BULLETIN OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4874, ISSN (o) 2303-4955
www.imvibl.org /JOURNALS / BULLETIN
Vol. 8(2018), 509-515

DOI: 10.7251/BIMVI1803509Ö

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

# AN INVESTIGATION OF SOLUTION OF FIRST ORDER COMPLEX COEFFICIENTS COMPLEX EQUATION BY USING ABOODH TRANSFORM

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Abstract. In this paper, we have studied the application of Aboodh transform to complex differendial equations with complex coefficients.This reliable transform has a wide application area for linear operator equations.

## 1. Introduction

Linear and nonlinear partial differential equations play an important role in wide variety scientific applications, physics, engineering and applied mathematics because many important mathematical models can be expressed through these equations. In recent years, many researchers such as Elzaki [**4, 5**], Aboodh [**1, 2**] have worked to find the solutions of linear and nonlinear differential equations by using various methods. One of those is Aboodh transform which is given by Khalid Aboodh to facilitate the process of solving ordinary and partial differential equations in the time domain [**1, 2**].

In this work we give a formulation for general first order complex equations with complex coefficients by using Aboodh transform.

## 2. Basic theorems of Aboodh transform

In this section, we summarized some basic theorems of Aboodh transform. Let $f(t)$ be a function for $t > 0$. The Aboodh transform which is denoted by the

2010 *Mathematics Subject Classification.* 65J10.

*Key words and phrases.* Aboodh transform, linear operator equations.

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operator $A[.]$ defined by the integral equations

$$A[f(t)] = \frac{1}{s}\int_0^\infty e^{-st} f(t)dt.$$

It is shown by Aboodh [**1**] that this transform has deeper connection with the Laplace and Elzaki transform.

Theorem 2.1. *Aboodh transform of some basic functions are given as*

**(i):** *If* $f(t) = 1$ *then* $A[1] = \frac{1}{s^2}$,

**(ii):** *If* $f(t) = t$ *then* $A[t] = \frac{1}{s^3}$,

**(iii):** *If* $f(t) = t^n$ *then* $A[t^n] = \frac{n!}{s^{n+2}}$,

**(iv):** *If* $f(t) = e^{at}$ *then* $A[e^{at}] = \frac{1}{s(s-a)}$,

**(v):** *If* $f(t) = \sin(at)$ *then* $A[\sin at] = \frac{a}{s(s^2+a^2)}$,

**(vi):** *If* $f(t) = \cos(at)$ *then* $A[\cos at] = \frac{1}{s^2+a^2}$.

Theorem 2.2. *Let* $A[f(x,t)] = K(x,s)$. *The Aboodh transforms of partial derivative of* $f(x,t)$ *are given as*

**(i):** $A[\frac{\partial f}{\partial t}] = sK(x,s) - \frac{1}{s} f(x,0)$,

**(ii):** $A[\frac{\partial f}{\partial x}] = \frac{\partial K(x,s)}{\partial x}$.

## 3. Solution of a general complex differential equation with complex constant coefficients

Theorem 3.1. *Let* $\tilde{A}, B, C, E$ *are complex constants and* $F(z,\overline{z})$ *is a polynomial of* $z, \overline{z}$. *Then the real and imaginary parts of solution of*

$$\begin{gathered} \tilde{A}\frac{\partial w}{\partial z} + B\frac{\partial w}{\partial \overline{z}} + Cw + E\overline{w} = F(z,\overline{z}) \\ w(x,0) = g(x) \end{gathered} \tag{3.1}$$

