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INTUITIONISTIC FUZZY WEAK BI-IDEALS OF Γ-NEAR-RINGS

K. Arulmozhi, V. Chinnadurai and A. Swaminathan

Abstract. In this paper, we introduce the notion of intuitionistic fuzzy weak bi-ideals of Γ-near-rings. We also investigate some of its properties with examples.

1. Introduction

Zadeh [**19**] introduced the concept of fuzzy sets in 1965. Near-ring was introduced by Pilz [**12**]. Gamma -near-ring was introduced by Satyanarayana [**15**] in 1984. The concept of bi-ideals of near-ring was applied to Γ-near-rings by Tamizh Chelvam *et al.* [**16**], and he developed weak bi-ideals of near-rings. Atanassov [**1**] introduced the intuitionistic fuzzy set. The idea of fuzzy ideals of near-rings was first proposed by Kim *et al.* [**8**]. Fuzzy ideals in Gamma- near-rings was proposed by Jun *et al.* [**7**] in 1998. Chinnadurai *et al* [**4**] studied the characteristics of fuzzy weak bi-ideals of Γ-near-rings. Moreover, Manikantan [**9**] introduced the notion of fuzzy bi-ideals of near-rings and discussed some of its properties. Meenakumari *et al.* [**10**] studied the fuzzy bi-ideals in gamma -near-rings. Cho *et al.* [**18**] introduced the concept of weak bi-ideals applied to near-rings. Chinnadurai *et al.* [**3**] studied the fuzzy weak bi-ideals of near rings. In this paper, we define a new notion of intuitionistic fuzzy weak bi-ideals of Γ- near-rings. We also investigate some of its properties with examples.

2. Preliminaries

In this section, we listed some basic definitions.

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DEFINITION 2.1. [**12**] A *near-ring* is an algebraic system $(R, +, \cdot)$ consisting of a non empty set R together with two binary operations called $+$ and $\cdot$ such that $(R, +)$ is a group not necessarily abelian and $(R, \cdot)$ is a semigroup connected by the following distributive law: $(x + z) \cdot y = x \cdot y + z \cdot y$ valid for all $x, y, z \in R$. We use the word 'near-ring 'to mean 'right near-ring '. We denote xy instead of $x \cdot y$.

DEFINITION 2.2. [**15**] A Γ- near-ring is a triple $(M, +, \Gamma)$ where
(i) $(M, +)$ is a group,
(ii) Γ is a nonempty set of binary operators on M such that for each $\alpha \in \Gamma$, $(M, +, \alpha)$ is a near-ring,
(iii) $x\alpha(y\beta z) = (x\alpha y)\beta z$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$.

DEFINITION 2.3. [**10**] A Γ-near-ring M is said to be zero-symmetric if $x\alpha 0 = 0$ for all $x \in M$ and $\alpha \in \Gamma$.

Throughout this paper M denotes a zero-symmetric right Γ- near-ring with atleast two elements.

DEFINITION 2.4. [**15**] A subset A of a Γ-near-ring M is called a left(resp. right) ideal of M if
(i) $(A, +)$ is a normal subgroup of $(M, +)$, (i.e) $x - y \in A$ for all $x, y \in A$ and $y + x - y \in A$ for $x \in A, y \in M$
(ii)$u\alpha(x + v) - u\alpha v \in A$ (resp. $x\alpha u \in A$) for all $x \in A, \alpha \in \Gamma$ and $u, v \in M$.

DEFINITION 2.5. [**16**] Let M be a Γ-near-ring. Given two subsets A and B of M, we define $A\Gamma B = \{a\alpha b | a \in A, b \in B \text{ and } \alpha \in \Gamma\}$.

DEFINITION 2.6. [**16**] A subgroup B of $(M, +)$ is a bi-ideal if and only if $B\Gamma M\Gamma B \subseteq B$.

The characteristic function of M is denoted by $\mathbf{M}$, that means, $\mathbf{M} : M \to [0, 1]$ mapping every element of M to 1.

DEFINITION 2.7. [**11**] A function η from a nonempty set M to the unit interval $[0, 1]$ is called a fuzzy subset of M. Let η be any fuzzy subset of M, for $t \in [0, 1]$ the set $\eta_t = \{x \in M | \eta(x) \geqslant t\}$ is called a level subset of η.

DEFINITION 2.8. [**11, 17**] Let η and λ be any two fuzzy subsets of M. Then $\eta \cap \lambda,\ \eta \cup \lambda,\ \eta + \lambda$ and $\eta * \lambda$ are fuzzy subsets of M defined by:

$(\eta \cap \lambda)(x) = \min\{\eta(x),\ \lambda(x)\}$.

