leqslant j \leqslant \ell \leqslant 6,\ \nu > -1/2)$.

## 1. Introduction

The Gegenbauer polynomials $C_k^\nu$ $(k \geqslant 0, \nu > -1/2)$ are natural and key ingredients in the analysis of the Laplacians on $\mathbb{S}^n$ and $\mathbf{P}^n(\mathbb{R})$, most especially, in the representations of the heat kernels $K_{\mathbb{S}^n}(t,\theta)$ and $K_{\mathbf{P}^n(\mathbb{R})}(t,\theta)$. It is well-known that the spherical functions (orthonormalised eigenfunctions) of the Laplacians on these symmetric spaces are given in terms of the Gegenbauer polynomials, and recently, it has been observed that the Gegenbauer coefficients $c_j^\ell(\nu)$ $(1 \leqslant j \leqslant \ell,\ \nu > -1/2)$ describe the Maclaurin heat coefficients $b_{2\ell}^n(t)$ $(\ell \geqslant 0, t > 0)$ (i.e., the coefficients associated with the Maclaurin expansion of the heat kernels on these spaces) (see, e.g., [**1, 2**]). In this note, we give another proof of a spectral identity associated with these coefficients, and compute the higher coefficients $c_j^\ell(\nu)$ for $(5 \leqslant j \leqslant \ell \leqslant 6$, $\nu > -1/2)$; a proof of this identity as well as the computation of the coefficients $c_j^\ell(\nu)$ $(1 \leqslant j \leqslant \ell \leqslant 4)$ has been given in [**1**].

For this reason we outline some of the key properties and features of the Gegenbauer polynomials needed. (For more on the Gegenbauer polynomials the interested

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reader is referred to [**3**], [**4**] and [**5**].) The Gegenbauer or ultraspherical polynomial $C_k^\nu(t)$ $(k \geqslant 0, \nu > -1/2)$ is a natural generalisation of the Legendre polynomial $P_k(t)$ (coincides when $\nu = 1/2$) and is defined by the coefficient of $\alpha^k$ in the generating function

$$(1.1)\qquad \left(1 - 2t\alpha + \alpha^2\right)^{-\nu} = \sum_{k=0}^{\infty} C_k^\nu(t)\alpha^k.$$

For $\nu > -1/2$ the Gegenbauer polynomial $C_k^\nu(t)$ has a nice truncated series representation resulting from the series solution to the Gegenbauer differential equation (*see* below) in the form

$$(1.2)\qquad C_k^\nu(t) = \sum_{0 \leqslant l \leqslant \frac{k}{2}} (-1)^l \frac{\Gamma(k - l + \nu)}{\Gamma(\nu) l!(k - 2l)!} (2t)^{k-2l},$$

with the derivatives satisfying the recursive relation

$$(1.3)\qquad \frac{d^m}{dt^m} C_k^\nu(t) = 2^m \frac{\Gamma(\nu + m)}{\Gamma(\nu)} C_{k-m}^{\nu+m}(t).$$

The Gegenbauer polynomial $y = C_k^\nu(t)$ satisfies the second-order homogenous differential equation

$$(1.4)\qquad \left(1 - t^2\right) \frac{d^2 y}{dt^2} - (2\nu + 1)t \frac{dy}{dt} + k(k + 2\nu)y = 0.$$

The pair form a so-called *regular* Sturm-Liouville system with the corresponding Gegenbauer operator a positive selfadjoint second order differential operator in $L^2[-1, 1; (1 - t^2)^{\nu-1/2} dt]$ having the discrete spectrum $(\lambda_k^\nu = k(k + 2\nu) : k \geqslant 0)$ and associated eigenfunctions $y = C_k^\nu(t)$. In particular we have the orthogonality relations

$$\begin{aligned}(C_k^\nu, C_m^\nu)_{L^2[-1,1;(1-t^2)^{\nu-1/2} dt]} &= \int_{-1}^{1} C_k^\nu(t)\, C_m^\nu(t) \left(1 - t^2\right)^{\nu - \frac{1}{2}} dt \\ (1.5)\qquad &= \frac{\pi 2^{1-2\nu} \Gamma(2\nu + m)}{m!(m + \nu)\Gamma(\nu)^2} \delta_{km}, \qquad k, m \geqslant 0,\end{aligned}$$

