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COMMON FIXED POINT THEOREMS IN GP-METRIC SPACES AND APPLICATIONS

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Abstract. A generalized condition (B) is introduced in the context of GP-metric spaces to establish coincidence and common fixed point of discontinuous mappings and utilized to solve the integral equation and the functional equation arising in dynamic programming. Our results are absolutely novel and provide a new dimension in fixed point theory and can not be attained from the available results in the literature. Conclusively two explanatory examples are also furnished for the sake of clarity.

1. Introduction

Motivated by the usefulness of the notion of a metric space introduced by French mathematician Frèchet [**13**] in the natural development of mathematics in general and to functional analysis in particular, numerous researchers tried different generalizations of this notion in the recent past. One of such generalizations called a generalized partial metric space (GP-metric space) is introduced by Zand and Nezhad [**23**] by combining the notion of a generalized metric space (G-metric space) due to Mustafa and Sims [**18**] and a partial metric space introduced by Matthews [**16**]. Aydi et al. [**7**] gave first fixed point result in GP-metric spaces. On the other hand Abbas et al. [**3**] introduced generalized condition (B) for a pair of mappings in a metric space and recently Tomar et al. [**21**] introduced it in a quasi-partial metric space.

Acknowledging the notions of Zand and Nezhad [**23**], Abbas et al. [**3**] and Tomar et al. [**21**] we introduce generalized condition (B) in GP-metric spaces to obtain coincidence and common fixed point of discontinuous mappings and provide favourable answers to two open problems presented by Abbas et al. [**1**]. We compare our results with many results existing in the literature ([**1**], [**3**], [**4–6**, **8**], [**10**, **11**], [**14**], [**16**, **17**] and so on) to elucidate the importance of generalized condition (B)

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in GP-metric spaces and apply them to solve integral equation and functional equation arising in dynamic programming. Finally two explanatory examples are furnished to illustrate the work.

2. Preliminaries

DEFINITION 2.1. ([**23**]) Let X be a nonempty set. A mapping $Gp : X \times X \times X \to \mathbb{R}^+$ is said to be a Gp-metric on X if it satisfies the following assumptions:

(1) $x = y = z$ if $Gp(x, y, z) = Gp(x, x, x) = Gp(y, y, y) = Gp(z, z, z)$;
(2) $0 \leqslant Gp(x, x, x) \leqslant Gp(x, x, y) \leqslant Gp(x, y, z)$;
(3) $Gp(x, y, z) = Gp(x, z, y) = Gp(y, z, x) = ...$,(symmetry in all three variables);
(4) $Gp(x, y, z) \leqslant Gp(x, a, a) + Gp(a, y, z) - Gp(a, a, a)$;

for all $x, y, z, a \in X$. Then the pair (X, Gp) is called a GP-metric space.

EXAMPLE 2.1. Let $X = [0, +\infty)$ and let $Gp : X \times X \times X \to [0, +\infty)$, be defined by $Gp(x, y, z) = d(x, y) + d(y, z) + d(z, x)$. Clearly, (X, Gp) is a GP-metric space but not a G-metric space.

EXAMPLE 2.2. Let $X = \{a, b, c\}$ and $Gp : X \times X \times X \to [0, +\infty)$, be defined by $Gp(x, y, z) = 1$, if $x = y = z$; $Gp(a, b, b) = Gp(b, a, a) = 10$; $Gp(a, c, c) = Gp(c, a, a) = 15$; $Gp(b, c, c) = Gp(c, b, b) = 17$ and $Gp(a, b, c) = 20$. Clearly, (X, Gp) is a GP-metric space but not a G-metric space.

EXAMPLE 2.3. Let $X = [0, +\infty)$ and $Gp(x, y, z) = \max\{x, y, z\}$, for all $x, y, z \in X$. Then (X, Gp) is a GP-metric space. Clearly, $d_{Gp} = |x - y|$ is a Gp metric on X.

PROPOSITION 2.1 ([**23**]). *Let (X, Gp) be a GP-metric space. The function $d_{Gp} : X \times X \to \mathbb{R}^+$, such that for all $x, y \in X$, we have:*

$$d_{Gp}(x, y) = Gp(x, y, y) + Gp(y, x, x) - Gp(x, x, x) - Gp(y, y, y)$$

defines a metric on X.

DEFINITION 2.2. ([**23**]) Let (X, Gp) be a GP-metric space and let $\{x_n\}$ be a sequence in X. A sequence $\{x_n\}$ is convergent to a point $x \in X$ if

$$\lim_{n,m\to\infty} Gp(x_n, x_m, x) = Gp(x, x, x).$$

PROPOSITION 2.2 ([**23**]). *Let (X, Gp) be a GP-metric space. For any sequence $\{x_n\}$ in X and a point $x \in X$, the following assumptions are equivalent:*

(1) *$\{x_n\}$ is Gp-convergent to x.*
(2) *$Gp(x_n, x_n, x) \to Gp(x, x, x)$ as $n \to \infty$.*
(3) *$G(x_n, x, x) \to 0$ as $n \to \infty$.*

LEMMA 2.1 ([**7**]). *Let (X, Gp) be a GP-metric space. Then*

- *If $Gp(x, y, z) = 0$, then $x = y = z$.*
- *If $x \neq y$, then $Gp(x, y, y) > 0$.*

DEFINITION 2.3. ([**8**]) A self mapping S of a metric space (X,d) satisfies the condition (B) if there exist $\delta \in (0,1)$, $L \geqslant 0$ and for all $x,y \in X$ we have:

$$d(Sx,Sy) \leqslant \delta d(x,y) + L\min(d(x,Sx), d(y,Sy), d(x,Sy), d(y,Sx)).$$

Abbas et al. [**1**] extended the notion of condition (B) to a pair of mappings as generalized condition (B) and Abbas and Ilic [**2**] independently extended it as generalized almost S-contraction.

