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WEAKLY $(1,2)^*$-$\ddot{g}$-OPEN AND WEAKLY $(1,2)^*$-$\ddot{g}$-CLOSED FUNCTIONS

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Abstract. In this paper, the concepts of weakly $(1,2)^*$-$\ddot{g}$-continuous functions, weakly $(1,2)^*$-$\ddot{g}$-compact spaces and weakly $(1,2)^*$-$\ddot{g}$-connected spaces are introduced and some of their properties are investigated.

1. Introduction

Ravi and Ganesan [**7**] have introduced the concept of $\ddot{g}$-closed sets and studied their most fundamental properties in topological spaces. In this paper, we introduce a new class of generalized closed sets called weakly $(1,2)^*$-$\ddot{g}$-closed sets which contains the above mentioned class. Also, we investigate the relationships among the related generalized closed sets.

2. Preliminaries

Throughout this paper, (X, τ_1, τ_2), (Y, τ_1, τ_2) and (Z, η_1, η_2) (briefly, X, Y and Z) will denote bitopological spaces.

Definition 2.1. Let S be a subset of X. Then S is said to be $\tau_{1,2}$-open [**8**] if $S = A \cup B$ where $A \in \tau_1$ and $B \in \tau_2$. The complement of $\tau_{1,2}$-open set is called $\tau_{1,2}$-closed.

Notice that $\tau_{1,2}$-open sets need not necessarily form a topology.

Definition 2.2. ([**8**]) Let S be a subset of a bitopological space X. Then

(1) the $\tau_{1,2}$-closure of S, denoted by $\tau_{1,2}$-$cl(S)$, is defined as $\cap\{F : S \subseteq F$ and F is $\tau_{1,2}$-closed $\}$.

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(2) the $\tau_{1,2}$-interior of S, denoted by $\tau_{1,2}$-$int(S)$, is defined as $\cup\{F : F \subseteq S$ and F is $\tau_{1,2}$-open $\}$.

Definition 2.3. A subset S of a bitopological space X is called

(1) $(1,2)^*$-semi-open set [**9**] if $S \subseteq \tau_{1,2}$-$cl(\tau_{1,2}$-$int(S))$;
(2) $(1,2)^*$-α-open set [**8**] if $S \subseteq \tau_{1,2}$-$int(\tau_{1,2}$-$cl(\tau_{1,2}$-$int(S)))$;
(3) regular $(1,2)^*$-open set [**10**] if $S = \tau_{1,2}$-$int(\tau_{1,2}$-$cl(S))$;
(4) $(1,2)^*$-π-open set [**10**] if S is the finite union of regular $(1,2)^*$-open sets.

The complements of the above mentioned open sets are called their respective closed sets.

The $(1,2)^*$-semi-closure [**9**] (resp. $(1,2)^*$-α-closure [**10**]) of a subset S of X, $(1,2)^*$-$scl(S)$ (resp. $(1,2)^*$-$\alpha cl(S)$), is defined to be the intersection of all $(1,2)^*$-semi-closed (resp. $(1,2)^*$-α-closed) sets of X containing S. It is known that $(1,2)^*$-$scl(S)$ (resp. $(1,2)^*$-$\alpha cl(S)$) is a $(1,2)^*$-semi-closed (resp. an $(1,2)^*$-α-closed) set.

Definition 2.4. A subset S of a bitopological space X is called

(1) $(1,2)^*$-generalized closed (briefly, $(1,2)^*$-g-closed) set [**11**] if $\tau_{1,2}$-$cl(S) \subseteq U$ whenever $S \subseteq U$ and U is $\tau_{1,2}$-open in X.
(2) $(1,2)^*$-semi-generalized closed (briefly, $(1,2)^*$-sg-closed) set [**9**] if $(1,2)^*$-$scl(S) \subseteq U$ whenever $S \subseteq U$ and U is $(1,2)^*$-semi-open in X.
(3) $(1,2)^*$-α-generalized closed (briefly, $(1,2)^*$-αg-closed) set [**12**] if $(1,2)^*$-$\alpha cl(S) \subseteq U$ whenever $S \subseteq U$ and U is $\tau_{1,2}$-open in X.
(4) $(1,2)^*$-$\ddot{g}$-closed set [**3**] if $\tau_{1,2}$-$cl(S) \subseteq U$ whenever $S \subseteq U$ and U is $(1,2)^*$-sg-open in X.
(5) $(1,2)^*$-πg-closed set [**10**] if $\tau_{1,2}$-$cl(S) \subseteq U$ whenever $S \subseteq U$ and U is $(1,2)^*$-π-open in X.