where $\delta_{km}$ is the usual Kronecker delta, that is, $\delta_{km} = 0$ when $k \neq m$ and $\delta_{km} = 1$ when $k = m$. The Gegenbauer polynomial can be expressed by the so-called Rodrigues' formula

$$(1.6)\qquad C_k^\nu(t) = \frac{(-1)^k \Gamma(\frac{1+2\nu}{2}) \Gamma(k + 2\nu) \left(1 - t^2\right)^{\frac{1}{2} - \nu}}{2^k k! \Gamma(2\nu) \Gamma\left(\frac{2\nu+1}{2} + k\right)} \frac{d^k}{dt^k} \left(1 - t^2\right)^{k + \nu - \frac{1}{2}},$$

and satisfies the pointwise value identities

$$(1.7)\qquad C_k^\nu(1) = \frac{(2\nu)_k}{k!}, \qquad C_k^\nu(-t) = (-1)^k C_k^\nu(t),$$

where $(x)_k = \Gamma(x + k)/\Gamma(x)$. In particular

$$(1.8)\qquad C_k^\nu(1) = \frac{\Gamma(k + 2\nu)}{\Gamma(2\nu) k!}.$$

## 2. Gegenbauer Coefficient and its Spectral Implications

In this section we show that the recursion formula (1.3) can be written as a spectral sum involving the powers of the eigenvalues ($\lambda_k^\nu = k(k+2\nu) : k \geqslant 0, \nu > -1/2$) of the Gegenbauer operator (see (1.4)), with nonzero coefficients ($c_j^\ell(\nu) : 1 \leqslant j \leqslant \ell, \nu > -1/2$).

THEOREM 2.1 (Spectral identity). *For the Gegenbauer polynomial $y = C_k^\nu(t)$ (with $k \geqslant 0, \nu > -1/2$) and any integer $\ell \geqslant 1$, the following relation holds:*

$$\frac{d^{2\ell}}{d\theta^{2\ell}} C_k^\nu(\cos\theta)\bigg|_{\theta=0} = C_k^\nu(1) \sum_{j=1}^{\ell} c_j^\ell(\nu)\,[k(k+2\nu)]^j\,. \tag{2.1}$$

*Here the (Gegenbauer) coefficients ($c_j^\ell(\nu) : 1 \leqslant j \leqslant \ell, \nu > -1/2$) depend on $\nu$ while ($k(k+2\nu) : k \geqslant 0$) are the eigenvalues of the Gegenbauer operator as in* (1.4).

PROOF. From the recursion formula (1.3) we have

$$\begin{aligned} \frac{d^{2\ell}}{d\theta^{2\ell}} C_k^\nu(\cos\theta)\bigg|_{\theta=0} &= \sum_{j=1}^{\ell} a_j^\ell 2^j \frac{\Gamma(\nu+j)}{\Gamma(\nu)} C_{k-j}^{\nu+j}(1) \\ &= \sum_{j=1}^{\ell} a_j^\ell \frac{\Gamma(\nu+j)}{\Gamma(\nu)} \frac{2^j \Gamma(k+2\nu+j)}{\Gamma(2\nu+2j)(k-j)!}, \end{aligned} \tag{2.2}$$

where ($c_j^\ell(\nu) : 1 \leqslant j \leqslant \ell, \nu > -1/2$) are integer coefficients. Then from (2.2) we have

$$\begin{aligned} \frac{d^{2\ell}}{d\theta^{2\ell}} C_k^\nu(\cos\theta)\bigg|_{\theta=0} &= \sum_{j=1}^{\ell} a_j^\ell 2^j \frac{(\nu)_j (k+2\nu)_j}{(2\nu)_{2j}} \frac{k!}{(k-j)!} C_k^\nu(1) \\ &= \sum_{j=1}^{\ell} a_j^\ell 2^j \frac{\nu(\nu+1)\cdots(\nu+j-1)(k+2\nu)(k+2\nu+1)\cdots}{2\nu(2\nu+1)(2\nu+2)\cdots(2\nu+2j-2)(2\nu+2j-1)} \times \\ &\quad \times \cdots (k+2\nu+j-1)k(k-1)(k-2)\cdots(k-j+1)C_k^\nu(1) \\ &= \sum_{j=1}^{\ell} a_j^\ell \frac{(k+2\nu+1)\cdots(k+2\nu+j-1)}{(2\nu+1)(2\nu+3)\cdots(2\nu+2j-1)} \times \\ &\quad \times (k-1)(k-2)\cdots(k-j+1)k(k+2\nu)C_k^\nu(1). \end{aligned}$$