DEFINITION 2.4. ([**1**]) Let A and S be two self mappings of a metric space (X,d). The mapping S satisfies generalized condition (B) associated with A if there exist $\delta \in (0,1)$ and $L \geqslant 0$ such that for all $x,y \in X$, we have:

$$d(Sx,Sy) \leqslant \delta(M(x,y)) + L\min\{d(Ax,Sx), d(Ay,Sy), d(Ax,Sy), d(Ay,Sx)\},$$

where

$$M(x,y) = \max\{d(Ax,Ay), d(Ax,Sx), d(Ay,Sy), \frac{d(Ax,Sy)+d(Ay,Sx)}{2}\}.$$

Evidently, for $A = I$, generalized condition (B) reduces to condition (B).

DEFINITION 2.5. ([**14**]) Let X be a nonempty set. Two mappings $A,S : X \to X$, are said to be weakly compatible if they commute at their coincidence point, i.e., if $Au = Su$ for some $u \in X$, then $ASu = SAu$.

3. Main Result

Following Tomar et al. [**21**] first we introduce generalized condition (B) in a GP-metric space for a pair and two pairs of self-mappings.

DEFINITION 3.1. Let A and S be two self mappings of a GP-metric space (X,Gp). The mapping S satisfies generalized condition (B) associated with A (S is a generalized almost A-contraction) if there exist $\delta \in (0,1)$ and $L \geqslant 0$ such that for all $x,y \in X$:

$$\begin{aligned}(3.1)\quad Gp(Sx,Sy,Sy) \leqslant \delta\max\{&Gp(Ax,Ay,Ay), Gp(Ax,Sx,Sx), Gp(Ay,Sy,Sy),\\ &\frac{1}{2}(Gp(Sx,Ay,Ay)+Gp(Ax,Sy,Sy))\} + L\min\{Gp(Ax,Sx,Sx), Gp(Ay,Sy,Sy),\\ &Gp(Ax,Sy,Sy), Gp(Sx,Ay,Ay)\}.\end{aligned}$$

If $A = id_X$, then S satisfies generalized condition (B) in a GP-metric space.

EXAMPLE 3.1. Let $X = [0,\infty)$ and the Gp metric: $Gp(x,y,z) = \max\{x,y,z\}$. Let two self mappings A and S be defined as:

$$Sx = \begin{cases} \frac{x}{6}, & 0 \leqslant x \leqslant 1 \\ 0, & x > 1, \end{cases} \quad Ax = \begin{cases} \frac{x}{2}, & 0 \leqslant x \leqslant 1 \\ 2, & x > 1. \end{cases}$$

For $x,y \in [0,1]$:

$$Gp(Sx,Sy,Sy) = \max\{\frac{x}{6}, \frac{y}{6}, \frac{y}{6}\} \leqslant \frac{1}{2}\max\{\frac{x}{2}, \frac{y}{2}, \frac{y}{2}\}.$$

For $x \in [0,1]$ and $y > 1$:

$$Gp(Sx, Sy, Sy) = \frac{x}{6} \leqslant \frac{1}{2} \max\{\frac{x}{2}, 2, 2\}.$$

For $x > 1$ and $y \in [0,1]$:

$$Gp(Sx, Sy, Sy) = \frac{y}{6} \leqslant \frac{1}{2} \max\{2, \frac{y}{2}, \frac{y}{2}\}.$$

Hence, S satisfies generalized condition (B) associated with A, for $\delta = \frac{1}{2}$ and $L = 0$.

Definition 3.2. Let A, B, S and T be four self mappings of a GP-metric space (X, Gp). The pair of mappings (A, S) satisfies generalized condition (B) associated with (B, T) ((A, S) is a generalized almost (B, T)-contraction) if there exist $\delta \in (0, 1)$ and $L \geqslant 0$ such that for all $x, y \in X$:

$$\begin{aligned}(3.2)\quad Gp(Sx, Ty, Ty) \leqslant \delta \max\{&Gp(Ax, By, By), Gp(Ax, Sx, Sx), Gp(By, Ty, Ty),\\ &\frac{1}{2}(Gp(Sx, By, By) + Gp(Ax, Ty, Ty))\} + L \min\{Gp(Ax, Sx, Sx), Gp(By, Ty, Ty),\\ &Gp(Ax, Ty, Ty), Gp(Sx, By, By)\}.\end{aligned}$$

Theorem 3.1. *Let A, B, S and T be self mappings of a GP-metric space (X, Gp). If a pair (A, S) satisfies generalized condition (B) associated with (B, T) such that for all $x, y \in X$:*
(1) *$TX \subset AX$ and $SX \subset BX$,*
(2) *AX or BX is closed,*
(3) *$\delta + L < 1$,*
then (A, S) and (B, T) have a coincidence point. Further, A, B, S and T have a unique common fixed point provided that (A, S) and (B, T) are weakly compatible.

Proof. Let $x_0 \in X$. Since $SX \subset BX$ there exists a point $x_1 \in X$ such that $y_1 = Bx_1 = Sx_0$. Let for this point y_1 there exists a point $y_2 \in Tx_1$. Also since $TX \subset AX$, there exists $x_2 \in X$ such that $y_2 = Ax_2 = Tx_1$. Continuing in this manner, we define a sequence $\{y_n\}$ in X as follows:

$$\begin{cases} y_{2n+1} = Bx_{2n+1} = Sx_{2n}, \\ y_{2n+2} = Ax_{2n+2} = Tx_{2n+1}. \end{cases}$$

Now
$Gp(y_{2n+1}, y_{2n+2}, y_{2n+2}) = Gp(Sx_{2n}, Tx_{2n+1}, Tx_{2n+1})$
$\leqslant \delta \max\{Gp(Ax_{2n}, Bx_{2n+1}, Bx_{2n+1}), Gp(Ax_{2n}, Sx_{2n}, Sx_{2n}),$
$Gp(Bx_{2n+1}, Tx_{2n+1}, Tx_{2n+1}), \frac{1}{2}(Gp(Sx_{2n}, Bx_{2n+1}, Bx_{2n+1})$
$+ Gp(Ax_{2n}, Tx_{2n+1}, Tx_{2n+1}))\} + L \min\{Gp(Ax_{2n}, Sx_{2n}, Sx_{2n}),$
$Gp(Bx_{2n+1}, Tx_{2n+1}, Tx_{2n+1}), Gp(Ax_{2n}, Tx_{2n+1}, Tx_{2n+1}),$
$Gp(Sx_{2n}, Bx_{2n+1}, Bx_{2n+1})\}.$
$= \delta \max\{Gp(y_{2n}, y_{2n+1}, y_{2n+1}), Gp(y_{2n}, y_{2n+1}, y_{2n+1}),$
$Gp(y_{2n+1}, y_{2n+2}, y_{2n+2}), \frac{1}{2}(Gp(y_{2n+1}, y_{2n+1}, y_{2n+1}) + Gp(y_{2n}, y_{2n+2}, y_{2n+2}))\}$
$+ L \min\{Gp(y_{2n}, y_{2n+1}, y_{2n+1}), Gp(y_{2n+1}, y_{2n+2}, y_{2n+2}), Gp(y_{2n}, y_{2n+2}, y_{2n+2}),$
$Gp(y_{2n+1}, y_{2n+1}, y_{2n+1})\}.$