The complements of the above mentioned open sets are called their respective closed sets.

The family of all $(1,2)^*$-$\ddot{g}$-open (resp. $(1,2)^*$-$\ddot{g}$-closed) sets in X is denoted by $(1,2)^*$-$\ddot{G}O(X)$ (resp. $(1,2)^*$-$\ddot{G}C(X)$).

Definition 2.5. ([**4**]) For every set $S \subseteq X$, we define the $(1,2)^*$-$\ddot{g}$-closure of S to be the intersection of all $(1,2)^*$-$\ddot{g}$-closed sets containing S. That is $(1,2)^*$-$\ddot{g}$-$cl(S) = \cap\{F : S \subseteq F \in (1,2)^*$-$\ddot{G}C(X)\}$.

Definition 2.6. Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be two bitopological spaces. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called

(1) completely $(1,2)^*$-continuous [**10**] (resp. $(1,2)^*$-R-map [**10**]) if $f^{-1}(V)$ is regular $(1,2)^*$-open in X for each $\sigma_{1,2}$-open (resp. regular $(1,2)^*$-open) set V of Y.
(2) perfectly $(1,2)^*$-continuous [**10**] if $f^{-1}(V)$ is both $\tau_{1,2}$-open and $\tau_{1,2}$-closed in X for each $\sigma_{1,2}$-open set V of Y.
(3) $(1,2)^*$-$\ddot{g}$-continuous [**4**] if $f^{-1}(V)$ is $(1,2)^*$-$\ddot{g}$-closed in X for every $\sigma_{1,2}$-closed set V of Y.

(4) $(1,2)^*$-$\ddot{g}$-irresolute [**4**] if $f^{-1}(V)$ is $(1,2)^*$-$\ddot{g}$-closed in X for every $(1,2)^*$-$\ddot{g}$-closed set V of Y.
(5) $(1,2)^*$-sg-irresolute [**13**] if $f^{-1}(V)$ is $(1,2)^*$-sg-open in X for every $(1,2)^*$-sg-open set V of Y.
(6) $(1,2)^*$-$\ddot{g}$-closed [**5**] if the image of every $\tau_{1,2}$-closed set in X is $(1,2)^*$-$\ddot{g}$-closed in Y.

Definition 2.7. A subset S of a bitopological space X is called
(1) weakly $(1,2)^*$-g-closed (briefly, $(1,2)^*$-wg-closed) set [**14**] if $\tau_{1,2}$-$cl(\tau_{1,2}$-$int(S)) \subseteq U$ whenever $S \subseteq U$ and U is $\tau_{1,2}$-open in X.
(2) weakly $(1,2)^*$-πg-closed (briefly, $(1,2)^*$-$w\pi g$-closed) set [**14**] if $\tau_{1,2}$-$cl(\tau_{1,2}$-$int(S)) \subseteq U$ whenever $S \subseteq U$ and U is $(1,2)^*$-πg-open in X.
(3) regular weakly $(1,2)^*$-generalized closed (briefly, $(1,2)^*$-rwg-closed) set [**14**] if $\tau_{1,2}$-$cl(\tau_{1,2}$-$int(S)) \subseteq U$ whenever $S \subseteq U$ and U is regular $(1,2)^*$-open in X.

Definition 2.8. ([**5**]) A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be an $(1,2)^*$-$\ddot{g}$-open map if the image $f(S)$ is $(1,2)^*$-$\ddot{g}$-open in Y for each $\tau_{1,2}$-open set S of X.

Remark 2.1. ([**4**]) Every $\tau_{1,2}$-open set is $(1,2)^*$-sg-open but not conversely.

