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COMMON FIXED POINTS FOR S-WEAKLY B-CONTRACTION MAPPINGS

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Abstract. The purpose of this paper is to prove common fixed point theorems for S-weakly B-contraction mappings in complete metric space which generalize and unify many fixed point theorems available in the literature.

1. Introduction and Preliminaries

Throughout this paper $\mathbb{R}$ and $\mathbb{N}$ represent the set of real numbers and the set of natural numbers, respectively. It is well known that Banach's contraction principle is one of the pivotal results of metric fixed point theory.

Banach contraction principle [**2**] states that if (X, d) is a complete metric space and $T : X \to X$ is a self-mapping such that

$$d(Tx, Ty) \leqslant \alpha d(x, y),$$

for all $x, y \in X$, where $0 \leqslant \alpha < 1$, then T has a unique fixed point. This theorem ensures the existence and uniqueness of fixed points of certain self-maps of metric spaces, and it gives a useful constructive method to find those fixed points. The traditional Banach contraction principle has been extended and generalized in wide directions. Al-Thagafi [**1**] proved common fixed point theorem, which generalize the results of Jungck [**8**], Dotson [**5**] and Habiniak [**6**]. Using this result Al-Thagafi [**1**] generalized and unified well-known results of Singh [**13**], Hicks and Humphries [**7**], Habiniak [**6**] on fixed points and common fixed points of best approximation. Vijayaraj et al [**14**] proved fixed point theorem for S-nonexpansive and asymptotically S-nonexpansive mapping that generalized results of Al-Thagafi [**1**].

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Bright V. S. et al [**11**] proved fixed point theorem for weakly B-contraction that unified my fixed point theorems including Banach contraction principle [**2**], Kannan fixed point theorem and Chatterjee's fixed point theorem.

In this paper we introduce S-weakly B-contraction mappings and prove common fixed point theorem for S-weakly B-contraction mappings that generalize the results of Bright V.S etal [**11**], Al-Thagafi [**1**] and Vijayaraj et al [**14**]. Also we obtain common fixed point theorem for three mappings.

The following results will be used in the sequel.

Definition 1.1. Let K be a nonempty subset of a normed linear space X, and let T be a self-mapping of K. Then K is q-starshaped if there exists a point $q \in K$ such that $(1-\lambda)q + \lambda x \in K$ for all $x \in K$.

Definition 1.2. Let K be a nonempty subset of a normed linear space X, and let T be a self-mapping of K. Then T is demicompact if whenever $\{x_n\}$ is a bounded sequence of points of K such that $x_n - T(x_n)$ converges, then $\{x_n\}$ has a convergent subsequence.

Example 1.1. Let $K = \{x = \{x_n\} \in l_2 : \sum x_n^2 < \infty \ and \ \|x\| \leqslant 1\}$, where $\|x\| = \sum x_n^2$. Define $T : K \to K$ by $T(x) = x/2$. Then T is demicompact but not compact.

It is known that every compact mapping is a demicompact mapping but the converse need not be true.

Definition 1.3. Let K be a nonempty convex subset of a normed linear space X. A self-mapping T of K is said to be affine if

$$T(\lambda x + (1-\lambda)y) = \lambda T(x) + (1-\lambda)T(y) \ for \ all \ x, y \in K \ and \ \lambda \in (0,1).$$

Definition 1.4. Let K be a nonempty convex subset of a normed linear space X. A self- mapping T of K is said to be affine with respect to q if

$$T(\lambda x + (1-\lambda)q) = \lambda T(x) + (1-\lambda)T(q) \ for \ all \ x \in K \ and \ \lambda \in (0,1).$$

The following example shows that an affine mapping with respect to a point need not be affine.

Example 1.2. Let $X = \mathbb{R}$ and $K = [0,1]$. Define T on K by

$$Tx = \begin{cases} 1 \ if \ x \in [0,1) \\ 0 \ if \ x = 1 \end{cases}$$

Then we have

$$T(\lambda x + (1-\lambda)1/2) = \begin{cases} 1 \ if \ x \in [0,1) \ and \ \lambda \in (0,1) \\ 0 \ if \ x = 1 \ and \ \lambda = 1 \end{cases}$$

If $x \in [0,1)$ and $\lambda \in (0,1)$, then $T(x) = 1 = T(1/2)$ and therefore

$$T(\lambda x + (1-\lambda)1/2) = 1 = \lambda T(x) + (1-\lambda)T(1/2).$$

If $x = 1$ and $\lambda = 1$, then

$$T(\lambda x + (1-\lambda)1/2) = 0 = \lambda T(x) + (1-\lambda)T(1/2).$$

Therefore T is affine with respect to $1/2$. If $x = 1$ and $\lambda = 1/2$, then

$$T(\lambda x + (1-\lambda)1/2) = T(3/4) = 1 \neq 1/2 = T(1) + (1-)T(1/2).$$

Hence T is not affine.