$\leqslant \delta \max\{Gp(y_{2n}, y_{2n+1}, y_{2n+1}), Gp(y_{2n+1}, y_{2n+2}, y_{2n+2}),$
$\frac{1}{2}(Gp(y_{2n}, y_{2n+1}, y_{2n+1}) + Gp(y_{2n+1}, y_{2n+2}, y_{2n+2}))\}$
$+ L \min\{Gp(y_{2n+1}, y_{2n+1}, y_{2n+1}), Gp(y_{2n}, y_{2n+2}, y_{2n+2})\}.$

Now we have the following cases:

Case I: Let $Gp(y_{2n}, y_{2n+1}, y_{2n+1}) \leqslant Gp(y_{2n+1}, y_{2n+2}, y_{2n+2})$
and $Gp(y_{2n+1}, y_{2n+1}, y_{2n+1}) \leqslant Gp(y_{2n}, y_{2n+2}, y_{2n+2})$. Then
$Gp(y_{2n+1}, y_{2n+2}, y_{2n+2}) \leqslant \delta Gp(y_{2n+1}, y_{2n+2}, y_{2n+2}) + LGp(y_{2n+1}, y_{2n+1}, y_{2n+1}),$
i.e.,$Gp(y_{2n+1}, y_{2n+2}, y_{2n+2}) \leqslant \delta Gp(y_{2n+1}, y_{2n+2}, y_{2n+2}) + LGp(y_{2n+1}, y_{2n+2}, y_{2n+2}),$
i.e., $Gp(y_{2n+1}, y_{2n+2}, y_{2n+2}) \leqslant (\delta + L)Gp(y_{2n+1}, y_{2n+2}, y_{2n+2}).$
Since, $\delta + L < 1$, $Gp(y_{2n+1}, y_{2n+2}, y_{2n+2}) < Gp(y_{2n+1}, y_{2n+2}, y_{2n+2})$, a contradiction.

Case II: Let $Gp(y_{2n}, y_{2n+1}, y_{2n+1}) \leqslant Gp(y_{2n+1}, y_{2n+2}, y_{2n+2})$
and $Gp(y_{2n}, y_{2n+2}, y_{2n+2}) \leqslant Gp(y_{2n+1}, y_{2n+1}, y_{2n+1})$.
Then
$Gp(y_{2n+1}, y_{2n+2}, y_{2n+2}) \leqslant \delta Gp(y_{2n+1}, y_{2n+2}, y_{2n+2}) + LGp(y_{2n}, y_{2n+2}, y_{2n+2}),$
i.e.,$Gp(y_{2n+1}, y_{2n+2}, y_{2n+2}) \leqslant \delta Gp(y_{2n+1}, y_{2n+2}, y_{2n+2}) + LGp(y_{2n+1}, y_{2n+2}, y_{2n+2}),$
i.e., $Gp(y_{2n+1}, y_{2n+2}, y_{2n+2}) \leqslant (\delta + L)Gp(y_{2n+1}, y_{2n+2}, y_{2n+2}).$
Since, $\delta + L < 1$, $Gp(y_{2n+1}, y_{2n+2}, y_{2n+2}) < Gp(y_{2n+1}, y_{2n+2}, y_{2n+2})$, a contradiction.

Case III: Let $Gp(y_{2n+1}, y_{2n+2}, y_{2n+2}) \leqslant Gp(y_{2n}, y_{2n+1}, y_{2n+1})$
and $Gp(y_{2n}, y_{2n+2}, y_{2n+2}) \leqslant Gp(y_{2n+1}, y_{2n+1}, y_{2n+1})$.
Then
$Gp(y_{2n+1}, y_{2n+2}, y_{2n+2}) \leqslant \delta Gp(y_{2n}, y_{2n+1}, y_{2n+1}) + LGp(y_{2n}, y_{2n+2}, y_{2n+2}),$
i.e.,$Gp(y_{2n+1}, y_{2n+2}, y_{2n+2}) \leqslant \delta Gp(y_{2n+1}, y_{2n+2}, y_{2n+2}) + LGp(y_{2n+1}, y_{2n+2}, y_{2n+2}),$
i.e., $Gp(y_{2n+1}, y_{2n+2}, y_{2n+2}) \leqslant (\delta + L)Gp(y_{2n+1}, y_{2n+2}, y_{2n+2}).$
Since, $\delta + L < 1$, $Gp(y_{2n+1}, y_{2n+2}, y_{2n+2}) < Gp(y_{2n+1}, y_{2n+2}, y_{2n+2})$, a contradiction.

