A NOTE ON COHEN-MACAULAY SKELETONS OF CYCLES AND VERTEX-DECOMPOSABILITY

Tahsin Öner and Alper Ülker

Abstract. Let $G$ be a graph and $I(G)$ be its edge ideal then $k[\text{Ind}(G)] = k[x_1, ..., x_n]/I(G)$ is the Stanley-Reisner ring of $G$. We compute the depth invariant of $k[\text{Ind}(G)]$ for independence complexes of cycle graphs. In addition to this we introduce an operation that allows us to generate vertex-decomposable graphs.

1. Introduction

Let $G$ be a simple undirected graph on the vertex set $V(G) = \{x_1, ..., x_n\}$. Let $R = k[x_1, ..., x_n]$ be a polynomial ring on $n$ variables corresponding to $V(G)$. We define $I(G) = \{x_i x_j : \{x_i, x_j\} \in E(G)\}$ where $E(G)$ is the edge set of $G$. Then the set $I(G)$ is called edge ideal of $G$ and introduced by Villarreal [10]. The independence complex of $G$ is a simplicial complex with vertex set $V(G)$ and the faces are the independent sets of $G$ and denoted by $\text{Ind}(G)$. The Stanley-Reisner ring over a field $k$ of $\text{Ind}(G)$ is the quotient ring $R = k[x_1, ..., x_n]/I(G)$ and denoted by $k[\text{Ind}(G)]$. Krull dimension of $k[\text{Ind}(G)]$ is the supremum of the longest chain of the strict inclusions of prime ideals of $k[\text{Ind}(G)]$ and denoted by $\text{dim}(k[\text{Ind}(G)])$. Depth of a ring or module is very important invariant in commutative algebra. For a Stanley-Reisner ring of an independence complex of a graph $G$, the depth($k[\text{Ind}(G)]$) is the longest homogeneous sequence $f_1, ..., f_k$ such that $f_i$ is not a zero-divisor of $k[x_1, ..., x_n]/(I, f_1, ..., f_{i-1})$ for all $1 \leq i \leq k$ and studied in [7, 6, 5]. The projective dimension of $k[\text{Ind}(G)]$ is the shortest length of a projective free resolution of $k[x_1, ..., x_n]/I(G)$, denoted by $\text{pd}(k[\text{Ind}(G)])$. The projective dimension of Stanley-Reisner rings is also recently well-studied in many papers i.e [4, 5]. Moreover,

2010 Mathematics Subject Classification. Primary 13F20; Secondary 05C69.

Key words and phrases. Vertex-Decomposability, Independence Complex, Depth.
by Auslander-Buchsbaum formula \[1\] computing depth of a ring gives information about its projective dimension.

Let \( G \) be a graph and \( \text{Ind}(G) \) be the independence complex of \( G \) such that \( \text{depth}(k[\text{Ind}(G)]) = \dim(k[\text{Ind}(G)]) \), then we call \( \text{Ind}(G) \) a Cohen-Macaulay complex and \( G \) a Cohen-Macaulay graph. We study vertex-decomposability of pure skeletons of independence complexes of cycles to determine in which dimension its skeletons are Cohen-Macaulay. Because, if a pure simplicial complex vertex-decomposable then it is a Cohen-Macaulay complex \[8\]. Vertex-decomposability is firstly introduced for pure complexes by Provan and Billera \[8\] and later extended to non-pure complexes by Björner and Wachs in \[3\]. Vertex-decomposability is well studied object i.e. for chordal graphs \[11\], bipartite graphs \[9\] and codismantlable graphs \[2\].

In section 3, we compute the depth of independence complexes of cycle graphs via vertex-decomposability. In section 4, we introduce an operation to find vertex-decomposable simple graphs.

2. Preliminaries

Let \( G = (V, E) \) be a undirected simple graph without multiple edges. The set \( N_G(v) = \{ u \in V(G) : uv \in E(G) \} \) is called open neighborhood of \( v \) and the closed neighborhood of \( v \) is the set \( N_G[v] = N_G(v) \cup \{ v \} \). A graph \( G \) is said to be well-covered, if all maximal independent sets of \( G \) has the same cardinality. If \( G \) is well-covered graph, then \( \text{Ind}(G) \) is pure simplicial complex.

