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A NOTE ON COHEN-MACAULAY SKELETONS OF CYCLES AND VERTEX-DECOMPOSABILITY

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ABSTRACT. Let G be a graph and $I(G)$ be its edge ideal then $k[\mathrm{Ind}(G)] = k[x_1, ..., x_n]/I(G)$ is the Stanley-Reisner ring of G. We compute the depth invariant of $k[\mathrm{Ind}(G)]$ for independence complexes of cycle graphs. In addition to this we introduce an operation that allows us to generate vertex-decomposable graphs.

1. Introduction

Let G be a simple undirected graph on the vertex set $V(G) = \{x_1, ..., x_n\}$. Let $R = k[x_1, ..., x_n]$ be a polynomial ring on n variables corresponding to $V(G)$. We define $I(G) = \{x_i x_j : \{x_i x_j\} \in E(G)\}$ where $E(G)$ is the edge set of G. Then the set $I(G)$ is called edge ideal of G and introduced by Villarreal [**10**]. The independence complex of G is a simplicial complex with vertex set $V(G)$ and the faces are the independent sets of G and denoted by $\mathrm{Ind}(G)$. The Stanley-Reisner ring over a field k of $\mathrm{Ind}(G)$ is the quotient ring $R = k[x_1, ..., x_n]/I(G)$ and denoted by $k[\mathrm{Ind}(G)]$. Krull dimension of $k[\mathrm{Ind}(G)]$ is the supremum of the longest chain of the strict inclusions of prime ideals of $k[\mathrm{Ind}(G)]$ and denoted by $\dim(k[\mathrm{Ind}(G)])$. Depth of a ring or module is very important invariant in commutative algebra. For a Stanley-Reisner ring of an independence complex of a graph G, the $\mathrm{depth}(k[\mathrm{Ind}(G)])$ is the longest homogeneous sequence $f_1, ..., f_k$ such that f_i is not a zero-divisor of $k[x_1, ..., x_n]/(I, f_1, ..., f_{i-1})$ for all $1 \leqslant i \leqslant k$ and studied in [**7, 6, 5**]. The projective dimension of $k[\mathrm{Ind}(G)]$ is the shortest length of a projective free resolution of $k[x_1, ..., x_n]/I(G)$, denoted by $\mathrm{pd}(k[\mathrm{Ind}(G)])$. The projective dimension of Stanley-Reisner rings is also recently well-studied in many papers i.e [**4, 5**]. Moreover,

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by Auslander-Buchsbaum formula [**1**] computing depth of a ring gives information about its projective dimension.

Let G be a graph and $\mathrm{Ind}(G)$ be the independence complex of G such that $\mathrm{depth}(k[\mathrm{Ind}(G)]) = \dim(k[\mathrm{Ind}(G)])$, then we call $\mathrm{Ind}(G)$ a Cohen-Macaulay complex and G a Cohen-Macaulay graph. We study vertex-decomposability of pure skeletons of independence complexes of cycles to determine in which dimension its skeletons are Cohen-Macaulay. Because, if a pure simplicial complex vertex-decomposable then it is a Cohen-Macaulay complex [**8**]. Vertex-decomposability is firstly introduced for pure complexes by Provan and Billera [**8**] and later extended to non-pure complexes by Björner and Wachs in [**3**]. Vertex-decomposability is well studied object i.e. for chordal graphs [**11**], bipartite graphs [**9**] and codismantlable graphs [**2**].

In section 3, we compute the depth of independence complexes of cycle graphs via vertex-decomposability. In section 4, we introduce an operation to find vertex-decomposable simple graphs.

2. Preliminaries

Let $G = (V, E)$ be a undirected simple graph without multiple edges. The set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ is called open neighborhood of v and the closed neighborhood of v is the set $N_G[v] = N_G(v) \cup \{v\}$. A graph G is said to be well-covered, if all maximal independent sets of G has the same cardinality. If G is well-covered graph, then $\mathrm{Ind}(G)$ is pure simplicial complex.