$(\eta \cup \lambda)(x) = \max\{\eta(x),\ \lambda(x)\}$.

$$(\eta + \lambda)(x) = \begin{cases} \sup\limits_{x=y+z} \min\{\eta(y),\ \lambda(z)\} & \text{if } x \text{ can be expressed as } x = y + z \\ 0 & \text{otherwise.} \end{cases}$$

$$(\eta * \lambda)(x) = \begin{cases} \sup\limits_{x=y\alpha z} \min\{\eta(y),\ \lambda(z)\} & \text{if } x \text{ can be expressed as } x = yz \\ 0 & \text{otherwise.} \end{cases}$$

for $x, y, z \in M$ and $\alpha \in \Gamma$.

Definition 2.9. [7] A fuzzy set η of a Γ-near-ring M is called a fuzzy left(resp. right)ideal of M if

(i) $\eta(x-y) \geqslant \min\{\eta(x), \eta(y)\}$, for all $x, y \in M$,

(ii) $\eta(y+x-y) \geqslant \eta(x)$, for all $x, y \in M$,

(iii) $\eta(u\alpha(x+v) - u\alpha v) \geqslant \eta(x)$,(resp. $\eta(x\alpha u) \geqslant \eta(x)$) for all $x, u, v \in M$ and $\alpha \in \Gamma$.

Definition 2.10. [10] A fuzzy set η of M is called a fuzzy bi-ideal of M if

(i) $\eta(x-y) \geqslant \min\{\eta(x), \eta(y)\}$ for all $x, y \in M$,

(ii) $\eta(x\alpha y\beta z) \geqslant \min\{\eta(x), \eta(z)\}$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$.

Definition 2.11. [1] An intuitionistic fuzzy set (briefly IFS) A in a non-empty set X is an object having the form $A = \{x, (\mu_A(x), \eta_A(x)) : x \in X\}$ where the functions $\mu_A : X \to [0,1]$ and $\eta_A : X \to [0,1]$ define the degree of membership and the degree of non-membership of the element $x \in X$ to the set A, which is a subset of X respectively $0 \leqslant \mu_A(x) + \eta_A(x) \leqslant 1$ we use the simple $A = (\mu_A, \eta_A)$.

3. Intuitionistic fuzzy weak bi-ideals of Γ-near-rings

In this section, we introduce the notion of intuitionistic fuzzy weak bi-ideal of M and discuss some of its properties.

Definition 3.1. [16] A subgroup W of $(M,+)$ is said to be a weak bi-ideal of M if $W\Gamma W\Gamma W \subseteq W$.

Definition 3.2. An intuitionistic fuzzy set $B = (\eta_B, \lambda_B)$ of M is called a intuitionistic fuzzy weak bi-ideal of M, if

(i) $\eta(x-y) \geqslant \min\{\eta(x), \eta(y)\}$

(ii) $\lambda(x-y) \leqslant \max\{\lambda(x), \lambda(y)\}$

(iii) $\eta(x\gamma y\gamma z) \geqslant \min\{\eta(x), \eta(y), \eta(z)\}$

(iv) $\lambda(x\gamma y\gamma z) \leqslant \max\{\lambda(x), \lambda(y), \lambda(z)\}$ for all $x, y, z \in M$ and for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$.

Example 3.1. Let $M = \{0, a, b, c\}$ be a non-empty set with binary operation $+$ and $\Gamma = \{\gamma\}$ be a nonempty set of binary operations as shown in the following tables:

$+$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

γ	0	a	b	c
0	0	a	0	a
a	0	a	0	a
b	0	a	b	c
c	0	a	b	c

Let $\eta : M \to [0,1]$ be a fuzzy subset defined by $\eta(0) = 0.7, \eta(a) = 0.6, \eta(b) = \eta(c) = 0.3$. and $\lambda(0) = 0.3, \lambda(a) = 0.4, \lambda(b) = 0.6 = \lambda(c)$ Then $B = (\eta_B, \lambda_B)$ is an intuitionistic fuzzy weak bi-ideal of M.

Theorem 3.1. *Let $B = (\eta_B, \lambda_B)$ be a intuitionistic fuzzy subgroup of M. Then $B = (\eta_B, \lambda_B)$ is a intuitionistic fuzzy weak bi-ideal of M if and only if $\eta * \eta * \eta \subseteq \eta$ and $\lambda * \lambda * \lambda \supseteq \lambda$.*

PROOF. Assume that $B = (\eta_B, \lambda_B)$ be a intuitionistic fuzzy weak bi-ideal of M. Let $x, y, z, y_1, y_2 \in M$ and $\alpha, \beta \in \Gamma$ such that $x = y\alpha z$ and $y = y_1\beta y_2$. Then

$$\begin{aligned}(\eta * \eta * \eta)(x) &= \sup_{x=y\alpha z} \{\min\{(\eta * \eta)(y), \eta(z)\}\} \\ &= \sup_{x=y\alpha z} \{\min\{ \sup_{y=y_1\beta y_2} \min\{\eta(y_1), \eta(y_2)\}, \eta(z)\}\} \\ &= \sup_{x=y\alpha z} \sup_{y=y_1\beta y_2} \{\min\{\min\{\eta(y_1), \eta(y_2)\}, \eta(z)\}\} \\ &= \sup_{x=y_1\beta y_2\alpha z} \{\min\{\eta(y_1), \eta(y_2), \eta(z)\}\} \\ &\text{since } \eta \text{ is a fuzzy weak bi-ideal of } M, \\ &\eta(y_1\beta y_2\alpha z) \geqslant \min\{\eta(y_1), \eta(y_2), \eta(z)\} \\ &\leqslant \sup_{x=y_1\beta y_2\alpha z} \eta(y_1\beta y_2\alpha z) \\ &= \eta(x).\end{aligned}$$