**Definition 2.1.** A vertex \( x \) is called shedding vertex of \( G \) if no independent set of \( G \setminus N_G[x] \) is a maximal independent set of \( G \setminus x \). A graph \( G \) is said to be vertex-decomposable, if \( G \) is an edgeless graph or for any shedding vertex \( x \), \( G \setminus x \) and \( G \setminus N_G[x] \) are vertex-decomposable.

The above definition of vertex-decomposability for graphs is originated from independence complexes \[8\]. So vertex-decomposability for independence complexes of \( G \) is given:

**Definition 2.2.** A vertex \( x \) is a shedding vertex for \( \text{Ind}(G) \) if for any face \( \sigma \in \text{Ind}(G) \) with \( x \in \sigma \), there exists a vertex \( y \in V(G) \) such that \( (\sigma \setminus \{ x \}) \cup \{ y \} \) is a face of \( \text{Ind}(G) \). Independence complex of \( \text{Ind}(G) \) of \( G \) is called vertex-decomposable if \( \text{Ind}(G) \) is a simplex or has a shedding vertex \( x \) such that both \( \text{Ind}(G \setminus x) \) and \( \text{Ind}(G \setminus N_G[x]) \) are vertex-decomposable.

Let \( G \) be a graph, recall that a set of vertices \( A \subseteq V(G) \) is a dominating set if every vertex of \( V(G) \setminus A \) is adjacent at least one vertex of \( A \). Independence domination number of \( G \) is as follows.

**Definition 2.3.** For any graph \( G \),

\[
i(G) = \min\{|A| : A \subseteq V(G) \text{ is independent and a dominating set of } G\}
\]

is called independence domination number of \( G \).
Definition 2.4. (Auslander-Buchsbaum) Let $R = k[x_1, \ldots, x_n]$ be polynomial ring over a field $k$ and $G$ be a graph on vertex set $V(G) = \{x_1, \ldots, x_n\}$ and $I(G)$ be its edge ideal. Then,

$$\text{depth}(R/I(G)) + \text{pd}(R/I(G)) = n.$$  

3. Depth of Cycles

In this section, we calculate the depth of Stanley-Reisner rings of cycle graphs. The $i$-skeleton of complex $\Delta$ of dimension $d$ is the simplical complex consists of $i$-dimensional faces of $\Delta$ such that $i \leq d$. The depth of a simplicial complex $\Delta$ is the maximum $i$ such that $i$-skeleton of $\Delta$ is Cohen-Macaulay [12].

Lemma 3.1. If $G$ is a vertex-decomposable graph with independence domination number $i(G)$, then $\text{depth}(k[\text{Ind}(G)]) = i(G)$.

Proof. Since any vertex-decomposable complex is Cohen-Macaulay in the dimension of minimal facet, an independence complex has minimal facet in the dimension of $i(G)$ of graph $G$. □

Lemma 3.2. Let $C_n$ be a cycle on $n$ vertices. If $n \equiv 0$ or $n \equiv 2 \mod 3$, then $i(C_n) = i(C_n - v)$ for any $v \in V(C_n)$.

Remark 3.1. Let $C_n$ and $P_n$ be cycle and path graphs on $n$ vertices respectively. The followings are their independence domination numbers.

$$i(P_n) = \begin{cases} \frac{n}{3}, & \text{if } n \equiv 0 \mod 3 \\ \frac{n+2}{3}, & \text{if } n \equiv 1 \mod 3 \\ \frac{n+1}{3}, & \text{if } n \equiv 2 \mod 3 \end{cases}$$

$$i(C_n) = \begin{cases} \frac{n}{3}, & \text{if } n \equiv 0 \mod 3 \\ \frac{n-1}{3}, & \text{if } n \equiv 1 \mod 3 \\ \frac{n+1}{3}, & \text{if } n \equiv 2 \mod 3 \end{cases}$$

Lemma 3.3. Let $G$ be a graph with independence domination number $i(G)$, then any $v \in V(G)$ is a shedding vertex for $(i(G) - 1)$-dimensional skeleton of $\text{Ind}(G)$.