The concept of weakly compatiable maps was introduced by Jungck [**8**]

Definition 1.5. Let (X, d) be a complete metric space and T, S be two mappings. Then T and S are said to be weakly compatible if they commute at their coincidence point x, that is, $Tx = Sx$ implies $TSx = STx$.

Definition 1.6. [**11**] Let $T : X \to X$, where (X, d) is a complete metric space, is called a B-contraction if there exists positive real number α, β, γ such that $0 \leqslant \alpha + 2\beta + 2\gamma < 1$ for all $x, y \in X$ the following inequality holds:

$$d(Tx, Ty) \leqslant \alpha d(x, y) + \beta[d(x, Tx) + d(y, Ty)] + \gamma[d(x, Ty) + d(y, Tx)] \tag{1.1}$$

Definition 1.7. [**3**] Let (X, d) be a metric space. A mapping $T : X \to X$ is said sequentially convergent if, for each sequence $\{y_n\}$ in X the following holds true:

if $\{Ty_n\}$ convergences, then $\{y_n\}$ also convergences.

Theorem 1.1. [**11**] *Let $T : X \to X$, where (X, d) is a complete metric space, be a weak B-contraction. Then T has a unique fixed point.*

The following theorem, due to Al-Thagafi [**1**], has played a vital role to prove some results in this paper.

Theorem 1.2. [**1**] *Let K be a nonempty closed subset of a metric space K and let T, S be self-mappings of K with $T(K) \subset S(K)$. If $\overline{T(K)}$ is complete, S is continuous, T and S are commuting and T is S-contraction, then T and S have a unique common fixed point in K.*

In this paper, let $F(T, S)$ denote the set of common fixed points of T and S and $F(S)$ the set of fixed points of S.

2. Main Results

Definition 2.1. Let (X, d) be a metric space, T and S be a self mappings of X. Then T is S-weakly B-contraction, if

$$\begin{aligned} d(Tx, Ty) \leqslant\ & \alpha d(Sx, Sy) + \beta[d(Sx, Tx) + d(Sy, Ty)] + \\ & \gamma[d(Sx, Ty) + d(Sy, Tx)]. \end{aligned} \tag{2.1}$$

for all $x, y \in X$, there exist positive real numbers α, β, γ such that $\alpha + 2\beta + 2\gamma < 1$.

Theorem 2.1. *Let D be a nonempty closed subset of a metric space X and let T and S be a self maps of D with $T(D) \subset S(D)$. If $\overline{T(D)}$ is complete, S is continuous, T and S are commuting and T is S-weakly B-contraction, then T and S have a unique common fixed point in D.*

Proof. Let us take an element $x_0 \in D$. Since $T(D) \subset S(D)$ we get an element $x_1 \in D$ such that $Tx_0 = Sx_1$. Again, since $T(D) \subset S(D)$, there exists an element $x_2 \in D$ such that $Tx_1 = Sx_2$. Continuing in this manner inductively, we obtain a sequence $\{x_n\}$ in D with

$$Tx_{n-1} = Sx_n$$

for all positive integers n. Since T is S-weakly B-contraction,

$$\begin{aligned}
d(Tx_{n+1}, Tx_n) &\leqslant \alpha d(Sx_{n+1}, Sx_n) \\
&\quad + \beta[d(Sx_{n+1}, Tx_{n+1}) + d(Sx_n, Tx_n)] + \\
&\quad \gamma[d(Sx_{n+1}, Tx_n) + d(Sx_n, Tx_{n+1})] \\
&= \alpha d(Tx_n, Tx_{n-1}) \\
&\quad + \beta[d(Tx_n, Tx_{n+1}) + d(Tx_{n-1}, Tx_n)] + \\
&\quad \gamma[d(Tx_n, Tx_n) + d(Tx_{n-1}, Tx_{n+1})] \\
(1-\beta-\gamma) d(Tx_{n+1}, Tx_n) &\leqslant (\alpha+\beta+\gamma) d(Tx_n, Tx_{n-1}) \\
d(Tx_{n+1}, Tx_n) &\leqslant \left(\frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\right) d(Tx_n, Tx_{n-1}) \\
&\quad \vdots \\
&\leqslant \left(\frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\right)^n d(Tx_n, Tx_{n-1})
\end{aligned}$$