Hence,

$$\frac{d^{2\ell}}{d\theta^{2\ell}} C_k^\nu(\cos\theta)\bigg|_{\theta=0} = \sum_{j=1}^{\ell} a_j^\ell \frac{(k+2\nu+1)_{j-1}(k-j+1)_{j-1}}{(2\nu+1)(2\nu+3)\cdots(2\nu+2j-1)} k(k+2\nu)C_k^\nu(1).$$

Turning to the coefficients $\left(a_j^\ell : 1 \leqslant j \leqslant \ell\right)$ we note that (1.3) (with $m=1$) gives

$$\frac{d^2}{d\theta^2} y(\cos\theta) = y''(\cos\theta)\sin^2\theta - y'(\cos\theta)\cos\theta, \tag{2.3}$$

where we have set $y(t) = C_k^\nu(t)$. Then $\left(a_j^\ell : 1 \leqslant j \leqslant \ell\right)$ are the coefficients of the derivatives $y^{(j)}(1)$ $(j \geqslant 1)$ in the formula

(2.4)
$$\frac{d^{2q}}{d\theta^{2q}}\left(\frac{d^2}{d\theta^2} y(\cos\theta)\right)\Big|_{\theta=0} = \frac{d^{2q}}{d\theta^{2q}}\left(y''(\cos\theta)\sin^2\theta - y'(\cos\theta)\cos\theta\right)\Big|_{\theta=0} \qquad q \geqslant 0.$$

□

REMARK 2.1. We can also obtain the derivatives $y^{(j)}(1)$ $(j \geqslant 1)$ by repeated differentiations of the Gegenbauer differential equation (1.4).

## 3. The Higher Gegenbauer Coefficients $\left(c_j^\ell : 5 \leqslant j \leqslant \ell \leqslant 6\right)$

We now present in detail the explicit computation of the Gegenbauer coefficients $\left(c_j^\ell : 5 \leqslant j \leqslant \ell \leqslant 6\right)$. Here also we set $y(t) = C_k^\nu(t)$.

$(\ell = 5)$ Indeed from (2.2) we have

$$\begin{aligned}
\frac{d^{10}}{d\theta^{10}} y(\cos\theta)\Big|_{\theta=0} &= a_1^5 y'(1) + a_2^5 y''(1) + a_3^5 y'''(1) + a_4^5 y^{(4)}(1) + a_5^5 y^{(5)}(1) = \\
&= a_1^5 \frac{\Gamma(\nu+1)}{\Gamma(\nu)} \frac{2\Gamma(k+2\nu+1)}{\Gamma(2\nu+2)(k-1)!} + a_2^5 \frac{\Gamma(\nu+2)}{\Gamma(\nu)} \frac{4\Gamma(k+2\nu+2)}{\Gamma(2\nu+4)(k-2)!} + \\
&+ a_3^5 \frac{\Gamma(\nu+3)}{\Gamma(\nu)} \frac{8\Gamma(k+2\nu+3)}{\Gamma(2\nu+6)(k-3)!} + a_4^5 \frac{\Gamma(\nu+4)}{\Gamma(\nu)} \frac{16\Gamma(k+2\nu+4)}{\Gamma(2\nu+8)(k-4)!} + \\
&+ a_5^5 \frac{\Gamma(\nu+5)}{\Gamma(\nu)} \frac{32\Gamma(k+2\nu+5)}{\Gamma(2\nu+10)(k-5)!},
\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}
y^{(5)}(1) =& \frac{\Gamma(\nu+5)}{\Gamma(\nu)} \frac{32\Gamma(k+2\nu+5)}{\Gamma(2\nu+10)(k-5)!} \\
=& \frac{(k+2\nu)(k+2\nu+1)(k+2\nu+2)(k+2\nu+3)(k+2\nu+4)\Gamma(k+2\nu)}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)(2\nu+9)\Gamma(2\nu)(k-5)!} \\
=& \frac{(k-1)(k-2)(k-3)(k-4)(k+2\nu+1)(k+2\nu+2)}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)(2\nu+9)} \times \\
& \times (k+2\nu+3)(k+2\nu+4)k(k+2\nu)y(1).
\end{aligned} \tag{3.2}$$