Case IV: Let $Gp(y_{2n+1}, y_{2n+2}, y_{2n+2}) \leqslant Gp(y_{2n}, y_{2n+1}, y_{2n+1})$
and $Gp(y_{2n+1}, y_{2n+1}, y_{2n+1}) \leqslant Gp(y_{2n}, y_{2n+2}, y_{2n+2})$.
Then
$Gp(y_{2n+1}, y_{2n+2}, y_{2n+2}) \leqslant \delta Gp(y_{2n}, y_{2n+1}, y_{2n+1}) + LGp(y_{2n+1}, y_{2n+1}, y_{2n+1}),$
i.e., $Gp(y_{2n+1}, y_{2n+2}, y_{2n+2}) \leqslant \delta Gp(y_{2n}, y_{2n+1}, y_{2n+1}) + LGp(y_{2n}, y_{2n+1}, y_{2n+1}),$
i.e., $Gp(y_{2n+1}, y_{2n+2}, y_{2n+2}) \leqslant (\delta + L)Gp(y_{2n}, y_{2n+1}, y_{2n+1}).$
$\leqslant (\delta + L)^2 Gp(y_{2n-1}, y_{2n}, y_{2n}) \leqslant ... \leqslant (\delta + L)^{n+1} Gp(y_0, y_1, y_1) \to 0$ as $n \to \infty$.
Therefore, $\{y_n\}$ is convergent and hence, its subsequences $\{y_{2n+2}\} = \{Ax_{2n+2}\}$ and $\{y_{2n+1}\} = \{Bx_{2n+1}\}$ are also convergent to z. Since AX is closed, $z \in AX$, i.e., there exists $u \in X$ such that $z = Au$. We claim that $z = Su$. If not, by using inequality (3.2), we get:
$Gp(Su, Tx_{2n+1}, Tx_{2n+1}) \leqslant \delta \max\{Gp(Au, Bx_{2n+1}, Bx_{2n+1}), Gp(Au, Su, Su),$
$Gp(Bx_{2n+1}, Tx_{2n+1}, Tx_{2n+1}), \frac{1}{2}(Gp(Su, Bx_{2n+1}, Bx_{2n+1}) +$
$Gp(Au, Tx_{2n+1}, Tx_{2n+1}))\} + L \min\{Gp(Au, Su, Su), Gp(Bx_{2n+1}, Tx_{2n+1}, Tx_{2n+1}),$
$Gp(Au, Tx_{2n+1}, Tx_{2n+1}), Gp(Su, Bx_{2n+1}, Bx_{2n+1})\}.$

Letting $n \to \infty$,
$Gp(Su,z,z) \leqslant \delta \max\{Gp(z,z,z), Gp(z,Su,Su), Gp(z,z,z), \frac{1}{2}(Gp(Su,z,z) + Gp(z,z,z))\} + L\min\{Gp(z,Su,Su), Gp(z,z,z), Gp(z,z,z), Gp(Su,z,z)\}$,
i.e., $Gp(Su,z,z) \leqslant \delta Gp(Su,z,z) + LGp(z,z,z)$,
i.e., $Gp(Su,z,z) \leqslant (\delta + L)Gp(Su,z,z)$, a contradiction to (3).
Hence, $G(Su,z,z) = 0$, i.e., $Su = z$. So $Au = Su$, i.e., A and S have a coincidence point.
Since $SX \subset BX$, there exists $v \in X$ such $z = Su = Bv$.

We claim that $Tv = z$. If not, by using inequality (3.2) we get:
$Gp(Su,Tv,Tv) \leqslant \delta \max\{Gp(Au,Bv,Bv), Gp(Au,Su,Su), Gp(Bv,Tv,Tv), \frac{1}{2}(Gp(Su,Bv,Bv) + Gp(Au,Tv,Tv))\} + L\min\{Gp(Au,Su,Su), Gp(Bv,Tv,Tv), Gp(Au,Tv,Tv), Gp(Su,Bv,Bv)\}$,
i.e., $Gp(z,Tv,Tv) \leqslant \delta \max\{Gp(z,z,z), Gp(z,z,z), Gp(z,Tv,Tv), \frac{1}{2}(Gp(z,z,z) + Gp(z,Tv,Tv))\} + L\min\{Gp(z,z,z), Gp(z,Tv,Tv), Gp(z,Tv,Tv), Gp(z,z,z)\}$,
i.e., $Gp(z,Tv,Tv) \leqslant \delta Gp(z,Tv,Tv) + LGp(z,z,z)$,
i.e., $Gp(z,Tv,Tv) \leqslant (\delta + L)Gp(z,Tv,Tv)$, a contradiction to (3).
So, $Gp(z,Tv,Tv) = 0$, i.e., $Tv = z$. Hence, $Bv = Tv$, i.e., B and T have a coincidence point. If we assume that BX is closed, then argument analogous to the previous argument establishes that the pairs (A,S) and (B,T) have a coincidence point. Since, (A,S) and (B,T) are weakly compatible, $Az = ASu = SAu = Sz$ and $Bz = BTv = TBv = Tz$.

Now we show that $z = Az$. If not, by using inequality (3.2) we get:
$Gp(Sz,Tv,Tv) \leqslant \delta \max\{Gp(Az,Bv,Bv), Gp(Az,Sz,Sz), Gp(Bv,Tv,Tv), \frac{1}{2}(Gp(Sz,Bv,Bv) + Gp(Az,Tv,Tv))\} + L\min\{Gp(Az,Sz,Sz), Gp(Bv,Tv,Tv), Gp(Az,Tv,Tv), Gp(Sz,Bv,Bv)\}$. Letting $n \to \infty$, we get
$Gp(Az,z,z) \leqslant \delta \max\{Gp(Az,z,z), Gp(z,z,z), \frac{1}{2}(Gp(Az,z,z) + Gp(Az,z,z))\} + L\min\{Gp(Sz,Sz,Sz), Gp(z,z,z), Gp(Az,z,z), Gp(Az,z,z)\}$,
i.e., $Gp(Az,z,z) \leqslant \delta Gp(Az,z,z) + LGp(z,z,z)$,
i.e., $Gp(Az,z,z) \leqslant (\delta + L)Gp(Az,z,z)$, a contradiction to (3).
So, $Gp(Az,z,z) = 0$, i.e., $z = Az$. Similarly we can prove that $z = Bz$. Hence, $z = Az = Bz = Sz = Tz$, i.e., z is a common fixed point for A, B, S and T. Uniqueness of the common fixed point is an easy consequence of inequality (3.2). $\square$

Now we furnish example to demonstrate the validity of Theorem 3.1.