for all $n \in \mathbb{N}$. Hence $\{Tx_n\}$ is a Cauchy sequence. Since $\overline{T(D)}$ is complete, there exists $y \in \overline{T(D)}$ such that $T(x_n) \to y$ as $n \to \infty$.

Since S is continuous, we have $S(Tx_n) \to Sy$ as $n \to \infty$. Since T and S are commute, we get $T(Sx_n) = S(Tx_n)$ for all n, and hence

$$\begin{aligned}
T(Sx_n) &\to Sy \text{ as } n \to \infty \\
i.e., T(Tx_{n-1}) &\to Sy \text{ as } n \to \infty
\end{aligned}$$

Now we claim that $Ty = Sy$.

$$\begin{aligned}
d(Ty, Sy) &\leqslant d(Ty, T(Tx_n)) + d(T(Tx_n), Sy) \\
&\leqslant \alpha d(Sy, S(Tx_n)) + \beta[d(Sy, Ty) + d(S(Tx_n), T(Tx_n))] + \\
&\quad \gamma[d(Sy, T(Tx_n)) + d(S(Tx_n), Ty)] + d(T(Tx_n), Sy) \\
&= \alpha d(Sy, S(Tx_n)) + \beta[d(Sy, Ty) + d(S(Tx_n), T(Sx_{n+1}))] + \\
&\quad \gamma[d(Sy, T(Sx_{n+1})) + d(S(Tx_n), Ty)] + d(T(Sx_{n+1}), Sy) \\
&= \alpha d(Sy, S(Tx_n)) + \beta[d(Sy, Ty) + d(S(Tx_n), S(Tx_{n+1}))] + \\
&\quad \gamma[d(Sy, S(Tx_{n+1})) + d(S(Tx_n), Ty)] + d(S(Tx_{n+1}), Sy).
\end{aligned}$$

Taking the limit as $n \to \infty$ on right side, we get

$$\begin{aligned}
d(Ty, Sy) &\leqslant \alpha d(Sy, Sy) + \beta[d(Sy, Ty) + d(Sy, Sy)] \\
&\quad + \gamma[d(Sy, Sy) + d(Sy, Ty)] + d(Sy, Sy) \\
(1-\beta-\gamma) d(Ty, Sy) &\leqslant 0.
\end{aligned}$$

Since $1-\beta-\gamma>0$, $d(Tx,Sy)=0$ and hence $Ty=Sy$. Hence $T(Ty)=S(Ty)=S(Sy)$ by commutativity. We can therefore infer

$$\begin{aligned}
d(Ty,T(Ty)) &\leqslant \alpha d(Sy,S(Ty)+\beta[d(Sy,Ty)+d(S(Ty),Ty)]+\\
&\quad \gamma[d(Sy,T(Ty))+d(S(Ty),Ty)]\\
&= \alpha d(Ty,T(Ty))+\beta[d(Ty,Ty)+d(T(Ty),Ty)]+\\
&\quad \gamma[d(Ty,T(Ty))+d(T(Ty),Ty)]\\
&= (\alpha+2\gamma)d(Ty,T(Ty))\\
(1-\alpha-2\gamma)d(Ty,T(Ty)) &\leqslant 0.
\end{aligned}$$

Since $\alpha+2\gamma<1$, we have $Ty=T(Ty)$. We now have $Ty=T(Ty)=S(Ty)$, that is Ty is a common fixed point of T and S.