Definition 2.1. A vertex x is called shedding vertex of G if no independent set of $G \backslash N_G[x]$ is a maximal independent set of $G \backslash x$. A graph G is said to be vertex-decomposable, if G is an edgeless graph or for any shedding vertex x, $G \setminus x$ and $G \backslash N_G[x]$ are vertex-decomposable.

The above definition of vertex-decomposability for graphs is originated from independence complexes [**8**]. So vertex-decomposability for independence complexes of G is given:

Definition 2.2. A vertex x is a shedding vertex for $\mathrm{Ind}(G)$ if for any face $\sigma \in \mathrm{Ind}(G)$ with $x \in \sigma$, there exists a vertex $y \in V(G)$ such that $(\sigma \backslash \{x\}) \cup \{y\}$ is a face of $\mathrm{Ind}(G)$. Independece complex of $\mathrm{Ind}(G)$ of G is called vertex-decomposable if $\mathrm{Ind}(G)$ is a simplex or has a shedding vertex x such that both $\mathrm{Ind}(G \backslash x)$ and $\mathrm{Ind}(G \backslash N_G[x])$ are vertex-decomposable.

Let G be a graph, recall that a set of vertices $A \subseteq V(G)$ is a dominating set if every vertex of $V(G) \backslash A$ is adjacent at least one vertex of A. Independence domination number of G is as follows.

Definition 2.3. For any graph G,

$$i(G) = \min\{|A| : A \subseteq V(G) \text{ is independent and a dominating set of } G\}$$

is called independence domination number of G.

Definition 2.4. (Auslander-Buchsbaum) Let $R = k[x_1, ..., x_n]$ be polynomial ring over a field k and G be a graph on vertex set $V(G) = \{x_1, ..., x_n\}$ and $I(G)$ be its edge ideal. Then,

$$\operatorname{depth}(R/I(G)) + \operatorname{pd}(R/I(G)) = n.$$

3. Depth of Cycles

In this section, we calculate the depth of Stanley-Reisner rings of cycle graphs. The i-skeleton of complex Δ of dimension d is the simplical complex consists of i-dimensional faces of Δ such that $i \leqslant d$. The depth of a simplicial complex Δ is the maximum i such that i-skeleton of Δ is Cohen-Macaulay [**12**].

Lemma 3.1. *If G is a vertex-decomposable graph with independence domination number $i(G)$, then* $\operatorname{depth}(k[\operatorname{Ind}(G)]) = i(G)$.

Proof. Since any vertex-decomposable complex is Cohen-Macaulay in the dimension of minimal facet, an independence complex has minimal facet in the dimension of $i(G)$ of graph G. □

Lemma 3.2. *Let C_n be a cycle on n vertices. If $n \equiv 0$ or $n \equiv 2 \mod 3$, then $i(C_n) = i(C_n - v)$ for any $v \in V(C_n)$.*

Remark 3.1. Let C_n and P_n be cycle and path graphs on n vertices respectively. The followings are their independence domination numbers.

$$i(P_n) = \begin{cases} \frac{n}{3} & ,\text{if } n \equiv 0 \mod 3 \\ \frac{n+2}{3} & ,\text{if } n \equiv 1 \mod 3 \\ \frac{n+1}{3} & ,\text{if } n \equiv 2 \mod 3 \end{cases}$$

$$i(C_n) = \begin{cases} \frac{n}{3} & ,\text{if } n \equiv 0 \mod 3 \\ \frac{n-1}{3} & ,\text{if } n \equiv 1 \mod 3 \\ \frac{n+1}{3} & ,\text{if } n \equiv 2 \mod 3 \end{cases}$$

Lemma 3.3. *Let G be a graph with independece domination number $i(G)$, then any $v \in V(G)$ is a shedding vertex for $(i(G) - 1)$-dimensional skeleton of* $\operatorname{Ind}(G)$.