and

$$\begin{aligned}(\lambda * \lambda * \lambda)(x) &= \inf_{x=y\alpha z} \{\min\{(\lambda * \lambda)(y), \lambda(z)\}\} \\ &= \inf_{x=y\alpha z} \{\max\{ \inf_{y=y_1\beta y_2} \min\{\lambda(y_1), \lambda(y_2)\}, \lambda(z)\}\} \\ &= \inf_{x=y\alpha z} \sup_{y=y_1\beta y_2} \{\max\{\max\{\lambda(y_1), \lambda(y_2)\}, \lambda(z)\}\} \\ &= \inf_{x=y_1\beta y_2\alpha z} \{\max\{\lambda(y_1), \lambda(y_2), \lambda(z)\}\} \\ &\text{since } \lambda \text{ is a intuitionistic fuzzy weak bi-ideal of} M, \\ &\eta(y_1\beta y_2\alpha z) \leqslant \max\{\lambda(y_1), \lambda(y_2), \lambda(z)\} \\ &\geqslant \inf_{x=y_1\beta y_2\alpha z} \eta(y_1\beta y_2\alpha z) \\ &= \lambda(x).\end{aligned}$$

If x can not be expressed as $x = y\alpha z$, then $(\eta * \eta * \eta)(x) = 0 \leqslant \eta(x)$ and $(\lambda * \lambda * \lambda)(x) = 0 \geqslant \lambda(x)$ In both cases $\eta * \eta * \eta \subseteq \eta$, and $\lambda * \lambda * \lambda \supseteq \lambda$.

Conversely, assume that $\eta * \eta * \eta \subseteq \eta$. For $x', x, y, z \in M$ and $\alpha, \beta, \alpha_1, \beta_1 \in \Gamma$. Let x' be such that $x' = x\alpha y\beta z$. Then

$$\begin{aligned}\eta(x\alpha y\beta z) = \eta(x') &\geqslant (\eta * \eta * \eta)(x') \\ &= \sup_{x'=p\alpha_1 q} \{\min\{(\eta * \eta)(p), \eta(q)\}\} \\ &= \sup_{x'=p\alpha_1 q} \{\min\{ \sup_{p=p_1\beta_1 p_2} \min\{\eta(p_1), \eta(p_2)\}, \eta(q)\}\} \\ &= \sup_{x'=p_1\beta_1 p_2\alpha_1 q} \{\min\{\eta(p_1), \eta(p_2), \eta(q)\}\} \\ &\geqslant \min\{\eta(x), \eta(y), \eta(z)\}.\end{aligned}$$

$$
\begin{aligned}
\lambda(x\alpha y\beta z) = lambda(x') &\leqslant (\lambda * \lambda * \lambda)(x') \\
&= \inf_{x'=p\alpha_1 q} \{\max\{(\lambda * \lambda)(p), \lambda(q)\}\} \\
&= \inf_{x'=p\alpha_1 q} \{\max\{ \inf_{p=p_1\beta_1 p_2} \min\{\eta(p_1), \eta(p_2)\}, \eta(q)\}\} \\
&= \inf_{x'=p_1\beta_1 p_2 \alpha_1 q} \{\max\{\lambda(p_1), \lambda(p_2), \lambda(q)\}\} \\
&\leqslant \max\{\lambda(x), \lambda(y), \lambda(z)\}.
\end{aligned}
$$

Hence $\eta(x\alpha y\beta z) \geqslant \min\{\eta(x), \eta(y), \eta(z)\}$, and $\lambda(x\alpha y\beta z) \leqslant \max\{\lambda(x), \lambda(y), \lambda(z)\}$. □

LEMMA 3.1. *Let η and λ be fuzzy weak bi-ideals of M. Then the products $\eta * \lambda$ and $\lambda * \eta$ are also fuzzy weak bi-ideals of M.*

PROOF. Let η and λ be fuzzy weak bi-ideals of M and let $\alpha, \alpha_1, \alpha_2 \in \Gamma$. Then

$$
\begin{aligned}
(\eta * \lambda)(x - y) &= \sup_{x-y=a\alpha b} \min\{\eta(a), \lambda(b)\} \\
&\geqslant \sup_{x-y=a_1\alpha_1 b_1 - a_2\alpha_2 b_2 < (a_1-a_2)(b_1-b_2)} \min\{\eta(a_1 - a_2), \lambda(b_1 - b_2)\} \\
&\geqslant \sup \min\{\min\{\eta(a_1), \eta(a_2)\}, \min\{\lambda(b_1), \lambda(b_2)\}\} \\
&= \sup \min\{\min\{\eta(a_1), \lambda(b_1)\}, \min\{\eta(a_2), \lambda(b_2)\}\} \\
&\geqslant \min\{ \sup_{x=a_1\alpha_1 b_1} \min\{\eta(a_1), \lambda(b_1)\}, \sup_{y=a_2\alpha_2 b_2} \min\{\eta(a_2), \lambda(b_2)\}\} \\
&= \min\{(\eta * \lambda)(x), (\eta * \lambda)(y)\}.
\end{aligned}
$$