Finally, we show that T and S have a unique common fixed point. Suppose x and y are two common fixed points for T and S, i.e. $x=T(x)=S(x)$ and $y=T(y)=S(y)$, then using (2.1), we see that

$$\begin{aligned}
d(x,y)=d(Tx,Ty) &\leqslant \alpha d(Sx,Sy)+\beta[d(Sx,Tx)+d(Sy,Ty)]\\
&\quad +\gamma[d(Sx,Ty)+d(Sy,Tx)]\\
&= \alpha d(x,y)+\beta[d(x,x)+d(y,y)]+\gamma[d(x,y)+d(y,x)]\\
(1-\alpha-2\gamma)d(x,y) &\leqslant 0.
\end{aligned}$$

Since $\alpha+2\gamma<1$, we have $x=y$. Hence T and S have a unique common fixed point. □

Definition 2.2. Let (X,d) be a metric space, T and S be a self mappings of X. Then T is S-weakly B-nonexpansive, if

$$\begin{aligned}
d(Tx,Ty) &\leqslant \alpha d(Sx,Sy)+\beta[d(Sx,Tx)+d(Sy,Ty)]+\\
&\quad \gamma[d(Sx,Ty)+d(Sy,Tx)].
\end{aligned} \tag{2.2}$$

for all $x,y\in X$, there exist positive real numbers α,β,γ such that $\alpha+2\beta+2\gamma\leqslant 1$.

Theorem 2.2. *Let K be a nonempty closed subset of a normed linear space X, T and S be self-mappings of K with $T(K)\subset S(K)$ and $q\in F(S)$. If K is q-starshapped, $\overline{T(K)}$ is complete bounded, T is demicompact, S is continuous and affine with respect to q, T and S are commuting and S-weakly B-nonexpansive, then T and S have a common fixed point in K.*

Proof. For each $n\in\mathbb{N}$, define T_n by

$$T_n(x)=(1-t_n)q+t_nT(x)\ for\ each\ x\in K,$$

where $\{t_n\}$ is a sequence in $(0,1)$ such that $t_n\to 1$ as $n\to\infty$. Since K is q-starshapped, S is affine with respect to q and $T(K)\subset S(K)$, it follows that

$$\begin{aligned}
T_n(x) &= (1-t_n)q+t_nT(x)\\
&= (1-t_n)S(q)+t_nT(x)\in S(K).
\end{aligned}$$

Hence $T_n(K) \subset S(K)$ for each n. Since T and S commute and that S is affine with respect to q, it follows that

$$\begin{aligned} T_n(S(x)) &= (1-t_n)q + t_n T(S(x)) \\ &= (1-t_n)S(q) + t_n S(T(x)) \\ &= S((1-t_n)q + t_n T(x)) \\ &= S(T_n(x)). \end{aligned}$$

Hence T_n and S commute. Since S-weakly B-nonexpansive, it follows that

$$\begin{aligned} \|T_n(x) - T_n(y)\| &= t_n \|T(x) - T(y)\| \\ &\leqslant t_n(\alpha\|Sx - Sy\| + \beta[\|Sx - Tx\| + \|Sy - Ty\|] \\ &\quad + \gamma[\|Sx - Ty\| + \|Sy - Tx\|]). \end{aligned}$$

Now $t_n(\alpha + 2\beta + 2\gamma) \leqslant t_n < 1$, T_n is S-weakly B-contraction. Since $\overline{T_n(K)} = (1-t_n)q + t_n\overline{T(K)}$ and $\overline{T(K)}$ is complete, it follows that $\overline{T_n(K)}$ is complete. Hence by Theorem 2.1, there exists $x_n \in K$ such that $\{x_n\} = F(T_n, S)$ for each n. Since $\overline{T(K)}$ is bounded and $t_n \to 1$ as $n \to \infty$, it follows that

$$x_n - T(x_n) = (1-t_n)(q - T(x_n)) \to 0 \ as \ n \to \infty.$$

Since T is demicompact, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ such that