Proof. Assume that σ is a face of $(i(G) - 1)$-dimensional skeleton of $\operatorname{Ind}(G)$ with $v \in \sigma$. Then $\mid \sigma \mid \leqslant i(G) - 1$. Since any maximal independent set of G has cardinality at least $i(G)$, it follows that there exist some vertices $w \in V(G)$ such that $(\sigma \setminus v) \cup \{w\}$ is independent. □

Theorem 3.1. *If $n = 3k + 2$ or $n = 3k$ for $k \in \mathbb{Z}^+$, then* $\operatorname{depth}(k[\operatorname{Ind}(C_n))] = i(C_n) - 1$.

Proof. By Lemma 3.3, we know that any vertex v is a shedding vertex for $(i(C_n) - 1)$-dimensional skeleton of $\operatorname{Ind}(C_n)$. Since $\operatorname{Ind}(C_n \setminus v) = \operatorname{Ind}(P_{n-1})$ and $\operatorname{Ind}(C_n \setminus N_{C_n}[v]) = \operatorname{Ind}(P_{n-3})$, from Remark 3.1 and Lemma 3.2 it is easy to see that $i(C_n) = i(C_n \setminus v) = i(P_{n-1})$ and $i(C_n \setminus N_{C_n}[v]) = i(P_{n-3}) = i(C_n) - 1$. From Lemma 3.1, $\operatorname{Ind}(P_{n-1})$ is $i(P_{n-1}) = i(C_n)$ dimensional Cohen-Macaulay graph and $\text{Ind}(P_{n-3})$ is $i(P_{n-3}) = i(C_n) - 1$ dimensional Cohen-Macaulay graph, because they are both vertex-decomposable. Therefore, $\text{Ind}(C_n)$ is $(C_n - 1)$ dimensional and vertex-decomposable and $i(P_{n-3})$ is $i(C_n) - 1$ dimensional vertex decomposable comlexes. Hence using Lemma 3.1 and lowering the dimension of skeleton to find a shedding vertex results that $(i(C_n) - 1)$-dimensional skeleton of $\text{Ind}(C_n \setminus v)$ and $(i(C_n) - 2)$-dimensional skeleton of $\text{Ind}(C_n \setminus N_{C_n}[v])$ are pure and vertex-decomposable by induction. Therefore $(i(C_n) - 1)$-dimensional skeleton of $\text{Ind}(C_n)$ is Cohen-Macaulay. Since $\text{depth}(k[\text{Ind}(G)]) = r$ if and only if r-dimensional skeletons of $\text{Ind}(G)$ is pure and vertex-decomposable, we conclude that $\text{depth}(k[\text{Ind}(C_n))] = i(C_n) - 1$. □

4. Directed Graphs and Vertex-Decomposability

In [**2**], the authors introduced an operation on directed graphs that allows to construct a simple graph which is vertex-decomposable. Let $\overrightarrow{G} = (V, E)$ be a directed graph, they called this simple graph arising from $\overrightarrow{G}$ a common-enemy graph of $\overrightarrow{G}$ and denoted by $CE(\overrightarrow{G})$. And $x \Longrightarrow y$ means a directed path starting from x ending at y. They define enemy set of a vertex u of V by $A(u) = \{v \in V : v \Longrightarrow u\}$ and set $A[u] = A(u) \cup \{u\}$. Let $E(CE(\overrightarrow{G}))$ be edge set of $CE(\overrightarrow{G})$, so $xy \in E(CE(\overrightarrow{G}))$ if and only if $x \neq y$ and $A[x] \cap A[y] \neq \emptyset$. They proved that if $\overrightarrow{G}$ is acyclic directed graph, then $CE(\overrightarrow{G})$ is vertex-decomposable. In this section, we introduce a new operation called *close gap operation* on directed graphs. Close-gap graph of a directed graph $\overrightarrow{G}$ is a simple graph arising from this $\overrightarrow{G}$ and denoted by $CP(\overrightarrow{G})$. We define enemy set and edge set of $CP(\overrightarrow{G})$ in similar way of $CE(\overrightarrow{G})$. In particular, if $\overrightarrow{G}$ is a directed graph and $x_0, x_1, x_2, ..., x_n$ is a directed path on $\overrightarrow{G}$, then in this operation deleting a vertex x_1 from directed path $x_0, \hat{x_1}, x_2, ..., x_n$ yields a new directed path $x_0, x_2, ..., x_n$.