Proof. Assume that $\sigma$ is a face of $(i(G) - 1)$-dimensional skeleton of $\text{Ind}(G)$ with $v \in \sigma$. Then $|\sigma| \leq i(G) - 1$. Since any maximal independent set of $G$ has cardinality at least $i(G)$, it follows that there exist some vertices $w \in V(G)$ such that $(\sigma \setminus v) \cup \{w\}$ is independent. □

Theorem 3.1. If $n = 3k + 2$ or $n = 3k$ for $k \in \mathbb{Z}^+$, then $\text{depth}(k[\text{Ind}(C_n)])] = i(C_n) - 1$.

Proof. By Lemma 3.3, we know that any vertex $v$ is a shedding vertex for $(i(C_n) - 1)$-dimensional skeleton of $\text{Ind}(C_n)$. Since $\text{Ind}(C_n \setminus v) = \text{Ind}(P_{n-1})$ and $\text{Ind}(C_n \setminus N_{C_n}[v]) = \text{Ind}(P_{n-3})$, from Remark 3.1 and Lemma 3.2 it is easy to see that $i(C_n) = i(C_n \setminus v) = i(P_{n-1})$ and $i(C_n \setminus N_{C_n}[v]) = i(P_{n-3}) = i(C_n) - 1$. From Lemma 3.1, $\text{Ind}(P_{n-1})$ is $i(P_{n-1}) = i(C_n)$ dimensional Cohen-Macaulay
graph and \( \text{Ind}(P_{n-3}) \) is \( i(P_{n-3}) = i(C_n) - 1 \) dimensional Cohen-Macaulay graph, because they are both vertex-decomposable. Therefore, \( \text{Ind}(C_n) \) is \( (C_n - 1) \) dimensional and vertex-decomposable and \( i(P_{n-3}) \) is \( i(C_n) - 1 \) dimensional vertex decomposable complexes. Hence using Lemma 3.1 and lowering the dimension of skeleton to find a shedding vertex results that \( (i(C_n) - 1) \)-dimensional skeleton of \( \text{Ind}(C_n \setminus v) \) and \( (i(C_n) - 2) \)-dimensional skeleton of \( \text{Ind}(C_n \setminus N_{C_n}[v]) \) are pure and vertex-decomposable by induction. Therefore \( (i(C_n) - 1) \)-dimensional skeleton of \( \text{Ind}(C_n) \) is Cohen-Macaulay. Since depth\( (k|\text{Ind}(C_n)) = r \) if and only if \( r \)-dimensional skeletons of \( \text{Ind}(G) \) is pure and vertex-decomposable, we conclude that depth\( (k|\text{Ind}(C_n)) \) = 1.

\[ \square \]

4. Directed Graphs and Vertex-Decomposability

In [2], the authors introduced an operation on directed graphs that allows to construct a simple graph which is vertex-decomposable. Let \( \overrightarrow{G} = (V, E) \) be a directed graph, they called this simple graph arising from \( 
\overrightarrow{G} \) a common-enemy graph of \( \overrightarrow{G} \) and denoted by \( CE(\overrightarrow{G}) \). And \( x \Rightarrow y \) means a directed path starting from \( x \) ending at \( y \). They define enemy set of a vertex \( u \) of \( \overrightarrow{V} \) by \( A(u) = \{ v \in \overrightarrow{V} : v \Rightarrow u \} \) and set \( A(u) = A(u) \cup \{ u \} \). Let \( E(CE(\overrightarrow{G})) \) be edge set of \( CE(\overrightarrow{G}) \), so \( xy \in E(CE(\overrightarrow{G})) \) if and only if \( x \neq y \) and \( A[x] \cap A[y] \neq \emptyset \). They proved that if \( \overrightarrow{G} \) is acyclic directed graph, then \( CE(\overrightarrow{G}) \) is vertex-decomposable. In this section, we introduce a new operation called close gap operation on directed graphs. Close-gap graph of a directed graph \( \overrightarrow{G} \) is a simple graph arising from this \( \overrightarrow{G} \) and denoted by \( CP(\overrightarrow{G}) \). We define enemy set and edge set of \( CP(\overrightarrow{G}) \) in similar way of \( CE(\overrightarrow{G}) \).

In particular, if \( \overrightarrow{G} \) is a directed graph and \( x_0, x_1, x_2, \ldots, x_n \) is a directed path on \( \overrightarrow{G} \), then in this operation deleting a vertex \( x_1 \) from directed path \( x_0, x_1, x_2, \ldots, x_n \) yields a new directed path \( x_0, x_1, x_2, \ldots, x_n \).