*are*

$$\begin{aligned} w_1 = -2A^{-1}\{P^{-1}\{\frac{\partial}{\partial x}[\,(a_1b_2 - a_2b_1)\,w_1(x,0) + \left(b_1^2 + b_2^2 - a_1^2 - a_2^2\right)\frac{w_2(x,0)}{2} \\ -sK_3(a_1+b_1) - sK_4(a_2+b_2)\,] \\ + \left[(b_2-a_2)(c_1-e_1) + (b_1-a_1)(e_2-c_2) - \frac{s}{2}\left((a_1-b_1)^2 + (a_2-b_2)^2\right)\right] w_1(x,0) \\ + [(a_2-b_2)(e_2-c_2) + (a_1-b_1)(e_1-c_1)]\,w_2(x,0) \\ -s\,(s\,(a_2-b_2) + 2c_1 - 2e_1)\,K_3 + s\,(s\,(a_1-b_1) + 2e_2 - 2c_2)\,K_4\}\} \end{aligned}$$

*and*

$$w_2 = -2A^{-1}\{P^{-1}\{\frac{\partial}{\partial x}[\left(a_1^2 + a_2^2 - b_1^2 - b_2^2\right)\frac{w_1(x,0)}{2} + (a_1b_2 - a_2b_1)\, w_2(x,0) + sK_3(a_2+b_2) - sK_4(a_1+b_1)]$$
$$+\left[(b_2-a_2)(c_1+e_1) + (a_1-b_1)(e_2+c_2) - \frac{s}{2}\left((a_1-b_1)^2 + (a_2-b_2)^2\right)\right] w_2(x,0)$$
$$+\left[(a_2-b_2)(e_2+c_2) + (a_1-b_1)(e_1+c_1)\right] w_1(x,0)$$
$$+s\left(s(b_1-a_1) + 2c_2 + 2e_2\right)K_3 + s\left(s(b_2-a_2) - 2e_1 - 2c_1\right)K_4\}\} \tag{3.2}$$

*where*

$P =$
$s\left((a_1+b_1)^2 + (a_2+b_2)^2\right)D^2 + 4s\left((b_1a_2 - a_1b_2)s + (a_1+b_1)c_1 + (a_2+b_2)c_2\right)D$
$+ s(s^2\left((a_1-b_1)^2 + (a_2-b_2)^2\right) + 4s\left((a_2-b_2)c_1 + (b_1-a_1)c_2\right)$
$+ 4\left(c_1^2 - e_1^2 + c_2^2 - e_2^2\right))$
*here* $a_j, b_j, c_j, e_j, w_j, f_j$ *is the real and imaginary part of* $\tilde{A}, B, C, E, W, F$ *for* $j = 1$ *and* $j = 2$*, respectively.*

Proof. By using of first derivative according to $z$ and $\overline{z}$

$$\frac{\partial w}{\partial z} = \frac{1}{2}\left(\frac{\partial w}{\partial x} - i\frac{\partial w}{\partial y}\right)$$
$$\frac{\partial w}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + i\frac{\partial w}{\partial y}\right)$$

Eq. (3.1) can be written due to this derivatives as

$$\tilde{A}\frac{1}{2}\left(\frac{\partial w}{\partial x} - i\frac{\partial w}{\partial y}\right) + B\frac{1}{2}\left(\frac{\partial w}{\partial x} + i\frac{\partial w}{\partial y}\right) + Cw + E\overline{w} = F(z,\overline{z}). \tag{3.3}$$

If we write terms of Eq. (3.3) explicitly and separate to real and imaginer part, then we obtain

$$(a_1+b_1)\tfrac{\partial w_1}{\partial x} + (a_2-b_2)\tfrac{\partial w_1}{\partial y} - (a_2+b_2)\tfrac{\partial w_2}{\partial x} + (a_1-b_1)\tfrac{\partial w_2}{\partial y} + 2(c_1+e_1)w_1 + 2(e_2-c_2)w_2 = 2f_1$$

and

$$(a_2+b_2)\tfrac{\partial w_1}{\partial x} + (b_1-a_1)\tfrac{\partial w_1}{\partial y} + (a_1+b_1)\tfrac{\partial w_2}{\partial x} + (a_2-b_2)\tfrac{\partial w_2}{\partial y} + 2(c_2+e_2)w_1 + 2(c_1-e_1)w_2 = 2f_2.$$