Remark 4.1. Let x be a vertex of $V(\overrightarrow{G})$ with zero out-degree and positive in-degree. Since there is no directed path starting from x ending at w for any $w \in V(\overrightarrow{G})$, then we have $CE(\overrightarrow{G} - x) \cong CP(\overrightarrow{G} - x)$ and $CE(\overrightarrow{G} - N_{CE(\overrightarrow{G})}[x]) \cong CP(\overrightarrow{G} - N_{CP(\overrightarrow{G})}[x])$.

A vertex x of G is called codominated vertex if $N_G[y] \subseteq N_G[x]$ for some vertex y in $N_G(y)$. The next lemma shows the relation between shedding and codominated vertices.

Lemma 4.1. [**2**] *If x is codominated vertex of G, then x is a shedding vertex.*

Remark 4.2. A vertex x of G is called if induced graph on $N_G(x)$ is a clique. It is clear that every vertex of $N_G(x)$ is codominated hence shedding vertex.

Theorem 4.1. *Let $\overrightarrow{G}$ be a directed graph, then $CP(\overrightarrow{G})$ is vertex-decomposable.*

Proof. Let $\overrightarrow{G}$ be an acyclic directed graph. Then its common-enemy graph is vertex decomposable by [**2**], and from Remark 4.1, $CP(\overrightarrow{G})$ is vertex-decomposable as well. Hence assume that $\overrightarrow{G}$ has at least one directed cycle. Let $\{x_1, x_2, ..., x_n, x_1\}$ be a directed cycle in $\overrightarrow{G}$. Since $xy \in E(CE(\overrightarrow{G}))$ if and only if $x \neq y$ and $A[x] \cap A[y] \neq \emptyset$, then $\{x_1, x_2, ..., x_n\}$ can induce a clique in $CP(\overrightarrow{G})$. So every vertex in $\{x_1, x_2, ..., x_n\}$ is codominated by Remark 4.2, hence it is shedding vertex. Without loss of generality, we assume that x_1 is a shedding vertex in $CP(\overrightarrow{G})$. It is sufficient to show that $CP(\overrightarrow{G} - x_1)$ and $CP(\overrightarrow{G} - N_{CP(\overrightarrow{G})}[x_1])$ are vertex-decomposable graphs. Assume that $ab \in E(CP(\overrightarrow{G} - x_1))$. Then $v \in A[a] \cap A[b]$ for some $v \in V(\overrightarrow{G})$. If $v = x_1$ then for $i = \{0, 2, ..., n\}$, $x_i \in A[a] \cap A[b]$ in $E(CP(\overrightarrow{G}) - x_1)$, since $\{x_0, x_1, ..., x_n\}$ is a directed path. Assume that there is a directed path from v to a and b. Let ab be in $E(CP(\overrightarrow{G} - N_{CP(\overrightarrow{G})}[x_1])$. So $w \in A[a] \cap A[b]$ for some $w \in V(\overrightarrow{G})$. If $w \in N[x_1]$, then clearly a and b both must be in $N[x_1]$, which is impossible. So $ab \in E(CP(\overrightarrow{G} - N_{CP(\overrightarrow{G})}[x_1]))$. Therefore, we have both $CP(\overrightarrow{G} - x_1)$ and $CP(\overrightarrow{G} - N_{CP(\overrightarrow{G})}[x_1])$ are vertex-decomposable graphs by induction. □

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