$$x_{n_k} \to x \ as \ k \to \infty.$$

Since S is continuous, $S(x_{n_k}) \to S(x)$. Hence

$$Sx = \lim_{n\to\infty} S(x_{n_k}) = \lim_{n\to\infty} x_{n_k} = x.$$

Hence we have $x \in F(S)$ and

$$\begin{aligned} \|x - T(x)\| &\leqslant \|x - x_{n_k}\| + \|x_{n_k} - T(x_{n_k})\| + \|T(x_{n_k}) - T(x)\| \\ &\leqslant \|x - x_{n_k}\| + \|x_{n_k} - T(x_{n_k})\| \\ &\quad + \alpha\|S(x_{n_k}) - S(x)\| + \beta[\|S(x_{n_k}) - T(x_{n_k})\| + \\ &\quad \|Sx - Tx\|] + \gamma[\|S(x_{n_k}) - Tx\| + \|Sx - Tx_{n_k}\|] \\ &\leqslant \|x - x_{n_k}\| + \|x_{n_k} - T(x_{n_k})\| \\ &\quad + \alpha\|Sx_{n_k} - S(x)\| + \beta[\|x_{n_k} - T(x_{n_k})\| + \\ &\quad \|x - Tx\|] + \gamma[\|x_{n_k} - Tx\| + \|x - Tx_{n_k}\|] \\ &\leqslant \|x - x_{n_k}\| + \|x_{n_k} - T(x_{n_k})\| \\ &\quad + \alpha\|Sx_{n_k} - S(x)\| + \beta[\|x_{n_k} - T(x_{n_k})\| + \|x - Tx\|] \\ &\quad + \gamma[\|x_{n_k} - x\| + \|x - Tx\| + \|x - x_{n_k}\| \\ &\quad + \|x_{n_k} - Tx_{n_k}\|] \\ (1 - \beta - \gamma)\|x - T(x)\| &\leqslant \|x - x_{n_k}\| + \|x_{n_k} - T(x_{n_k})\| \\ &\quad + \alpha\|Sx_{n_k} - S(x)\| + \beta\|x_{n_k} - T(x_{n_k})\| \\ &\quad + \gamma[\|x_{n_k} - x\| + \|x - x_{n_k}\| + \|x_{n_k} - Tx_{n_k}\|] \\ &\to 0 \ as \ k \to \infty. \end{aligned}$$

Thus $T(x) = x$. Hence T and S have a common fixed point in K. □

Corollary 2.1. *Let K be a nonempty complete and bounded subset of a normed linear space X, T and S be self-mappings of K with $T(K) \subset S(K)$ and $q \in F(S)$. If K is q-starshaped, T is demicompact, S is continuous and affine with respect to q, T and S are commuting and T is S-nonexpansive, then T and S have a common fixed point in K.*

Proof. $\overline{T(K)}$, being a closed subset of a complete and bounded set of K, is complete and bounded. Hence the proof follows from Theorem 2.2. □

The preceding corollary subsumes the following result due to Al-Thagafi [**1**].

Corollary 2.2. *Let K be a nonempty closed subset of a normed linear space X and let T, S be self- mappings of K with $T(K) \subset S(K)$ and $q \in F(S)$. If K is starshaped, $T(K)$ is compact, S is continuous and linear, T and S are commuting and T is S-nonexpansive, then T and S have a common fixed point in K.*

Proof. Since $\overline{T(K)}$ is compact, it is complete and bounded and the mapping T is compact. Hence T is demicompact. Hence by Theorem 2.2, T and S have a common fixed point in K. □

Corollary 2.3. *Let K be a nonempty closed starshaped subset of a normed linear space X. If T is a nonexpansive and demicompact self-mapping of K and $T(K)$ is complete bounded, then T has a fixed point in K.*

Proof. The proof of this corollary follows from Theorem 2.2 by putting S = I, the identity mapping. □

The following result due to Habiniak [**6**] is a special case of preceding corollary.

Corollary 2.4. *Let K be a nonempty closed starshaped subset of a normed linear space X and let T be a nonexpansive self-mapping of K such that $\overline{T(K)}$ is compact. Then T has a fixed point in K.*

Proof. Since $\overline{T(K)}$ is compact, it is complete bounded and the mapping T is compact. Hence T is demicompact. Hence by Corollary 2.3, T has a fixed point in K. □

The following result due to Dotson [**5**] is a special case of preceding corollary.

Corollary 2.5. *Let K be a nonempty compact starshaped subset of a normed linear space X. Let T be a nonexpansive self-mapping of K. Then T has a fixed point in K.*

Proof. Since K is compact and T is continuous, it follows that $\overline{T(K)}$ is compact. Therefore by Corollary 2.4, T has a fixed point in K. □

The following theorem, is the common fixed point theorem of three types of mappings T, S_1 and S_2, where T is a sequentially convergent mapping.