It follows that $\eta * \lambda$ is a fuzzy subgroup of M. Further,

$$
\begin{aligned}
(\eta * \lambda) * (\eta * \lambda) * (\eta * \lambda) &= \eta * \lambda * (\eta * \lambda * \eta) * \lambda \\
&\subseteq \eta * \lambda * (\lambda * \lambda * \lambda) * \lambda \\
&\subseteq \eta * (\lambda * \lambda * \lambda), \text{ since } \lambda \text{ is a fuzzy weak bi-ideal of } M \\
&\subseteq \eta * \lambda.
\end{aligned}
$$

Therefore $\eta * \lambda$ is a fuzzy weak bi-ideal of M. Similarly $\lambda * \eta$ is a fuzzy weak bi-ideal of M. □

LEMMA 3.2. *Every intuitionistic fuzzy ideal of M is a intuitionistic fuzzy bi-ideal of M.*

PROOF. Let $B = (\eta_B, \lambda_B)$ be an intuitionistic fuzzy ideal of M. Then

$$\eta * \mathbf{M} * \eta \subseteq \eta * \mathbf{M} * \mathbf{M} \subseteq \eta * \mathbf{M} \subseteq \eta$$

$$\lambda * \mathbf{M} * \lambda \supseteq \lambda * \mathbf{M} * \mathbf{M} \supseteq \lambda * \mathbf{M} \supseteq \lambda$$

since $B = (\eta_B, \lambda_B)$ be an intuitionistic fuzzy ideal of M. This implies that $\eta * \mathbf{M} * \eta \subseteq \eta$ and $\lambda * \mathbf{M} * \lambda \supseteq \lambda$. Therefore $B = (\eta_B, \lambda_B)$ be an intuitionistic fuzzy bi-ideal of M. □

THEOREM 3.2. *Every intuitionistic fuzzy bi-ideal of M is an intuitionistic fuzzy weak bi-ideal of M.*

PROOF. Assume that $B = (\eta_B, \lambda_B)$ be an intuitionistic fuzzy bi-ideal of M. Then $\eta * \mathbf{M} * \eta \subseteq \eta$ and $\lambda * \mathbf{M} * \lambda \supseteq \lambda$. We have $\eta * \eta * \eta \subseteq \eta * \mathbf{M} * \eta$ and $\lambda * \lambda * \lambda \supseteq \lambda * \mathbf{M} * \lambda$. This implies that $\eta * \eta * \eta \subseteq \eta * \mathbf{M} * \eta \subseteq \eta$ and $\lambda * \lambda * \lambda \supseteq \lambda * \mathbf{M} * \lambda \supseteq \lambda$.Therefore $B = (\eta_B, \lambda_B)$ is an intuitionistic fuzzy weak bi-ideal of M. □

THEOREM 3.3. *Every intuitionistic fuzzy ideal of M is intuitionistic fuzzy weak bi-ideal of M.*

PROOF. By Lemma 3.2, every intuitionistic fuzzy ideal of M is intuitionistic fuzzy bi-ideal of M. By Theorem 3.2, every intuitionistic fuzzy bi-ideal of M is intuitionistic fuzzy weak bi-ideal of M. Theorefore $B = (\eta_B, \lambda_B)$ is intuitionistic fuzzy weak bi-ideal of M. □

However the converse of the Theorems 3.2 and 3.3 is not true in general which is demonstrated by the following example.

EXAMPLE 3.2. Let $M = \{0, a, b, c\}$ be a non-emptyset with binary operation $+$ and $\Gamma = \{\alpha, \beta\}$ be a nonempty set of binary operations as shown in the following tables:

$+$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

α	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
c	0	a	b	c

Let $\eta : M \to [0,1]$ be a fuzzy subset defined by $\eta(0) = 0.9, \eta(a) = 0.4 = \eta(b)$ and $\eta(c) = 0.6$, and $\lambda(0) = 0.1, \lambda(a) = 0.5 = \lambda(b), \lambda(c) = 0.3$. Then η is a fuzzy weak bi-ideal of M. But η is not a fuzzy ideal and bi-ideal of $M$and $\eta(c\gamma b\gamma c) = \eta(b) = 0.4 \ngeqslant 0.6 = \min\{\eta(c), \eta(c)\}$ and $\lambda(a\alpha(c+0) - a\alpha 0) \leqslant \lambda(c) = 0.5 \nleqslant 0.3$ and $\lambda(c\gamma a\gamma c) = \eta(a) = 0.5 \nleqslant 0.3 = \min\{\eta(c), \eta(c)\}$.