EXAMPLE 3.2. Let $X = [0,2]$ and the Gp-metric: $Gp(x,y,z) = \max\{x,y,z\}$. Let self mappings A, B, S and T be defined by:

$$Ax = \begin{cases} \frac{x}{2}, & 0 \leqslant x \leqslant 1 \\ \frac{5}{4}, & 1 < x \leqslant 2, \end{cases} \quad Bx = \begin{cases} \frac{3x}{2}, & 0 \leqslant x \leqslant 1 \\ \frac{3}{2}, & 1 < x \leqslant 2, \end{cases}$$

$$Sx = \begin{cases} \frac{x}{6}, & 0 \leqslant x \leqslant 1 \\ \frac{1}{2}, & 1 < x \leqslant 2, \end{cases} \quad Tx = \begin{cases} \frac{x}{4}, & 0 \leqslant x \leqslant 1 \\ \frac{1}{4}, & 1 < x \leqslant 2. \end{cases}$$

Clearly, $AX = [0, \frac{1}{2}] \cup \{\frac{5}{4}\}$, $BX = [0, \frac{3}{2}]$, $TX = [0, \frac{1}{4}] \subset AX$ and $SX = [0, \frac{1}{6}] \cup \{\frac{1}{2}\} \subset BX$. The point 0 is a coincidence point of A, B, S and T. Also $AS0 = SA0 = 0$ and $TB0 = BT0 = 0$, i.e., pairs (A, S) and (B, T) are weakly compatible.

Case 1. For $x, y \in [0, 1]$: $Gp(Sx, Ty, Ty) = \max\{\frac{x}{6}, \frac{y}{4}, \frac{y}{4}\}$
$\leqslant \frac{1}{2}\max\{Gp(\frac{x}{2}, \frac{3y}{2}, \frac{3y}{2}), Gp(\frac{x}{2}, \frac{x}{6}, \frac{x}{6}), Gp(\frac{3y}{2}, \frac{y}{4}, \frac{y}{4}), \frac{1}{2}(Gp(\frac{x}{2}, \frac{x}{4}, \frac{x}{4}) + Gp(\frac{3y}{2}, \frac{3y}{2}, \frac{x}{6}))\}$,
i.e.,

$$Gp(Sx, Ty, Ty) = \frac{y}{4} \leqslant \frac{1}{2}Gp(\frac{x}{2}, \frac{3y}{2}, \frac{3y}{2}).$$

Case 2. For $x \in [0, 1]$ and $y \in (1, 2]$: $Gp(Sx, Ty, Ty) = \max\{\frac{x}{6}, \frac{1}{4}, \frac{1}{4}\}$
$\leqslant \frac{1}{2}\max\{Gp(\frac{x}{2}, \frac{3}{2}, \frac{3}{2}), Gp(\frac{x}{2}, \frac{x}{6}, \frac{x}{6}), Gp(\frac{3}{2}, \frac{1}{4}, \frac{1}{4}), \frac{1}{2}(Gp(\frac{x}{2}, \frac{1}{4}, \frac{1}{4}) + Gp(\frac{3}{2}, \frac{3}{2}, \frac{x}{6}))\}$,
i.e.,

$$Gp(Sx, Ty, Ty) = \frac{1}{4} \leqslant \frac{1}{2}Gp(\frac{x}{2}, \frac{3}{2}, \frac{3}{2}).$$

Case 3. For $x \in (1, 2]$ and $y \in [0, 1]$: $Gp(Sx, Ty, Ty) = \max\{\frac{1}{2}, \frac{y}{4}, \frac{y}{4}\}$
$\leqslant \frac{1}{2}\max\{Gp(\frac{5}{4}, \frac{3y}{2}, \frac{3y}{2}), Gp(\frac{5}{4}, \frac{1}{2}, \frac{1}{2}), Gp(\frac{3y}{2}, \frac{y}{4}, \frac{y}{4}), \frac{1}{2}(Gp(\frac{5}{4}, \frac{y}{4}, \frac{y}{4}) + Gp(\frac{3y}{4}, \frac{3y}{4}, \frac{1}{2}))\}$,
i.e.,

$$Gp(Sx, Ty, Ty) = \frac{1}{2} \leqslant \frac{1}{2}Gp(\frac{5}{4}, \frac{3y}{2}, \frac{3y}{2}).$$

Case 4. For $x, y \in (1, 2]$: $Gp(Sx, Ty, Ty) = \max\{\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\}$
$\leqslant \frac{1}{2}\max\{Gp(\frac{5}{4}, \frac{3}{2}, \frac{3}{2}), Gp(\frac{5}{4}, \frac{1}{2}, \frac{1}{2}), Gp(\frac{3}{2}, \frac{1}{4}, \frac{1}{4}), \frac{1}{2}(Gp(\frac{5}{4}, \frac{1}{4}, \frac{1}{4}) + Gp(\frac{3}{2}, \frac{3}{2}, \frac{1}{2}))\}$,
i.e.,

$$Gp(Sx, Ty, Ty) = \frac{1}{2} \leqslant \frac{1}{2}Gp(\frac{5}{4}, \frac{3}{2}, \frac{3}{2}).$$

Hence, all the hypotheses of Theorem 3.1 are satisfied (for $\delta = \frac{1}{2}$ and $L = 0$) and 0 is the unique common fixed point of A, B, S and T. One may notice that the mappings AX and SX are discontinuous.

Since Banach contraction [**9**], Chatterjea contraction [**11**], Kannan contraction [**15**], Zamfirescu contraction [**22**], quasi-contractions (C̀irìc [**12**]) are all contained in generalized condition (B), Theorem 3.1 extends, generalizes and improves existing results ([**1**], [**3**], [**8**, **9**], [**11**, **12**], [**15**], [**22**] and references there in) in GP-metric spaces using more natural condition of closedness of range spaces and demonstrate the significance of generalized condition (B), in the existence of coincidence and common fixed points.

For $A = B$ and $S = T$, we get the following Corollary:

Corollary 3.1. *Let A and T be self mappings of a GP-metric space (X, Gp). If A satisfies generalized condition (B) associated with T such that for all $x, y \in X$:*

(1) $TX \subset AX$,
(2) *AX is closed,*
(3) $\delta + L < 1$,

then A and T have a coincidence point. Further, A and T have a unique common fixed point provided that the pair (A,T) is weakly compatible.