Let us apply the Aboodh transform to above equalities, then we have

$$(a_1+b_1)\frac{\partial K_1}{\partial x} + (a_2-b_2)\left(sK_1 - \frac{w_1(x,0)}{s}\right) - (a_2+b_2)\frac{\partial K_2}{\partial x} + (a_1-b_1)\left(sK_2 - \frac{w_2(x,0)}{s}\right) + 2(c_1+e_1)K_1 + 2(e_2-c_2)K_2 = 2K_3 \tag{3.4}$$

$$(a_2+b_2)\frac{\partial K_1}{\partial x}+(b_1-a_1)\left(sK_1-\frac{w_1(x,0)}{s}\right)+(a_1+b_1)\frac{\partial K_2}{\partial x} \\ +(a_2-b_2)\left(sK_2-\frac{w_2(x,0)}{s}\right)+2(c_2+e_2)K_1+2(c_1-e_1)K_2=2K_4. \tag{3.5}$$

Where $K_1, K_2, K_3, K_4$ are Aboodh transforms of $w_1, w_2, f_1, f_2$ respectively. By solving the equations (3.4) and (3.5) together, we get the solutions of this system as

$$K_1=-2P^{-1}\{\frac{\partial}{\partial x}[(a_1b_2-a_2b_1)w_1(x,0)+\left(b_1^2+b_2^2-a_1^2-a_2^2\right)\frac{w_2(x,0)}{2} \\ -sK_3(a_1+b_1)-sK_4(a_2+b_2)] \\ +\left[(b_2-a_2)(c_1-e_1)+(b_1-a_1)(e_2-c_2)-\frac{s}{2}\left((a_1-b_1)^2+(a_2-b_2)^2\right)\right]w_1(x,0) \\ +[(a_2-b_2)(e_2-c_2)+(a_1-b_1)(e_1-c_1)]w_2(x,0) \\ -s(s(a_2-b_2)+2c_1-2e_1)K_3+s(s(a_1-b_1)+2e_2-2c_2)K_4\}$$

and

$$K_2=-2P^{-1}\{\frac{\partial}{\partial x}[\left(a_1^2+a_2^2-b_1^2-b_2^2\right)\frac{w_1(x,0)}{2}+(a_1b_2-a_2b_1)w_2(x,0) \\ +sK_3(a_2+b_2)-sK_4(a_1+b_1)] \\ +\left[(b_2-a_2)(c_1+e_1)+(a_1-b_1)(e_2+c_2)-\frac{s}{2}\left((a_1-b_1)^2+(a_2-b_2)^2\right)\right]w_2(x,0) \\ +[(a_2-b_2)(e_2+c_2)+(a_1-b_1)(e_1+c_1)]w_1(x,0) \\ +s(s(b_1-a_1)+2c_2+2e_2)K_3+s(s(b_2-a_2)-2e_1-2c_1)K_4\}$$

where

$$P=s\left((a_1+b_1)^2+(a_2+b_2)^2\right)D^2+ \\ 4s((b_1a_2-a_1b_2)s+(a_1+b_1)c_1+(a_2+b_2)c_2)D+ \\ s(s^2\left((a_1-b_1)^2+(a_2-b_2)^2\right)+4s((a_2-b_2)c_1+(b_1-a_1)c_2)+ \\ 4\left(c_1^2-e_1^2+c_2^2-e_2^2\right))$$

where $D=\frac{\partial}{\partial x}$. Taking the inverse Aboodh transform we have the real and imaginer part of $w$ as