**Remark 4.1.** Let \( x \) be a vertex of \( V(\overrightarrow{G}) \) with zero out-degree and positive in-degree. Since there is no directed path starting from \( x \) ending at \( w \) for any \( w \in V(\overrightarrow{G}) \), then we have \( CE(\overrightarrow{G} - x) \cong CP(\overrightarrow{G} - x) \) and \( CE(\overrightarrow{G} - N_{CE(\overrightarrow{G})}[x]) \cong CP(\overrightarrow{G} - N_{CP(\overrightarrow{G})}[x]) \).

A vertex \( x \) of \( G \) is called codominated vertex if \( N_{G}[y] \subseteq N_{G}[x] \) for some vertex \( y \) in \( N_{G}(y) \). The next lemma shows the relation between shedding and codominated vertices.

**Lemma 4.1.** [2] If \( x \) is codominated vertex of \( G \), then \( x \) is a shedding vertex.

**Remark 4.2.** A vertex \( x \) of \( G \) is called if induced graph on \( N_{G}(x) \) is a clique. It is clear that every vertex of \( N_{G}(x) \) is codominated hence shedding vertex.

**Theorem 4.1.** Let \( \overrightarrow{G} \) be a directed graph, then \( CP(\overrightarrow{G}) \) is vertex-decomposable.

**Proof.** Let \( \overrightarrow{G} \) be an acyclic directed graph. Then its common-enemy graph is vertex decomposable by [2], and from Remark 4.1, \( CP(\overrightarrow{G}) \) is vertex-decomposable.
as well. Hence assume that \( \overrightarrow{G} \) has at least one directed cycle. Let \( \{x_1, x_2, \ldots, x_n, x_1\} \) be a directed cycle in \( \overrightarrow{G} \). Since \( xy \in E(CE(\overrightarrow{G})) \) if and only if \( x \neq y \) and \( A[x] \cap A[y] \neq \emptyset \), then \( \{x_1, x_2, \ldots, x_n\} \) can induce a clique in \( CP(\overrightarrow{G}) \). So every vertex in \( \{x_1, x_2, \ldots, x_n\} \) is codominated by Remark 4.2, hence it is shedding vertex. Without loss of generality, we assume that \( x_1 \) is a shedding vertex in \( CP(\overrightarrow{G}) \). It is sufficient to show that \( CP(\overrightarrow{G} - x_1) \) and \( CP(\overrightarrow{G} - N_{CP(\overrightarrow{G})}[x_1]) \) are vertex-decomposable graphs. Assume that \( ab \in E(\overrightarrow{G} - x_1) \). Then \( v \in A[a] \cap A[b] \) for some \( v \in V(\overrightarrow{G}) \). If \( v = x_1 \) then for \( i = \{0, 2, \ldots, n\}, x_i \in A[a] \cap A[b] \) in \( E(\overrightarrow{G} - x_1) \), since \( \{x_0, x_1, \ldots, x_n\} \) is a directed path. Assume that there is a directed path from \( v \) to \( a \) and \( b \). Let \( ab \) be in \( E(\overrightarrow{G} - N_{CP(\overrightarrow{G})}[x_1]) \). So \( w \in A[a] \cap A[b] \) for some \( w \in V(\overrightarrow{G}) \). If \( w \in N[x_1] \), then clearly \( a \) and \( b \) both must be in \( N[x_1] \), which is impossible. So \( ab \in E(\overrightarrow{G} - N_{CP(\overrightarrow{G})}[x_1]) \). Therefore, we have both \( CP(\overrightarrow{G} - x_1) \) and \( CP(\overrightarrow{G} - N_{CP(\overrightarrow{G})}[x_1]) \) are vertex-decomposable graphs by induction. \( \square \)

References


Received by editors 04.10.2017; Revised version 02.10.2018; Available online 08.10.2018.

DEPARTMENT OF MATHEMATICS, EGE UNIVERSITY, İZMİR, TURKEY
E-mail address: tahsin.oner@ege.edu.tr

DEPARTMENT OF MATHEMATICS, Ağrı İbrahim Çeçen University, Ağrı, TURKEY
E-mail address: aulker@agri.edu.tr