Substituting $L=0$ in Theorem 3.1 we get the following Corollary:

COROLLARY 3.2. *Let A, B, S and T be self mappings of a GP-metric space (X,Gp). If the pairs (A,S) and (B,T) satisfy*
$G_p(Sx,Ty,Ty) \leqslant \delta \max\{G_p(Ax,By,By), Gp(Ax,Sx,Sx), G_p(By,Ty,Ty),$
$\frac{1}{2}(G_p(Sx,By,By)+G_p(Ax,Ty,Ty))\}$, $\delta \in (0,1)$
such that for all $x,y \in X$:
(1) $TX \subset AX$ *and* $SX \subset BX$,
(2) AX *or* BX *is closed,*

then the pairs (A,S) and (B,T) have a coincidence point. Further, A, B, S and T have a unique common fixed point provided that the pairs (A,S) and (B,T) are weakly compatible.

Substituting $L=0, A=B$ and $S=T$ in Theorem 3.1 we get the following Corollary:

COROLLARY 3.3. *Let A and T be self mappings of a GP-metric space (X,Gp). If the pair (A,T) satisfies*
$G_p(Tx,Ty,Ty) \leqslant \delta \max\{G_p(Ax,Ay,Ay), G_p(Ax,Tx,Tx), G_p(Ay,Ty,Ty),$
$\frac{1}{2}(G_p(Tx,Ay,Ay)+G_p(Ax,Ty,Ty))\}$, $\delta \in (0,1)$
such that for all $x,y \in X$:
(1) $TX \subset AX$,
(2) AX *is closed,*

then the pair (A,T) has a coincidence point. Further, A and T have a unique common fixed point provided that the pair (A,T) is weakly compatible.

COROLLARY 3.4. *Let A and T be self mappings of a GP-metric space (X,Gp). If the pair (A,T) satisfies*
$G_p(Tx,Ty,Ty) \leqslant \delta G_p(Ax,Ay,Ay)$, $\delta \in (0,1)$
such that for all $x,y \in X$:
(1) $TX \subset AX$,
(2) AX *is closed,*

then the pair (A,T) has a coincidence point. Further, A and T have a unique common fixed point provided that the pairs (A,T) is weakly compatible.

Instead of assuming range space AX or BX to be closed, if we consider closures of range space TX or SX, we get slightly more interesting result.

THEOREM 3.2. *Let A, B, S and T be self mappings of a GP-metric space (X,Gp). If there exist $\delta \in (0,1)$, $L \geqslant 0$ and pairs of mappings (A,S) and (B,T)*

satisfy

$$(3.3)\quad Gp(Sx,Ty,Ty) \leqslant \delta \max\{Gp(Ax,By,By), Gp(Ax,Sx,Sx), Gp(By,Ty,Ty), Gp(Ax,Ty,Ty), Gp(Sx,By,By)\} + L\min\{Gp(Ax,Sx,Sx), Gp(By,Ty,Ty), Gp(Ax,Ty,Ty), Gp(Sx,By,By)\}$$

such that for all $x,y \in X$*:*

(1) $\overline{TX} \subset AX$ *or* $\overline{SX} \subset BX$,

(2) $\delta + L < 1$,

then the pairs (A,S) *and* (B,T) *have a coincidence point. Further,* A, B, S *and* T *have a unique common fixed point provided that the pairs* (A,S) *and* (B,T) *are weakly compatible.*

Proof. It can also be proved following the similar arguments to those given in the proof of Theorem 3.1. □

Example 3.3. Let $X = [0,\infty)$ and the Gp metric: $Gp(x,y,z) = \max\{x,y,z\}$. Let A, B, S and T be mappings defined by:

$$Ax = \begin{cases} 2x, & 0 \leqslant x \leqslant 1 \\ 4, & x > 1, \end{cases} \qquad Bx = \begin{cases} \frac{3}{2}x, & 0 \leqslant x \leqslant 1 \\ 2, & x > 1, \end{cases}$$

$$Sx = \begin{cases} \frac{x}{2}, & 0 \leqslant x \leqslant 1 \\ \frac{1}{4}, & x > 1, \end{cases} \qquad Tx = \begin{cases} 0, & 0 \leqslant x \leqslant 1 \\ \frac{1}{2}, & x > 1. \end{cases}$$

Clearly, $\overline{SX} = [0,\frac{1}{2}] \subset [0,\frac{3}{2}] \cup \{2\} = BX$ and $\overline{TX} = \{0,\frac{1}{2}\} \subset [0,4] = AX$. The point 0 is a coincidence point of A,B,S and T. Further, $AS0 = SA0 = 0$ and $TB0 = BT0 = 0$, i.e., pairs (A,S) and (B,T) are weakly compatible.

Case 1. For $x,y \in [0,1]$: $Gp(Sx,Ty,Ty) = \max\{\frac{x}{2},0,0\} \leqslant \frac{4x}{3} = \frac{2}{3}\max\{Gp(2x,\frac{3y}{2},\frac{3y}{2}), Gp(2x,\frac{y}{2},\frac{y}{2}), Gp(\frac{3x}{2},0,0), Gp(2x,0,0), Gp(\frac{x}{2},\frac{3y}{2},\frac{3y}{2})\}$ or

$$Gp(Sx,Ty,Ty) = \frac{x}{2} \leqslant \frac{4x}{3} = \frac{2}{3}\{Gp(2x,\frac{3y}{2},\frac{3y}{2})\}.$$

Case 2. For $x \in [0,1]$ and $y > 1$: $Gp(Sx,Ty,Ty) = \max\{\frac{x}{2},\frac{1}{2},\frac{1}{2}\} \leqslant \frac{4x}{3} = \frac{2}{3}\max\{Gp(2x,2,2), Gp(2x,\frac{3y}{2},\frac{3y}{2}), Gp(2,\frac{1}{2},\frac{1}{2}), Gp(2x,\frac{1}{2},\frac{1}{2}), Gp(\frac{3x}{2},2,2)\}$ or

$$Gp(Sx,Ty,Ty) = \frac{x}{2} \leqslant \frac{4x}{3} = \frac{2}{3}\{Gp(2x,2,2)\}.$$