Simplifying further we have

$$y^{(5)}(1) = \Bigg\{ \frac{[k(k+2\nu)]^5}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)(2\nu+9)} - \frac{10(2\nu+3)[k(k+2\nu)]^4}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)(2\nu+9)} + \frac{(140\nu^2+400\nu+273)[k(k+2\nu)]^3}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)(2\nu+9)} - \frac{\left(400\nu^3+1616\nu^2+2060\nu+820\right)[k(k+2\nu)]^2}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)(2\nu+9)} + \frac{96(\nu+1)(\nu+2)(2\nu+1)(2\nu+3)k(k+2\nu)}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)(2\nu+9)} \Bigg\} y(1). \tag{3.3}$$

It now follows that

$$\left.\frac{d^{10}}{d\theta^{10}} y(\cos\theta)\right|_{\theta=0} = \left\{ c_1^5(\nu)k(k+2\nu) + c_2^5(\nu)[k(k+2\nu)]^2 + c_3^5(\nu)[k(k+2\nu)]^3 + c_4^5(\nu)[k(k+2\nu)]^4 + c_5^5(\nu)[k(k+2\nu)]^5 \right\} y(1), \tag{3.4}$$

where

$$c_1^5(\nu) = \frac{a_1^5(2\nu+3)(2\nu+5)(2\nu+7)(2\nu+9)}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)(2\nu+9)} - \frac{a_2^5(2\nu+1)(2\nu+5)(2\nu+7)(2\nu+9)}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)(2\nu+9)} + \frac{a_3^5 4(\nu+1)(2\nu+1)(2\nu+7)(2\nu+9)}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)(2\nu+9)} - \frac{a_4^5 12(\nu+1)(2\nu+1)(2\nu+3)(2\nu+9)}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)(2\nu+9)} + \frac{a_5^5 96(\nu+1)(\nu+2)(2\nu+1)(2\nu+3)}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)(2\nu+9)},$$

$$c_2^5(\nu) = \frac{a_2^5(2\nu+5)(2\nu+7)(2\nu+9) - a_3^5(6\nu+5)(2\nu+7)(2\nu+9)}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)(2\nu+9)} + \frac{a_4^5\left(44\nu^2+96\nu+49\right)(2\nu+9) - a_5^5\left(400\nu^3+1616\nu^2+2060\nu+820\right)}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)(2\nu+9)}$$

$$c_3^5(\nu) = \frac{a_3^5(2\nu+7)(2\nu+9) - a_4^5(12\nu+14)(2\nu+9) + a_5^5(140\nu^2+400\nu+273)}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)(2\nu+9)},$$

$$c_4^5(\nu) = \frac{a_4^5(2\nu+9) - a_5^5 10(2\nu+3)}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)(2\nu+9)},$$

$$c_5^5(\nu) = \frac{a_5^5}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)(2\nu+9)}.$$

Clearly here we see that

$$a_1^5 = -1,\ a_2^5 = -255,\ a_3^5 = -2205,\ a_4^5 = -3150,\ a_5^5 = -945$$

and subsequently we obtain

$$c_1^5(\nu) = -\frac{126976\nu^4 + 172032\nu^3 + 88064\nu^2 + 16128\nu}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)(2\nu+9)}, \tag{3.5}$$

$$c_2^5(\nu) = \frac{151680\nu^3 + 120960\nu^2 + 175620\nu}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)(2\nu+9)}, \tag{3.6}$$

$$c_3^5(\nu) = -\frac{65520\nu^2 + 20160\nu}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)(2\nu+9)}, \tag{3.7}$$

$$c_4^5 = \frac{12600\nu}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)(2\nu+9)}, \tag{3.8}$$

$$c_5^5 = -\frac{945}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)(2\nu+9)}. \tag{3.9}$$