THEOREM 3.4. *Let $\{(\eta_i, \lambda_i) | i \in \Omega\}$ be family of intuitionistic fuzzy weak bi-ideals of a near-ring M, then $\bigcap_{i\in\Omega} \eta_i$ is also a intuitionistic fuzzy weak bi-ideal of M, where Ω is any index set.*

PROOF. Let $\{\eta_i\}_{i\in\Omega}$ be a family of intuitionistic fuzzy weak bi-ideals of M. Let $x, y, z \in M, \alpha, \beta \in \Gamma$ and $\eta = \bigcap_{i\in\Omega} \eta_i$. Then, $\eta(x) = \bigcap_{i\in\Omega} \eta_i(x) = \left(\inf_{i\in\Omega} \eta_i\right)(x) = \inf_{i\in\Omega} \eta_i(x)$.

$$\begin{aligned}
\eta(x-y) &= \inf_{i\in\Omega}\ \eta_i(x-y) \\
&\geqslant \inf_{i\in\Omega}\ \min\{\eta_i(x),\eta_i(y)\} \\
&= \min\left\{\inf_{i\in\Omega}\eta_i(x),\inf_{i\in\Omega}\eta_i(y)\right\} \\
&= \min\left\{\bigcap_{i\in\Omega}\eta_i(x),\bigcap_{i\in\Omega}\eta_i(y)\right\} \\
&= \min\{\eta(x),\eta(y)\}.
\end{aligned}$$

$$\begin{aligned}
\lambda(x-y) &= \sup_{i\in\Omega}\ \eta_i(x-y) \\
&\leqslant \sup_{i\in\Omega}\ \max\{\lambda_i(x),\lambda_i(y)\} \\
&= \max\left\{\sup_{i\in\Omega}\lambda_i(x),\sup_{i\in\Omega}\lambda_i(y)\right\} \\
&= \max\left\{\bigcap_{i\in\Omega}\lambda_i(x),\bigcap_{i\in\Omega}\lambda_i(y)\right\} \\
&= \max\{\lambda(x),\lambda(y)\}.
\end{aligned}$$

And,

$$\begin{aligned}
\eta(x\alpha y\beta z) &= \inf_{i\in\Omega}\ \eta_i(x\alpha y\beta z) \\
&\geqslant \inf_{i\in\Omega}\ \min\{\eta_i(x),\eta_i(y),\eta_i(z)\} \\
&= \min\left\{\inf_{i\in\Omega}\eta_i(x),\inf_{i\in\Omega}\eta_i(y),\inf_{i\in\Omega}\eta_i(z)\right\} \\
&= \min\left\{\bigcap_{i\in\Omega}\eta_i(x),\bigcap_{i\in\Omega}\eta_i(y),\bigcap_{i\in\Omega}\eta_i(z)\right\} \\
&= \min\{\eta(x),\eta(y),\eta(z)\}.
\end{aligned}$$

$$\begin{aligned}
\lambda(x\alpha y\beta z) &= \sup_{i\in\Omega}\ \lambda_i(x\alpha y\beta z) \\
&\leqslant \sup_{i\in\Omega}\ \max\{\lambda_i(x),\lambda_i(y),\lambda_i(z)\} \\
&= \max\left\{\sup_{i\in\Omega}\lambda_i(x),\sup_{i\in\Omega}\lambda_i(y),\sup_{i\in\Omega}\lambda_i(z)\right\} \\
&= \max\left\{\bigcap_{i\in\Omega}\lambda_i(x),\bigcap_{i\in\Omega}\lambda_i(y),\bigcap_{i\in\Omega}\lambda_i(z)\right\} \\
&= \max\{\lambda(x),\lambda(y),\lambda(z)\}.
\end{aligned}$$

□

THEOREM 3.5. *Let $B = (\eta_B, \lambda_B)$ be an intuitionistic fuzzy subset of M. Then $U(\eta_B; t)$ and $L(\delta; s)$ is an intuitionistic fuzzy weak bi-ideal of M if and only if η_t is a weak bi-ideal of M, for all $t \in [0, 1]$.*

PROOF. Assume that $B = (\eta_B, \lambda_B)$ is an intuitionistic fuzzy weak bi-ideal of R. Let $s, t \in [0, 1]$ such that $x, y \in U(\eta_B; t)$, Then $\eta_B(x) \geqslant t$ and $\eta_B(y) \geqslant t$,then $\eta_B(x-y) \geqslant \min\{\eta_B(x), \eta_B(y)\} \geqslant \min\{t, t\} = t$. and $\lambda_B(x - y) \leqslant \max\{\lambda_B(x), \lambda_B(y)\} \leqslant \max\{s, s\} = s$. Thus $x - y \in U(\eta_B; t)$. Let $x, y, z \in \eta_t$ and $\alpha, \beta \in \Gamma$. This implies that $\eta(x\alpha y\beta z) \geqslant \min\{\eta(x), \eta(y), \eta(z)\} \geqslant \min\{t, t, t\} = t$, and $\lambda_B(x\alpha y\beta z) \leqslant \max\{\lambda_B(x), \lambda_B(y), \lambda_B(z)\} \leqslant \max\{s, s, s\} = s$. Therefore $x\alpha y\beta z \in U(\eta_B; s)$. Hence $U(\eta_B; t)$ and $(\lambda_B; s)$ is a weak bi-ideal of M.