THEOREM 2.3. *Let (X, d) be a nonempty complete metric space and let $T : X \to X$ be a continuous, injective, sequentially convergent mapping. Let $S_1, S_2 : X \to X$ be self maps such that*

$$d(TS_1x, TS_2y) \leqslant \alpha d(Tx, Ty) + \beta[d(Tx, TS_1x) + d(Ty, TS_2y)] + \gamma[d(Ty, TS_1x) + d(Tx, TS_2y)] \tag{2.3}$$

where $\alpha > 0$ and $\beta, \gamma \geqslant 0$ with $\alpha + 2\beta + 2\gamma < 1$ and $d(Tx, S_1y) \leqslant d(x, y)$ (or) $d(Tx, S_2y) \leqslant d(x, y)$ for all $x, y \in X$, then T, S_1 and S_2 have a unique common fixed point.

PROOF. Let $x_0 \in X$. Define x_n by $x_{2n+1} = S_1x_{2n}$, $x_{2n+2} = S_2x_{2n+1}$, for $n = 0, 1, 2, \cdots$. Let n be odd.

$$\begin{aligned} d(Tx_n, Tx_{n+1}) &= d(TS_1x_{n-1}, TS_2x_n) \\ &\leqslant \alpha d(Tx_n, Tx_{n+1}) + \beta[d(Tx_{n-1}, TS_1x_{n-1}) + d(Tx_n, TS_2x_n)] \\ &\quad +\gamma[d(Tx_n, TS_1x_{n-1}) + d(Tx_{n-1}, TS_2x_n)] \\ &= \alpha d(Tx_n, Tx_{n+1}) + \beta[d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n+1})] \\ &\quad +\gamma[d(Tx_n, Tx_n) + d(Tx_{n-1}, Tx_{n+1})] \\ d(Tx_n, Tx_{n+1}) &\leqslant \left(\frac{\beta + \gamma}{1 - (\alpha + \beta + \gamma)}\right) d(Tx_{n-1}, Tx_n). \end{aligned}$$

Since $\alpha + 2\beta + 2\gamma < 1$, $\{Tx_n\}$ is a Cauchy sequence in X. Therefore $\{Tx_n\}$ is convergent in X. Since T is a sequentially convergent, $\{x_n\}$ is convergent, that is there exists $x \in X$ such that $x_n \to x$ as $n \to \infty$. Since T is continuous, $Tx_n \to Tx$ as $n \to \infty$. Now

$$\begin{aligned} d(Tx, TS_1x) &\leqslant d(Tx, Tx_{2n}) + d(Tx_{2n} + TS_1x) \\ &= d(Tx, Tx_{2n}) + d(TS_2x_{2n-1} + TS_1x) \\ &\leqslant d(Tx, Tx_{2n}) + +\alpha d(Tx_{2n-1}, Tx) \\ &\quad +\beta[d(Tx, TS_1x) + d(Tx_{2n-1}, TS_2x_{2n-1})] + \\ &\quad \gamma[d(Tx_{2n-1}, TS_1x) + d(Tx, TS_2x_{2n-1})] \\ &= d(Tx, Tx_{2n}) + \alpha d(Tx_{2n-1}, Tx) \\ &\quad +\beta[d(Tx, TS_1x) + d(Tx_{2n-1}, Tx_{2n})] + \\ &\quad \gamma[d(Tx_{2n-1}, TS_1x) + d(Tx, Tx_{2n})] \\ &\to (\beta + \gamma)d(Tx, TS_1x) \textit{ as } n \to \infty. \end{aligned}$$

Therefore $Tx = TS_1x$, since T is injective, $x = S_1x$. Similarly $x = S_2x$. Hence $x = S_1x = S_2x$ and

$$\begin{aligned} d(x, Tx) &\leqslant d(x, x_{2n}) + d(x_{2n}, Tx) \\ &= d(x, x_{2n}) + d(S_2x_{2n-1}, Tx) \\ &\leqslant d(x, x_{2n}) + d(x_{2n-1}, x) \\ &\to 0 \textit{ as } n \to \infty. \end{aligned}$$

Therefore $Tx = x$. Hence T, S_1, S_2 have a common fixed point.