$$w_1=-2A^{-1}\{P^{-1}\{\frac{\partial}{\partial x}[(a_1b_2-a_2b_1)w_1(x,0)+\left(b_1^2+b_2^2-a_1^2-a_2^2\right)\frac{w_2(x,0)}{2} \\ -sK_3(a_1+b_1)-sK_4(a_2+b_2)] \\ +\left[(b_2-a_2)(c_1-e_1)+(b_1-a_1)(e_2-c_2)-\frac{s}{2}\left((a_1-b_1)^2+(a_2-b_2)^2\right)\right]w_1(x,0) \\ +[(a_2-b_2)(e_2-c_2)+(a_1-b_1)(e_1-c_1)]w_2(x,0) \\ -s(s(a_2-b_2)+2c_1-2e_1)K_3+s(s(a_1-b_1)+2e_2-2c_2)K_4\}\}$$

and

$$w_2 = -2A^{-1}\{P^{-1}\{\frac{\partial}{\partial x}[(a_1^2 + a_2^2 - b_1^2 - b_2^2)\,\frac{w_1(x,0)}{2} + (a_1 b_2 - a_2 b_1)\,w_2(x,0)$$
$$+sK_3(a_2+b_2) - sK_4(a_1+b_1)]$$
$$+\left[(b_2-a_2)(c_1+e_1) + (a_1-b_1)(e_2+c_2) - \frac{s}{2}\left((a_1-b_1)^2 + (a_2-b_2)^2\right)\right] w_2(x,0)$$
$$+[(a_2-b_2)(e_2+c_2) + (a_1-b_1)(e_1+c_1)]\,w_1(x,0)$$
$$+s\,(s\,(b_1-a_1) + 2c_2 + 2e_2)\,K_3 + s\,(s\,(b_2-a_2) - 2e_1 - 2c_1)\,K_4\}\}.$$

□

REMARK 3.1. Our result generalized the case of complex equation with constant coefficients obtaind by Aboodh et al. [**3**].

EXAMPLE 3.1. Let solve the following problem by the help of Theorem 3.1

$$2\tfrac{\partial w}{\partial z} + 3\tfrac{\partial w}{\partial \bar{z}} - 2iw + 3\overline{w} = 0 \qquad (3.6)$$
$$w(x,0) = x.$$

Equation's coefficients are $\tilde{A} = 2, B = 3, C = -3i, E = 3$ and $F = 0$. Let substitute them into (3.2), we have

$$\begin{aligned}
w_1(x,y) &= A^{-1}\left[\frac{x\,(s-6)}{s\,(25D^2 + s^2 - 12s)}\right] \\
&= A^{-1}\left[\frac{(s-6)}{s\,(s^2-12s)\left(1 + \frac{25}{s^2-12s}D^2\right)}x\right] \\
&= A^{-1}\left[\frac{(s-6)}{s\,(s^2-12s)}\left(\left(1 - \frac{25}{s^2-12s}D^2 + ...\right)\right)x\right] \\
&= A^{-1}\left[\frac{(s-6)}{s\,(s^2-12s)}x\right] \\
&= \frac{x}{2}\left(1 + e^{12y}\right).
\end{aligned}$$

Similarly, imaginary part of solution can be obtained as

$$\begin{aligned}
w_2(x,y) &= A^{-1}\left[\frac{5+6x}{s\,(25D^2 + s^2 - 12s)}\right] \\
&= A^{-1}\left[\frac{1}{s\,(s^2-12s)}\left(\left(1 - \frac{25}{s^2-12s}D^2 + ...\right)\right)(5+6x)\right] \\
&= \frac{(5+6x)}{12}\left(-1 + e^{12y}\right).
\end{aligned}$$

Hence, the solution of (3.6) is

$$\begin{aligned}
w &= w_1 + iw_2 \\
&= \frac{x}{2}\left(1 + e^{12y}\right) + i\frac{(5+6x)}{12}\left(-1 + e^{12y}\right)
\end{aligned}$$

EXAMPLE 3.2. Consider the following problem

$$(1+i)\,\frac{\partial w}{\partial z}+\frac{\partial w}{\partial \overline{z}}+w+\overline{w}=0 \qquad (3.7)$$
$$w(x,0)=e^{2x}.$$