Case 3. For $x > 1$ and $y \in [0,1]$: $Gp(Sx,Ty,Ty) = \max\{\frac{1}{4},0,0\} \leqslant \frac{8}{3} = \frac{2}{3}\max\{Gp(4,\frac{3y}{2},\frac{3y}{2}), Gp(4,\frac{1}{4},\frac{1}{4}), Gp(\frac{3y}{2},0,0), Gp(4,0,0), Gp(\frac{1}{4},\frac{3y}{2},\frac{3y}{2})\}$ or

$$Gp(Sx,Ty,Ty) = \frac{1}{4} \leqslant \frac{8}{3} = \frac{2}{3}\{Gp(4,\frac{3y}{2},\frac{3y}{2})\}.$$

Case 4. For $x,y \in (1,\infty)$: $Gp(Sx,Ty,Ty) = \max\{\frac{1}{4},\frac{1}{2},\frac{1}{2}\} \leqslant \frac{4}{3} = \frac{2}{3}\max\{Gp(4,2,2), Gp(4,\frac{1}{4},\frac{1}{4}), Gp(2,\frac{1}{2},\frac{1}{2}), Gp(4,Ty,Ty), Gp(Sx,2,2)\}$ or

$$Gp(Sx,Ty,Ty) = \frac{1}{2} \leqslant \frac{4}{3} = \frac{2}{3}\{Gp(4,2,2)\}.$$

Hence, all hypotheses of Theorem 3.2 are satisfied (for $\delta = \frac{2}{3}$ and $L = 0$) and 0 is the unique common fixed point of A, B, S and T. One may notice that all the mappings are discontinuous.

For $A = B$ and $S = T$, Theorem 3.2 reduces to the following Corollary.

COROLLARY 3.5. *Let A and T be self mappings of a GP-metric space (X, Gp). If A satisfies generalized condition (B) associated with T, such that for all $x, y \in X$:*
(1) $\overline{TX} \subseteq AX$,
(2) $\delta + L < 1$,

then the pair (A, T) has a coincidence point. Further, A and T have a unique common fixed point provided that the pair (A, T) is weakly compatible.

Abbas et al. [**1**] presented two open problems:

I. Is Theorem 3.1 [**1**] valid for $\frac{1}{2} \leqslant \delta < 1$?
We provide two favourable answers in a non-complete GP-metric space assuming

(1) $TX \subset AX, SX \subset BX$, AX or BX to be closed and the pairs (A, S) and (B, T) to be weakly compatible. It is also demonstrated by an illustrative Example 3.1 that Theorem 3.1 is valid for $\delta = 1/2$.

(2) the closure of range space TX or SX ($\overline{TX} \subset AX$ or $\overline{SX} \subset BX$) and the pairs (A, S) and (B, T) to be weakly compatible. It is also demonstrated by an illustrative Example 3.2 that Theorem 3.2 is valid for $\delta = 2/3$.

II. Under what additional assumptions (Theorem 3.3, Berinde [**10**]), either on f and T or on the domain of f and T, do the mappings f and T have common fixed points ?
In a non-complete GP-metric space assuming

(1) $TX \subset AX$, AX to be closed and $\delta + L < 1$, the weakly compatible pair (A, T) of self mappings has a unique common fixed point (taking $f = A$ in Corollary 3.1).

(2) the closure of range space TX ($\overline{TX} \subset AX$) and $\delta + L < 1$, the weakly compatible pair (A, T) of self mappings has a unique common fixed point (taking $f = A$ in Corollary 3.5).

Hence, our both the Theorems 3.1 and 3.2 (Corollaries 3.1 and 3.5) extends the results of Berinde [**10**] to two pairs (a pair) of self mappings in GP-metric spaces.

REMARK 3.1. (i) Coincidence and unique common fixed point theorems have been established for two pairs of self mappings satisfying generalized condition (B) in a non-complete GP-metric space (X, Gp) without utilizing the notion of continuity or its variants (Tomar and Karapinar [**20**]). However, a more natural condition of closedness of the range space is assumed.

(ii) Generalized condition (B) does not reduce any metric condition as Gp is not a metric. Consequently, our results also do not reduce to the existing coincidence and common fixed point theorems in metric spaces.

4. Application to Integral Equations

Now we solve following integral equation using Corollary 3.4:

$$u(l) = \int_0^l K(l, s, u(s))ds + g(l), \tag{4.1}$$

where $l \in [0, L]$, $L > 0$, $K : [0, L] \times [0, L] \times \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$. Let $X = [0, L]$. Define $G_p : X \times X \times X \to \mathbb{R}^+$ by

$G_p(x, y, z) = sup_{l\in[0,L]}|x(l) - y(l)| + sup_{l\in[0,L]}|y(l) - z(l)| + sup_{l\in[0,L]}|z(l) - x(l)|$.

Then (X, G_p) is a GP-metric space.

Theorem 4.1. *Let T, $A : [0, L] \to [0, L]$ be self mappings of a GP-metric space (X, G_p) such that:*

1. *$K_1, K_2 : [0, L] \times [0, L] \times \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$,*
2. *there exists a function $G : [0, L] \times [0, L] \to [0, +\infty]$, such that*
$$|K_1(l, s, u(l)) - K_1(l, s, v(l)| \leqslant G(l, s)|Au - Av|,$$
for each $u, v \in \mathbb{R}$ and each l, $s \in [0, L]$,
3. *$sup_{l\in[0,L]} \int_0^l G(l, s)ds \leqslant \delta$ for some $\delta \in [0, 1)$,*
4. *$TX \subset AX$ and AX is closed,*
5. *$ATh = TAh$, whenever $Ah = Th$ for some $h \in [0, L]$.*