Uniqueness: Suppose there exists $y \in X$ such that $S_1y = S_2y = Ty = y$. Now

$$\begin{aligned} d(Tx,Ty) &= d(TS_1x,TS_2y) \\ &\leqslant \alpha d(Tx,Ty) + \beta[d(Tx,TS_1x) + d(Ty,TS_2y)] \\ &\quad +\gamma[d(Ty,TS_1x) + d(Tx,TS_2y)] \\ &= (\alpha + 2\gamma)d(Tx,Ty). \end{aligned}$$

Since $2\beta + \gamma < 1$, $Tx = Ty$ and hence $x = y$.
Hence T, S_1 and S_2 have a unique common fixed point. □

Corollary 2.6. *Let (X,d) be a complete metric space and let $T : X \to X$ be a continuous, injective, sequentially convergent mapping. Let $S_1, S_2 : X \to X$ be self maps such that $d(TS_1x,TS_2y) \leqslant kd(Tx,Ty)$, where $k \in [0,1)$ and $d(Tx,S_1y) \leqslant d(x,y)$ (or) $d(Tx,S_2y) \leqslant d(x,y)$ for all $x,y \in X$, then T, S_1 and S_2 have a unique common fixed point.*

Proof. The proof of the corollary follows from the Theorem 2.3 by putting $\beta = \gamma = 0$ and $\alpha = k$. □

Theorem 2.4. *Let K be a nonempty compact subset of a metric space (X,d). Let $T : K \to K$ be a continuous, injective, mapping and let $S_1, S_2 : K \to K$ be self maps such that*

$$\begin{aligned} d(TS_1x,TS_2y) &\leqslant \alpha d(Tx,Ty) + \beta[d(Tx,TS_1x) + d(Ty,TS_2y)] + \\ &\quad \gamma[d(Ty,TS_1x) + d(Tx,TS_2y)] \end{aligned} \tag{2.4}$$

where $\alpha > 0$ and $\beta,\gamma \geqslant 0$ with $\alpha + 2\beta + 2\gamma \leqslant 1$ and $d(Tx,S_1y) \leqslant d(x,y)$ (or) $d(Tx,S_2y) \leqslant d(x,y)$ for all $x,y \in X$, then T, S_1 and S_2 have a unique common fixed point.

Proof. For each $n \in \mathbb{N}$, let $x_{2n+1} = S_1x_{2n}$, $x_{2n+2} = S_2x_{2n+1}$. Then the sequence $\{x_n\} \subset K$. Since K is compact, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to x$ as $k \to \infty$. Therefore $Tx_{n_k} \to Tx$. Now

$$\begin{aligned} d(Tx,TS_1x) &\leqslant d(Tx,Tx_{2n_k}) + d(Tx_{2n_k} + TS_1x) \\ &= d(Tx,Tx_{2n_k}) + d(TS_2x_{2n_k-1} + TS_1x) \\ &\leqslant d(Tx,Tx_{2n_k}) + +\alpha d(Tx_{2n_k-1},Tx) \\ &\quad +\beta[d(Tx,TS_1x) + d(Tx_{2n_k-1},TS_2x_{2n_k-1})] + \\ &\quad \gamma[d(Tx_{2n_k-1},TS_1x) + d(Tx,TS_2x_{2n_k-1})] \\ &= d(Tx,Tx_{2n_k}) + \alpha d(Tx_{2n_k-1},Tx) \\ &\quad +\beta[d(Tx,TS_1x) + d(Tx_{2n_k-1},Tx_{2n_k})] + \\ &\quad \gamma[d(Tx_{2n_k-1},TS_1x) + d(Tx,Tx_{2n_k})] \\ &\to (\beta + \gamma)d(Tx,TS_1x) \text{ as } n \to \infty. \end{aligned}$$

Therefore $Tx = TS_1x$, since T is injective, $x = S_1x$. Similarly $x = S_2x$. Hence $x = S_1x = S_2x$ and

$$\begin{aligned} d(x, Tx) &\leqslant d(x, x_{2n+1}) + d(x_{2n+1}, Tx) \\ &= d(x, x_{2n+1}) + d(S_1x_{2n}, Tx) \\ &\leqslant d(x, x_{2n}) + d(x_{2n}, x) \\ &\to 0 \text{ as } k \to \infty. \end{aligned}$$

Therefore $Tx = x$. Hence T, S_1 and S_2 have a common fixed point.