Remark 2.2. ([**14**]) For a subset of a bitopological space, we have following implications:

$$\text{regular } (1,2)^*\text{-open} \to (1,2)^*\text{-}\pi\text{-open} \to \tau_{1,2}\text{-open}$$

Definition 2.9. A subset S of a bitopological space X is said to be nowhere dense if $\tau_{1,2}$-$int(\tau_{1,2}$-$cl(S)) = \phi$.

Definition 2.10. Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a function. Then f is said to be
(1) contra-$(1,2)^*$-$\ddot{g}$-continuous [**14**] if $f^{-1}(V)$ is $(1,2)^*$-$\ddot{g}$-closed in X for every $\sigma_{1,2}$-open set of Y.
(2) $(1,2)^*$-continuous [**14**] if $f^{-1}(V)$ is $\tau_{1,2}$-closed in X for every $\sigma_{1,2}$-closed set of Y.

Remark 2.3. ([**4**]) Every $(1,2)^*$-continuous function is $(1,2)^*$-$\ddot{g}$-continuous but not conversely.

3. WEAKLY $(1,2)^*$-$\ddot{g}$-CLOSED SETS

We introduce the definition of weakly $(1,2)^*$-$\ddot{g}$-closed sets in bitopological spaces and study the relationships of such sets.

Definition 3.1. A subset S of a bitopological space X is called a weakly $(1,2)^*$-$\ddot{g}$-closed (briefly, $(1,2)^*$-$w\ddot{g}$-closed) set if $\tau_{1,2}$-$cl(\tau_{1,2}$-$int(S)) \subseteq U$ whenever $S \subseteq U$ and U is $(1,2)^*$-sg-open in X.

Theorem 3.1. *Every $(1,2)^*$-$\ddot{g}$-closed set is $(1,2)^*$-$w\ddot{g}$-closed but not conversely.*

Example 3.1. Let $X = \{a_1, a_2, a_3\}$, $\tau_1 = \{\phi, X, \{a_1, a_2\}\}$ and $\tau_2 = \{\phi, X\}$. Then the sets in $\{\phi, X, \{a_1, a_2\}\}$ are called $\tau_{1,2}$-open and the sets in $\{\phi, X, \{a_3\}\}$ are called $\tau_{1,2}$-closed. Then the set $\{a_1\}$ is $(1,2)^*$-$w\ddot{g}$-closed set but it is not a $(1,2)^*$-$\ddot{g}$-closed in X.

Theorem 3.2. *Every $(1,2)^*$-$w\ddot{g}$-closed set is $(1,2)^*$-wg-closed but not conversely.*

Proof. Let S be any $(1,2)^*$-$w\ddot{g}$-closed set and V be any $\tau_{1,2}$-open set containing S. Then V is a $(1,2)^*$-sg-open set containing S. We have $\tau_{1,2}$-$cl(\tau_{1,2}$-$int(S)) \subseteq U$. Thus, S is $(1,2)^*$-wg-closed. □

Example 3.2. Let $X = \{a_1, a_2, a_3\}$, $\tau_1 = \{\phi, X, \{a_1\}\}$ and $\tau_2 = \{\phi, X\}$. Then the sets in $\{\phi, X, \{a_1\}\}$ are called $\tau_{1,2}$-open and the sets in $\{\phi, X, \{a_2, a_3\}\}$ are called $\tau_{1,2}$-closed. Then the set $\{a_1, a_2\}$ is $(1,2)^*$-wg-closed but it is not a $(1,2)^*$-$w\ddot{g}$-closed.

Theorem 3.3. *Every $(1,2)^*$-$w\ddot{g}$-closed set is $(1,2)^*$-$w\pi g$-closed but not conversely.*

Proof. Let S be any $(1,2)^*$-$w\ddot{g}$-closed set and V be any $(1,2)^*$-π-open set containing S. Then V is a $(1,2)^*$-sg-open set containing S. We have $\tau_{1,2}$-$cl(\tau_{1,2}$-$int(S)) \subseteq U$. Thus, S is $(1,2)^*$-$w\pi g$-closed. □

Example 3.3. In Example 3.2, the set $\{a_1, a_3\}$ is $(1,2)^*$-$w\pi g$-closed but it is not a $(1,2)^*$-$w\ddot{g}$-closed.