Conversely, assume that $U(\eta_B; t)$ and $(\lambda_B; s)$ is a weak bi-ideal of M, for all $s, t \in [0, 1]$. Let $x, y \in M$. Suppose $\eta(x - y) < \min\{\eta(x), \eta(y)\}$ and $\lambda(x - y) > \max\{\lambda(x), \lambda(y)\}$. Choose t such that $\eta(x-y) < t < \min\{\eta(x), \eta(y)\}$ and $\lambda(x-y) > s > \max\{\lambda(x), \lambda(y)\}$ This implies that $\eta(x) > t, \eta(y) > t$ and $\eta(x - y) < t$. Then we have $x, y \in \eta_t$ but $x - y \notin \eta_t$ and $\lambda(x) < s, \lambda(y) < s$ and $\lambda(x - y) > s$, we have $x, y \in \lambda_s$ but $x - y \notin \lambda_s$ a contradiction. Thus $\eta(x - y) \geqslant \min\{\eta(x), \eta(y)\}$ and $\lambda(x - y) \leqslant \max\{\lambda(x), \lambda(y)\}$. If there exist $x, y, z \in M$ and $\alpha, \beta \in \Gamma$ such that $\eta(x\alpha y\beta z) < \min\{\eta(x), \eta(y), \eta(z)\}$ and $\lambda(x\alpha y\beta z) > \min\{\eta(x), \eta(y), \eta(z)\}$. Choose t such that $\eta(x\alpha y\beta z) < t < \min\{\eta(x), \eta(y), \eta(z)\}$. Choose s such that $\lambda(x\alpha y\beta z) > s > \min\{\eta(x), \eta(y), \eta(z)\}$. Then $\eta(x) > t, \eta(y) > t, \eta(z) > t$ and $\lambda(x) < s, \lambda(y) < s, \lambda(z) < s$ and $\eta(x\alpha y\beta z) < t$. So, $x, y, z \in \eta_t$ but $x\alpha y\beta z \notin \eta_t$, and $x\alpha y\beta z \notin \lambda_s$, which is a contradiction. Hence $\eta(x\alpha y\beta z) \geqslant \min\{\eta(x), \eta(y), \eta(z)\}$, $\lambda(x\alpha y\beta z) \leqslant \max\{\lambda(x), \lambda(y), \lambda(z)\}$. Therefore $B = (\eta_B, \lambda_B)$ is a intuitionistic fuzzy weak bi-ideal of M. ☐

THEOREM 3.6. *Let $B = (\eta_B, \lambda_B)$ be intuitionistic fuzzy weak bi-ideal of M then the set $M_{\eta,\lambda} = \{x \in M |\ \eta(x) = \eta(0) = \lambda(x)\}$ is a weak bi-ideal of M.*

PROOF. Let $x, y \in M_{(\eta,\lambda)}$. Then $\eta(x) = \eta(0), \eta(y) = \eta(0), \lambda(x) = 0, \lambda(y) = 0$ and $\eta(x - y) \geqslant \min\{\eta(x), \eta(y)\} = \min\{\eta(0), \eta(0)\} = \eta(0)$, and $\lambda(x - y) \leqslant \max\{\lambda(x), \lambda(y)\} = \max\{\lambda(0), \lambda(0)\} = \lambda(0)$. So $\eta(x - y) = \eta(0), \lambda(x - y) = \lambda(0)$. Thus $x - y \in M_\eta$, $x - y \in M_\lambda$. For every $x, y, z \in M_\eta$ and $\alpha, \beta \in \Gamma$. We have $\eta(x\alpha y\beta z) \geqslant \min\{\eta(x), \eta(y), \eta(z)\} = \min\{\eta(0), \eta(0), \eta(0)\} = \eta(0)$ and $\lambda(x\alpha y\beta z) \leqslant \max\{\lambda(x), \lambda(y), \lambda(z)\} = \max\{\lambda(0), \lambda(0), \lambda(0)\} = \lambda(0)$. Thus $x\alpha y\beta z \in M_\eta, x\alpha y\beta z \in M_\lambda$. Hence $M_{(\eta,\lambda)}$ is a weak bi-ideal of M. ☐

4. Homomorphism of intuitionistic fuzzy weak bi-ideals of Γ-near-rings

In this section, we characterize fuzzy weak bi-ideals of Γ-near-rings using homomorphism.