Uniqueness: Suppose there exists $y \in X$ such that $S_1y = S_2y = Ty = y$. Now

$$\begin{aligned} d(Tx, Ty) &= d(TS_1x, TS_2y) \\ &\leqslant \alpha d(Tx, Ty) + \beta[d(Tx, TS_1x) + d(Ty, TS_2y)] \\ &\quad + \gamma[d(Ty, TS_1x) + d(Tx, TS_2y)] \\ &= (\alpha + 2\gamma) d(Tx, Ty). \end{aligned}$$

Since $2\beta + \gamma < 1$, $Tx = Ty$ and hence $x = y$.
Hence T, S_1 and S_2 have a unique common fixed point. □

THEOREM 2.5. *Let K be a nonempty compact convex subset of a Banach space X. Let $T : K \to K$ be continuous, injective, affine and let $S_1, S_2 : K \to K$ be self maps such that*

$$\text{(2.5)}\quad \begin{aligned} \|TS_1x - TS_2y\| &\leqslant \alpha\|Tx - Ty\| + \beta[\|Tx - TS_1x\| + \|Ty - TS_2y\|] + \\ &\quad \gamma[\|Ty - TS_1x\| + \|Tx - TS_2y\|] \end{aligned}$$

where $\alpha > 0$ and $\beta, \gamma \geqslant 0$ with $\alpha + 2\beta + 2\gamma \leqslant 1$ and $\|Tx - S_1y\| \leqslant \|x - y\|$ (or) $\|Tx - S_2y\| \leqslant \|x - y\|$ for all $x, y \in X$, then T, S_1 and S_2 have a common fixed point.

PROOF. Let $x_0 \in K$, $\alpha_n \in (0, 1)$ such that $\alpha_n \to 1$, as $n \to \infty$. Define $S_{1n}, S_{2n} : K \to K$ by

$$\begin{aligned} S_{1n}(x) &= (1 - \alpha_n)x_0 + \alpha_n S_1 x, \\ S_{2n}(x) &= (1 - \alpha_n)x_0 + \alpha_n S_2 x. \end{aligned}$$

Then

$$\begin{aligned} \|TS_{1n}x - TS_{2n}y\| &= \alpha_n\|TS_1x - TS_2y\| \\ &\leqslant \alpha_n\big(\alpha\|Tx - Ty\| + \beta[\|Tx - TS_1x\| + \|Ty - TS_2y\|] \\ &\quad + \gamma[\|Ty - TS_1x\| + \|Tx - TS_2y\|]\big). \end{aligned}$$

Now $\alpha_n(\alpha + 2\beta + 2\gamma) \leqslant \alpha_n < 1$. Then by Theorem 2.3, S_{1n}, S_{2n} have a common fixed point. Let $S_{1n}x_n = S_{2n}x_n = x_n$, for all $n \in \mathbb{N}$. Since K is compact, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to x$ as $k \to \infty$. Since T is continuous, $Tx_{n_k} \to Tx$.
Now

$$\begin{aligned} x_{n_k} = S_{1n_k}(x_{n_k}) &= (1 - \alpha_{n_k})x_0 + \alpha_{n_k} S_1 x_{n_k}, \\ S_1 x_{n_k} &\to x \text{ as } k \to \infty. \end{aligned}$$

Similarly $S_2x_{n_k} \to x$ as $k \to \infty$. Then

$$\begin{aligned}\|Tx - TS_1x\| &\leqslant \|Tx - TS_2x_{n_k}\| + \|TS_2x_{n_k} - TS_1x\| \\ &\leqslant \|Tx - TS_2x_{n_k}\| + \alpha\|Tx - Tx_{n_k}\| \\ &\quad +\beta[\|Tx - TS_1x\| + \|Tx_{n_k} - TS_2x_{n_k}\|] + \\ &\quad \gamma[\|Tx_{n_k} - TS_1x\| + \|Tx - TS_2x_{n_k}\|] \\ &\to (\beta+\gamma)\|Tx - TS_1x\| \ as\ k \to \infty.\end{aligned}$$

Therefore $Tx = TS_1x$, since T is injective, $x = S_1x$. Similarly $x = S_2x$. Hence $x = S_1x = S_2x$ and

$$\begin{aligned}\|x - Tx\| &\leqslant \|x - S_1x_{n_k}\| + \|S_1x_{n_k} - Tx\| \\ &= \|x - S_1x_{n_k}\| + \|x_{n_k} - x\| \\ &\to 0\ as\ k \to \infty.\end{aligned}$$

Therefore $Tx = x$. Hence T, S_1 and S_2 have a common fixed point.

□

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