The coefficients are $\tilde{A}=1+i, B=1, C=1, E=1$ and $F=0$. From Theorem 3.1, we have

$$\begin{aligned}
w_1(x,y) &= A^{-1}\left[\frac{\frac{\partial}{\partial x}\left(2e^{2x}\right)+se^{2x}}{s\left(5D^2+(8+4s)\,D+s^2+4s\right)}\right]\\
&= e^{2x}A^{-1}\left[\frac{4+s}{s\left(s^2+12s+36\right)}\right]\\
&= e^{2x}A^{-1}\left[\frac{1}{3\left(s+6\right)^2}+\frac{6}{9s\left(s+6\right)}\right]\\
&= e^{2x}A^{-1}\left[\frac{1}{3\left(s+6\right)^2}\right]+\frac{2}{3}e^{2x-6y}
\end{aligned}$$

and similarly

$$\begin{aligned}
w_2(x,y) &= A^{-1}\left[\frac{-2e^{2x}}{s\left(5D^2+(8+4s)\,D+s^2+4s\right)}\right]\\
&= -2e^{2x}A^{-1}\left[\frac{1}{s\left(s^2+12s+36\right)}\right]\\
&= -2e^{2x}A^{-1}\left[\frac{-1}{6\left(s+6\right)^2}+\frac{1}{6s\left(s+6\right)}\right]\\
&= 2e^{2x}A^{-1}\left[\frac{1}{6\left(s+6\right)^2}\right]-\frac{1}{3}e^{2x-6y}.
\end{aligned}$$

Therefore the solution of (3.7) is

$$\begin{aligned}
w &= w_1+iw_2\\
&= e^{2x}A^{-1}\left[\frac{1}{3\left(s+6\right)^2}\right]+\frac{2}{3}e^{2x-6y}+i\left(-2e^{2x}A^{-1}\left[\frac{1}{6\left(s+6\right)^2}\right]-\frac{1}{3}e^{2x-6y}.\right)\\
&= e^{2x}A^{-1}\left[\frac{1}{3\left(s+6\right)^2}\right](1+i)+\frac{1}{3}e^{2x-6y}\,(2-i)
\end{aligned}$$

REMARK 3.2. The result of this paper can be applied for matrix equation having following form

$$Aw_z+Bw_{\overline{z}}+Cw+E\overline{w}=F$$
$$w(x,0)=f(x)$$

where $w, F$ are $m \times s$ type complex matrix and coefficients $A, B, C, E$ are $m \times m$ type constant matrices in form

$$\begin{pmatrix} a_{11} & & & 0 \\ \vdots & a_{11} & & \\ & & \ddots & \\ & \cdots & & a_{11} \end{pmatrix}.$$

By using Theorem 3.1 , desired solution can be obtained succesively in component form.

## References


[1] K. S. Aboodh. The new integral transform "Aboodh transform". *Global J. Pure Appl. Math.* **9**(1)(2013), 35–43.

[2] K. S. Aboodh. Application of new transform "Aboodh transform" to partial differential equations. *Global J. Pure Appl. Math.* **10**(2)(2014),249–254.

[3] K. S. Aboodh, R. A. Farah, I. A. Almardy, F. A. ALmostafa. Some application of Aboodh Transform to first order constant coefficients complex equations. *Int. J. Math. And Appl.* **6**(1-A)(2018),1–6.

[4] T. M. Elzaki. The new integral transform Elzaki transform. *Global J. Pure Appl. Math.*, **7**(1)(2011), 57–64.

[5] T. M. Elzaki, S. M. Elzaki. Application of new transform Elzaki transform to partial differential equations. *Global J. Pure Appl. Math.*, **7**(1)(2011), 65–70.


Receibed by editors 26.03.2018; Revised version 30.09.2018; Available online 08.10.2018.

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