DEFINITION 4.1. [**8**] Let f be a mapping from a set M to a set S. Let η and δ be fuzzy subsets of M and S, respectively. Then $f(\eta)$, the image of η under f is a fuzzy subset of S defined by

$$f(\eta)(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \eta(x) & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

and the pre-image of η under f is a fuzzy subset of M defined by
$f^{-1}(\delta(x)) = \delta(f(x))$, for all $x \in M$ and $f^{-1}(y) = \{x \in M | f(x) = y\}$.

Definition 4.2. [8] Let M and S be Γ-near-rings. A map $\theta : M \to S$ is called a (Γ-near-ring) homomorphism if $\theta(x + y) = \theta(x) + \theta(y)$ and $\theta(x\alpha y) = \theta(x)\alpha\theta(y)$ for all $x, y \in M$ and $\alpha \in \Gamma$.

Theorem 4.1. *Let $f : M \to S$ be a homomorphism between Γ-near-rings M and S. If $B = (\delta_B, \eta_B)$ is a intuitionistic fuzzy weak bi-ideal of S, then $f^{-1}(B) = [f^{-1}(\delta_B, \eta_B)]$ is a intuitionistic fuzzy weak bi-ideal of M.*

Proof. Let δ be a fuzzy weak bi-ideal of S. Let $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Then

$$\begin{aligned}
f^{-1}(\delta)(x - y) &= \delta(f(x - y)) \\
&= \delta(f(x) - f(y)) \\
&\geqslant \min\{\delta(f(x)), \delta(f(y))\} \\
&= \min\{f^{-1}(\delta(x)), f^{-1}(\delta(y))\}. \\
f^{-1}(\eta)(x - y) &= \eta(f(x - y)) \\
&= \eta(f(x) - f(y)) \\
&\leqslant \max\{\eta(f(x)), \eta(f(y))\} \\
&= \max\{f^{-1}(\eta(x)), f^{-1}(\eta(y))\}. \\
f^{-1}(\delta)(x\alpha y\beta z) &= \delta(f(x\alpha y\beta z)) \\
&= \delta(f(x)\alpha f(y)\beta f(z)) \\
&\geqslant \min\{\delta(f(x)), \delta(f(y)), \delta(f(z))\} \\
&= \min\{f^{-1}(\delta(x)), f^{-1}(\delta(y)), f^{-1}(\delta(z))\}. \\
f^{-1}(\eta)(x\alpha y\beta z) &= \eta(f(x\alpha y\beta z)) \\
&= \eta(f(x)\alpha f(y)\beta f(z)) \\
&\leqslant \max\{\eta(f(x)), \eta(f(y)), \eta(f(z))\} \\
&= \max\{f^{-1}(\eta(x)), f^{-1}(\eta(y)), f^{-1}(\eta(z))\}.
\end{aligned}$$

Therefore $f^{-1}(B) = [f^{-1}(\delta_B, \eta_B)]$ is a intuitionistic fuzzy weak bi-ideal of M. □

We can also state the converse of the Theorem 4.3 by strengthening the condition on f as follows.

Theorem 4.2. *Let $f : M \to S$ be an onto homomorphism of Γ-near-rings M and S. Let $B = (\delta_B, \eta_B)$ be a intuitionistic fuzzy subset of S. If $f^{-1}(B) = [f^{-1}(\delta_B)f^{-1}(\eta_B)]$ is a intuitionistic fuzzy weak bi-ideal of M, then $B = (\delta_B, \eta_B)$ is a intuitionistic is a fuzzy weak bi-ideal of S.*

PROOF. Let $x, y, z \in S$. Then $f(a) = x, f(b) = y$ and $f(c) = z$ for some $a, b, c \in M$ and $\alpha, \beta \in \Gamma$. It follows that

$$\begin{aligned}\delta(x-y) &= \delta(f(a)-f(b)) \\ &= \delta(f(a-b)) \\ &= f^{-1}(\delta)(a-b) \\ &\geqslant \min\{f^{-1}(\delta)(a), f^{-1}(\delta)(b)\} \\ &= \min\{\delta(f(a)), \delta(f(b))\} \\ &= \min\{\delta(x), \delta(y)\}. \\ \eta(x-y) &= \eta(f(a)-f(b)) \\ &= \eta(f(a-b)) \\ &= f^{-1}(\eta)(a-b) \\ &\leqslant \max\{f^{-1}(\eta)(a), f^{-1}(\eta)(b)\} \\ &= \max\{\eta(f(a)), \eta(f(b))\} \\ &= \max\{\eta(x), \eta(y)\}.\end{aligned}$$

And

$$\begin{aligned}\delta(x\alpha y\beta z) &= \delta(f(a)\alpha f(b)\beta f(c)) \\ &= \delta(f(a\alpha b\beta c)) \\ &= f^{-1}(\delta)(a\alpha b\beta c) \\ &\geqslant \min\{f^{-1}(\delta)(a), f^{-1}(\delta)(b), f^{-1}(\delta)(c)\} \\ &= \min\{\delta(f(a)), \delta(f(b)), \delta(f(c))\} \\ &= \min\{\delta(x), \delta(y), \delta(z)\}. \\ \delta(x\alpha y\beta z) &= \delta(f(a)\alpha f(b)\beta f(c)) \\ &= \delta(f(a\alpha b\beta c)) \\ &= f^{-1}(\delta)(a\alpha b\beta c) \\ &\geqslant \min\{f^{-1}(\delta)(a), f^{-1}(\delta)(b), f^{-1}(\delta)(c)\} \\ &= \min\{\delta(f(a)), \delta(f(b)), \delta(f(c))\} \\ &= \min\{\delta(x), \delta(y), \delta(z)\}.\end{aligned}$$