($\ell = 6$) We now consider the lengthier case of $\ell = 6$. Indeed from (2.2) we have

$$\frac{d^{12}}{d\theta^{12}} y(\cos\theta)\Big|_{\theta=0} \tag{3.10}$$

$$\begin{aligned}
&= a_1^6 y'(1) + a_2^6 y''(1) + a_3^6 y'''(1) + a_4^6 y^{(4)}(1) + a_5^6 y^{(5)}(1) + a_6^6 y^{(6)}(1) \\
&= a_1^6 \frac{\Gamma(\nu+1)}{\Gamma(\nu)} \frac{2\Gamma(k+2\nu+1)}{\Gamma(2\nu+2)(k-1)!} + a_2^6 \frac{\Gamma(\nu+2)}{\Gamma(\nu)} \frac{4\Gamma(k+2\nu+2)}{\Gamma(2\nu+4)(k-2)!} + \\
&\quad + a_3^6 \frac{\Gamma(\nu+3)}{\Gamma(\nu)} \frac{8\Gamma(k+2\nu+3)}{\Gamma(2\nu+6)(k-3)!} + a_4^6 \frac{\Gamma(\nu+4)}{\Gamma(\nu)} \frac{16\Gamma(k+2\nu+4)}{\Gamma(2\nu+8)(k-4)!} + \\
&\quad + a_5^6 \frac{\Gamma(\nu+5)}{\Gamma(\nu)} \frac{32\Gamma(k+2\nu+5)}{\Gamma(2\nu+10)(k-5)!} + a_6^6 \frac{\Gamma(\nu+6)}{\Gamma(\nu)} \frac{64\Gamma(k+2\nu+6)}{\Gamma(2\nu+12)(k-6)!},
\end{aligned} \tag{3.11}$$

where

$$
\begin{aligned}
y^{(6)}(1) =& \frac{\Gamma(\nu+6)}{\Gamma(\nu)} \frac{64\Gamma(k+2\nu+6)}{\Gamma(2\nu+12)(k-6)!} \\
=& \frac{(k+2\nu)(k+2\nu+1)(k+2\nu+2)(k+2\nu+3)(k+2\nu+4)}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)(2\nu+9)(2\nu+11)\Gamma(2\nu)(k-6)!} \times \\
& \times (k+2\nu+5)\Gamma(k+2\nu) \\
=& \frac{(k-1)(k-2)(k-3)(k-4)(k-5)(k+2\nu+1)(k+2\nu+2)}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)(2\nu+9)(2\nu+11)} \times \\
& \times \frac{(k+2\nu+3)(k+2\nu+4)(k+2\nu+5)k(k+2\nu)}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)(2\nu+9)(2\nu+11)} y(1).
\end{aligned} \tag{3.12}
$$

Simplifying further we have

$$
\begin{aligned}
y^{(6)}(1) = \Bigg\{ & \frac{[k(k+2\nu)]^6}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)(2\nu+9)(2\nu+11)} - \\
& - \frac{5(6\nu+11)[k(k+2\nu)]^5}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)(2\nu+9)(2\nu+11)} + \\
& + \frac{(340\nu^2+1200\nu+1023)[k(k+2\nu)]^4}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)(2\nu+9)(2\nu+11)} - \\
& - \frac{\left(1800\nu^3+9116\nu^2+14790\nu+7645\right)[k(k+2\nu)]^3}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)(2\nu+9)(2\nu+11)} + \\
& + \frac{\left(4384\nu^4+28080\nu^3+64360\nu^2+62100\nu+21076\right)[k(k+2\nu)]^2}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)(2\nu+9)(2\nu+11)} - \\
& - \frac{480(\nu+1)(\nu+2)(2\nu+1)(2\nu+3)(2\nu+5)k(k+2\nu)}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)(2\nu+9)(2\nu+11)} \Bigg\} y(1).
\end{aligned} \tag{3.13}
$$

As a result it follows that

$$
\begin{aligned}
\frac{d^{12}}{d\theta^{12}} y(\cos\theta)\Big|_{\theta=0} = & \left\{ c_1^6(\nu)k(k+2\nu) + c_2^6(\nu)[k(k+2\nu)]^2 + \right. \\
& + c_3^6(\nu)[k(k+2\nu)]^3 + c_4^6(\nu)[k(k+2\nu)]^4 + \\
& \left. + c_5^6(\nu)[k(k+2\nu)]^5 + c_6^6(\nu)[k(k+2\nu)]^6 \right\} y(1),
\end{aligned} \tag{3.14}
$$