Theorem 3.4. *Every $(1,2)^*$-$w\ddot{g}$-closed set is $(1,2)^*$-rwg-closed but not conversely.*

Proof. Let S be any $(1,2)^*$-$w\ddot{g}$-closed set and V be any regular $(1,2)^*$-open set containing S. Then V is a $(1,2)^*$-sg-open set containing S. We have $\tau_{1,2}$-$cl(\tau_{1,2}$-$int(S)) \subseteq V$. Thus, S is $(1,2)^*$-rwg-closed. □

Example 3.4. In Example 3.2, the set $\{a_1\}$ is $(1,2)^*$-rwg-closed but it is not a $(1,2)^*$-$w\ddot{g}$-closed.

Theorem 3.5. *If a subset S of a bitopological space X is both $\tau_{1,2}$-closed and $(1,2)^*$-αg-closed, then it is $(1,2)^*$-$w\ddot{g}$-closed in X.*

Proof. Let S be an $(1,2)^*$-αg-closed set in X and V be any $\tau_{1,2}$-open set containing S. Then $V \supseteq (1,2)^*$-$\alpha cl(S) = S \cup \tau_{1,2}$-$cl(\tau_{1,2}$-$int(\tau_{1,2}$-$cl(S)))$. Since S is $\tau_{1,2}$-closed, $V \supseteq \tau_{1,2}$-$cl(\tau_{1,2}$-$int(S))$ and hence $(1,2)^*$-$w\ddot{g}$-closed in X. □

Theorem 3.6. *If a subset S of a bitopological space X is both $\tau_{1,2}$-open and $(1,2)^*$-$w\ddot{g}$-closed, then it is $\tau_{1,2}$-closed.*

Proof. Since S is both $\tau_{1,2}$-open and $(1,2)^*$-$w\ddot{g}$-closed, $S \supseteq \tau_{1,2}$-$cl(\tau_{1,2}$-$int(S))$ $= \tau_{1,2}$-$cl(S)$ and hence S is $\tau_{1,2}$-closed in X. □

Corollary 3.1. *If a subset S of a bitopological space X is both $\tau_{1,2}$-open and $(1,2)^*$-$w\ddot{g}$-closed, then it is both regular $(1,2)^*$-open and regular $(1,2)^*$-closed in X.*

THEOREM 3.7. *Let X be a bitopological space and $S \subseteq X$ be $\tau_{1,2}$-open. Then, S is $(1,2)^*$-$w\ddot{g}$-closed if and only if S is $(1,2)^*$-$\ddot{g}$-closed.*

PROOF. Let S be $(1,2)^*$-$\ddot{g}$-closed. By Theorem 3.1, it is $(1,2)^*$-$w\ddot{g}$-closed.

Conversely, let S be $(1,2)^*$-$w\ddot{g}$-closed. Since S is $\tau_{1,2}$-open, by Theorem 3.6, S is $\tau_{1,2}$-closed. Hence S is $(1,2)^*$-$\ddot{g}$-closed. □

THEOREM 3.8. *If a set S is $(1,2)^*$-$w\ddot{g}$-closed then $\tau_{1,2}$-$cl(\tau_{1,2}$-$int(S)) - S$ contains no non-empty $(1,2)^*$-sg-closed set.*

PROOF. Let F be a $(1,2)^*$-sg-closed set such that $F \subseteq \tau_{1,2}$-$cl(\tau_{1,2}$-$int(S)) - S$. Since F^c is $(1,2)^*$-sg-open and $S \subseteq F^c$, from the definition of $(1,2)^*$-$w\ddot{g}$-closedness, it follows that $\tau_{1,2}$-$cl(\tau_{1,2}$-$int(S)) \subseteq F^c$. That is $F \subseteq (\tau_{1,2}$-$cl(\tau_{1,2}$-$int(S)))^c$. This implies that $F \subseteq (\tau_{1,2}$-$cl(\tau_{1,2}$-$int(S))) \cap (\tau_{1,2}$-$cl(\tau_{1,2}$-$int(S)))^c = \phi$. □