Hence δ is a fuzzy weak bi-ideal of S. □

THEOREM 4.3. *Let $f : M \to S$ be an onto Γ-near-ring homomorphism. If $B = (\eta_B, \lambda_B)$ is an intuitionistic fuzzy weak bi-ideal of M, then $f(B) = f(\eta_B, \lambda_B)$ is an intuitionistic fuzzy weak bi-ideal of S.*

PROOF. Let η be a fuzzy weak bi-ideal of M. Since $f(\eta)(x') = \sup_{f(x)=x'} (\eta(x))$, for $x' \in S$ and hence $f(\eta)$ is nonempty. Let $x', y' \in S$ and $\alpha, \beta \in \Gamma$. Then we have $\{x | x \in f^{-1}(x' - y')\} \supseteq \{x - y | x \in f^{-1}(x')$ and $y \in f^{-1}(y')\}$

and $\{x|x \in f^{-1}(x'y')\} \supseteq \{x\alpha y|x \in f^{-1}(x')$ and $y \in f^{-1}(y')\}$.

$$\begin{aligned} f(\eta)(x'-y') &= \sup_{f(z)=x'-y'} \{\eta(z)\} \\ &\geqslant \sup_{f(x)=x',f(y)=y'} \{\eta(x-y)\} \\ &\geqslant \sup_{f(x)=x',f(y)=y'} \{\min\{\eta(x),\eta(y)\}\} \\ &= \min\{\sup_{f(x)=x'}\{\eta(x)\}, \sup_{f(y)=y'}\{\eta(y)\}\} \\ &= \min\{f(\eta)(x'), f(\eta)(y')\}. \end{aligned}$$

and

$$\begin{aligned} f(\delta)(x'-y') &= \inf_{f(z)=x'-y'} \{\delta(z)\} \\ &\leqslant \inf_{f(x)=x',f(y)=y'} \{\delta(x-y)\} \\ &\leqslant \inf_{f(x)=x',f(y)=y'} \{\max\{\delta(x),\delta(y)\}\} \\ &= \max\{\inf_{f(x)=x'}\{\delta(x)\}, \inf_{f(y)=y'}\{\delta(y)\}\} \\ &= \max\{f(\delta)(x'), f(\delta)(y')\}. \end{aligned}$$

Next,

$$\begin{aligned} f(\eta)(x'\alpha y'\beta z') &= \sup_{f(w)=x'\alpha y'\beta z'} \{\eta(w)\} \\ &\geqslant \sup_{f(x)=x',f(y)=y',f(z)=z'} \{\eta(x\alpha y\beta z)\} \\ &\geqslant \sup_{f(x)=x',f(y)=y',f(z)=z'} \{\min\{\eta(x),\eta(y),\eta(z)\}\} \\ &= \min\{\sup_{f(x)=x'}\{\eta(x)\}, \sup_{f(y)=y'}\{\eta(y)\}, \sup_{f(z)=z'}\{\eta(z)\}\} \\ &= \min\{f(\eta)(x'), f(\eta)(y'), f(\eta)(z')\}. \end{aligned}$$

and

$$\begin{aligned} f(\delta)(x'\alpha y'\beta z') &= \inf_{f(w)=x'\alpha y'\beta z'} \{\delta(w)\} \\ &\leqslant \inf_{f(x)=x',f(y)=y',f(z)=z'} \{\delta(x\alpha y\beta z)\} \\ &\leqslant \inf_{f(x)=x',f(y)=y',f(z)=z'} \{\max\{\delta(x),\delta(y),\delta(z)\}\} \\ &= \max\{\inf_{f(x)=x'}\{\delta(x)\}, \inf_{f(y)=y'}\{\delta(y)\}, \inf_{f(z)=z'}\{\delta(z)\}\} \\ &= \max\{f(\delta)(x'), f(\delta)(y'), f(\delta)(z')\}. \end{aligned}$$

Therefore $f(\eta)$ is a fuzzy weak bi-ideal of S. □

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K. Arulmozhi, Department of Mathematics, Vels Institute of Science, Technology and Advanced Studies (VISTAS), Chennai-600117.
E-mail address: arulmozhiems@gmail.com

V. Chinnadurai, Department of Mathematics, Annamalai University, Annamalainagar - 608002, India.
E-mail address: kv.chinnadurai@yahoo.com

A. Swaminathan, Department of Mathematics, Akshya college of Engineering, chennai.
E-mail address: nathanswamiin@gmail.com