Then the integral equation (4.1) *has a unique solution $u \in [0, L]$.*

Proof. Define T, $A : X \to X$ by

$$Tx(l) = \int_0^l K_1(l, s, x(s))ds + g(l) \quad \text{and} \quad Ax(l) = \int_0^l K_2(l, s, x(s))ds + g(l), l \in [0, L],$$

such that $TX \subset AX$ and AX is closed. So,

$$\begin{aligned} G_p(Tx, Ty, Ty) &= sup_{l\in[0,L]}|Tx(l) - Ty(l)| + sup_{l\in[0,L]}|Tx(l) - Ty(l)| \\ &= 2sup_{l\in[0,L]}|Tx(l) - Ty(l)| \\ &= 2\left|\int_0^l K_1(l, s, x(s))ds - \int_0^l K_1(l, s, y(s))ds\right| \\ &\leqslant 2\int_0^l |K_1(l, s, x(s)) - K_1(l, s, y(s))|ds \\ &\leqslant 2\int_0^l G(l, s)|Ax(s) - Ay(s)|ds \end{aligned}$$

$$\leqslant 2(sup_{l\in[0,L]}|Ax(s)-Ay(s)|)sup_{l\in[0,L]}\int_0^l G(l,s)ds$$

$$= G_p(Ax,Ay,Ay)sup_{l\in[0,L]}\int_0^l G(l,s)ds.$$

By hypotheses (3), we have $G_p(Tx,Ty,Ty) \leqslant \delta G_p(Ax,Ay,Ay)$. Also (A,T) is weakly compatible by (5). Thus, all the hypotheses of Corollary 3.4 are satisfied and hence, there exists a unique common fixed point $u \in [0,L]$ of A and T, i.e., there exists a unique solution $u \in X$ of the integral equation (4.1). □

5. Application to Functional Equations Arising in Dynamic Programming Problem

Let U and V be Banach spaces, $W \subset U$ be a state space, $D \subset V$ be a decision space and R be the field of real numbers. Let $X = B(W)$ denotes the set of all closed and bounded real valued functions on W. Consider the following functional equation

$$p(x) = sup_{y\in D}\{g(x,y)+M(x,y,p(\tau(x,y)))\}, x\in W \tag{5.1}$$

where $g : W \times D \to \mathbb{R}$ and $M : W \times D \times \mathbb{R} \to \mathbb{R}$ are bounded functions. $\tau : W \times D \to W$ represents transformation of the process and $p(x)$ represents the optimal return function with initial state x. Also, $(B(W), \|.\|)$ is a Banach space wherein convergence is uniform.

Define $G_p : X \times X \times X \to \mathbb{R}^+$ by $G_p(x,y,z) = \max\{d(x,y), d(y,z), d(z,x)\}$, where $d : X \times X \to \mathbb{R}^+$ is defined as $d(x,y) = sup_{t\in W}(|x(t)-y(t)|)$, then (X, G_p) are GP-metric spaces.

Now we prove the existence and uniqueness of the solution of the functional equation (5.1) in a GP-metric space using Corollary 3.4.

Theorem 5.1. *Let $T, A : B(W) \to B(W)$ be self mappings of a GP-metric space $(B(W), G_p)$. If there exists a $\delta \in [0,1)$ such that for every $(x,y) \in W \times D$, Ah_1, $Ah_2 \in B(W)$ and $t \in W$:*

1. *$|M(x,y,Ah_1(t)) - M(x,y,Ah_2(t))| \leqslant \delta|Ah_1(t) - Ah_2(t)|$ holds,*
2. *$g : W \times D \to R$ and $M : W \times D \times \mathbb{R} \to \mathbb{R}$ are bounded functions,*
3. *$ATh = TAh$, whenever $Ah = Th$, for some $h \in B(W)$,*
4. *$T\{B(W)\} \subset A\{B(W)\}$ and $A\{B(W)\}$ is closed,*

then the functional equation

$$Th_i(x) = \sup_{y\in D}\{g(x,y)+M(x,y,Ah_i(\tau(x,y)))\}, x,y\in W, i=1,2 \tag{5.2}$$

has a unique bounded solution in $B(W)$.

Proof. By hypothesis (3), the pair (A,T) is weakly compatible. Let λ be an arbitrary positive real number and $Ah_1, Ah_2 \in B(W)$. For $x \in W$, we choose $y_1, y_2 \in D$ so that

$$T(h_1(x)) < g(x,y_1) + M(x,y_1,Ah_1(\tau_1)) + \lambda, \tag{5.3}$$

$$T(h_2(x)) < g(x, y_2) + M(x, y_2, Ah_2(\tau_2)) + \lambda, \tag{5.4}$$

where $\tau_1 = \tau(x, y_1)$ and $\tau_2 = \tau(x, y_2)$. From the definition of the mapping T, we have

$$T(h_1(x)) \geqslant g(x, y_2) + M(x, y_2, Ah_1(\tau_2)), \tag{5.5}$$

$$T(h_2(x)) \geqslant g(x, y_1) + M(x, y_1, Ah_2(\tau_1)). \tag{5.6}$$

Now, from (5.3) and (5.6), we obtain

$$\begin{aligned} T(h_1(x)) - T(h_2(x)) &< M(x, y_1, Ah_1(\tau_1)) - M(x, y_1, Ah_2(\tau_1)) + \lambda \\ &\leqslant |M(x, y_1, Ah_1(\tau_1)) - M(x, y_1, Ah_2(\tau_1))| + \lambda \\ &\leqslant \delta|Ah_1(x) - Ah_2(x)| + \lambda. \end{aligned}$$

Similarly, from (5.4) and (5.5), we obtain

$$T(h_2(x)) - T(h_1(x)) \leqslant \delta|Ah_1(x) - Ah_2(x)| + \lambda.$$

Hence, we have

$$|T(h_1(x)) - T(h_2(x))| \leqslant \delta|Ah_1(x) - Ah_2(x)| + \lambda. \tag{5.7}$$

Since the inequality (5.7) is true for all $x \in W$ and arbitrary $\lambda > 0$, then we have

$$G_p(Th_1, Th_2, Th_2) \leqslant \delta G_p(Ah_1, Ah_2, Ah_2).$$

Thus, using (4) all the conditions of Corollary 3.4 are satisfied and hence, the mappings, A and T have a unique common fixed point, i.e., the functional equation (5.1) has a unique bounded solution. $\square$

Conclusion. In GP-metric spaces, the generalized condition (B) is introduced to establish coincidence and common fixed point for two discontinuous weakly compatible pairs using more natural condition of closedness of the range space. It is worth mentioning here that weak compatibility is still the minimal and most widely used notion among all weaker modifications of commutativity (Singh and Tomar [**19**]). Further, we postulated two more favorable answers to each of the two open problems presented by Abbas et al. [**1**] regarding the existence of common fixed point. In the end obtained results are exploited to establish the existence and uniqueness of a solution of the integral equation and the functional equation.

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