where

$$c_1^6(\nu) = \frac{a_1^6(2\nu+3)(2\nu+5)(2\nu+7)(2\nu+9)(2\nu+11)}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)(2\nu+9)(2\nu+11)} -$$
$$- \frac{a_2^6(2\nu+1)(2\nu+5)(2\nu+7)(2\nu+9)(2\nu+11)}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)(2\nu+9)(2\nu+11)} +$$
$$+ \frac{a_3^6 4(\nu+1)(2\nu+1)(2\nu+7)(2\nu+9)(2\nu+11)}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)(2\nu+9)(2\nu+11)} -$$
$$- \frac{a_4^6 12(\nu+1)(2\nu+1)(2\nu+3)(2\nu+9)(2\nu+11)}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)(2\nu+9)(2\nu+11)} +$$
$$+ \frac{a_5^6 96(\nu+1)(\nu+2)(2\nu+1)(2\nu+3)(2\nu+11)}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)(2\nu+9)(2\nu+11)} -$$
$$- \frac{a_6^6 480(\nu+1)(\nu+2)(2\nu+1)(2\nu+3)(2\nu+5)}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)(2\nu+9)(2\nu+11)},$$

$$c_2^6(\nu) = \frac{a_2^6(2\nu+5)(2\nu+7)(2\nu+9)(2\nu+11)}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)(2\nu+9)(2\nu+11)} -$$
$$- \frac{a_3^6(6\nu+5)(2\nu+7)(2\nu+9)(2\nu+11)}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)(2\nu+9)(2\nu+11)} +$$
$$+ \frac{a_4^6\left(44\nu^2+96\nu+49\right)(2\nu+9)(2\nu+11)}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)(2\nu+9)(2\nu+11)} -$$
$$- \frac{a_5^6\left(400\nu^3+1616\nu^2+2060\nu+820\right)(2\nu+11)}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)(2\nu+9)(2\nu+11)} +$$
$$+ \frac{a_6^6\left(4384\nu^4+28080\nu^3+64360\nu^2+62100\nu+21076\right)}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)(2\nu+9)(2\nu+11)},$$

$$c_3^6(\nu) = \frac{a_3^6(2\nu+7)(2\nu+9)(2\nu+11) - a_4^6(12\nu+14)(2\nu+9)(2\nu+11)}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)(2\nu+9)(2\nu+11)} +$$
$$+ \frac{a_5^6(140\nu^2+400\nu+273)(2\nu+11)}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)(2\nu+9)(2\nu+11)} -$$
$$- \frac{a_6^6(1800\nu^3+9116\nu^2+14790\nu+7645)}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)(2\nu+9)(2\nu+11)},$$

$$c_4^6(\nu) = \frac{a_4^6(2\nu+9)(2\nu+11) - a_5^6 10(2\nu+3)(2\nu+11) + a_6^6(340\nu^2+1200\nu+1023)}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)(2\nu+9)(2\nu+11)},$$

$$c_5^6(\nu) = \frac{a_5^6(2\nu+11) - a_6^6(30\nu+55)}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)(2\nu+9)(2\nu+11)},$$

$$c_6^6(\nu) = \frac{a_6^6}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)(2\nu+9)(2\nu+11)}.$$

Finally a further differentiation results in $a^6_1 = 1$, $a^6_2 = 1023$, $a^6_3 = 21120$, $a^6_4 = 65835$, $a^6_5 = 51975$, $a^6_6 = 10395$ and subsequently we obtain

(3.15)
$$c^6_1(\nu) = -\frac{11321344\nu^5 + 24838144\nu^4 + 22575104\nu^3 + 9703424\nu^2 + 1629233\nu}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)(2\nu+9)(2\nu+11)},$$

$$c^6_2(\nu) = \frac{14581248\nu^4 + 22099968\nu^3 + 12613260\nu^2 + 2838528\nu}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)(2\nu+9)(2\nu+11)}, \tag{3.16}$$

$$c^6_3(\nu) = -\frac{7149120\nu^3 + 6145920\nu^2 + 1657920\nu}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)(2\nu+9)(2\nu+11)}, \tag{3.17}$$

$$c^6_4(\nu) = \frac{1718640\nu^2 + 554400\nu}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)(2\nu+9)(2\nu+11)}, \tag{3.18}$$

$$c^6_5 = -\frac{207900\nu}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)(2\nu+9)(2\nu+11)}, \tag{3.19}$$

$$c^6_6 = \frac{10395}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)(2\nu+9)(2\nu+11)}. \tag{3.20}$$

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