THEOREM 3.9. *If a subset S of a bitopological space X is nowhere dense, then it is $(1,2)^*$-$w\ddot{g}$-closed.*

PROOF. Since $\tau_{1,2}$-$int(S) \subseteq \tau_{1,2}$-$int(\tau_{1,2}$-$cl(S))$ and S is nowhere dense, $\tau_{1,2}$-$int(S) = \phi$. Therefore $\tau_{1,2}$-$cl(\tau_{1,2}$-$int(S)) = \phi$ and hence S is $(1,2)^*$-$w\ddot{g}$-closed in X.

The converse of Theorem 3.9 need not be true as seen in the following example. □

EXAMPLE 3.5. Let $X = \{a_1, a_2, a_3\}$, $\tau_1 = \{\phi, X, \{a_1\}\}$ and $\tau_2 = \{\phi, X, \{a_2, a_3\}\}$. Then the sets in $\{\phi, X, \{a_1\}, \{a_2, a_3\}\}$ are called $\tau_{1,2}$-open and the sets in $\{\phi, X, \{a_1\}, \{a_2, a_3\}\}$ are called $\tau_{1,2}$-closed. Then the set $\{a_1\}$ is $(1,2)^*$-$w\ddot{g}$-closed set but not nowhere dense in X.

REMARK 3.1. The following examples show that $(1,2)^*$-$w\ddot{g}$-closedness and $(1,2)^*$-semi-closedness are independent.

EXAMPLE 3.6. In Example 3.1, we have the set $\{a_1, a_3\}$ is $(1,2)^*$-$w\ddot{g}$-closed set but not $(1,2)^*$-semi-closed in X.

EXAMPLE 3.7. Let $X = \{a_1, a_2, a_3\}$, $\tau_1 = \{\phi, X, \{a_1\}\}$ and $\tau_2 = \{\phi, X, \{a_2\}\}$. Then the sets in $\{\phi, X, \{a_1\}, \{a_2\}, \{a_1, a_2\}\}$ are called $\tau_{1,2}$-open and the sets in $\{\phi, X, \{a_3\}, \{a_1, a_3\}, \{a_2, a_3\}\}$ are called $\tau_{1,2}$-closed. Then the set $\{a_1\}$ is $(1,2)^*$-semi-closed set but not $(1,2)^*$-$w\ddot{g}$-closed in X.

REMARK 3.2. From the above discussions and known results in [**12**]. We obtain the following diagram, where $A \to B$ represents A implies B but not conversely.

Diagram

$$\tau_{1,2}\text{-closed} \to (1,2)^*\text{-}w\ddot{g}\text{-closed} \to (1,2)^*\text{-}wg\text{-closed} \to (1,2)^*\text{-}w\pi g\text{-closed} \to (1,2)^*\text{-}rwg\text{-closed}$$

None of the above implications is reversible as shown in the above examples and in the related paper [**14**].

DEFINITION 3.2. A subset S of a bitopological space X is called $(1,2)^*$-$w\ddot{g}$-open set if S^c is $(1,2)^*$-$w\ddot{g}$-closed in X.

PROPOSITION 3.1. (1) *Every* $(1,2)^*$-$\ddot{g}$-*open set is* $(1,2)^*$-$w\ddot{g}$-*open but not conversely.*
(2) *Every* $(1,2)^*$-g-*open set is* $(1,2)^*$-$w\ddot{g}$-*open but not conversely.*

THEOREM 3.10. *A subset* S *of a bitopological space* X *is* $(1,2)^*$-$w\ddot{g}$-*open if* $G \subseteq \tau_{1,2}\text{-}int(\tau_{1,2}\text{-}cl(S))$ *whenever* $G \subseteq S$ *and* G *is* $(1,2)^*$-sg-*closed.*

PROOF. Let S be any $(1,2)^*$-$w\ddot{g}$-open. Then S^c is $(1,2)^*$-$w\ddot{g}$-closed. Let G be a $(1,2)^*$-sg-closed set contained in S. Then G^c is a $(1,2)^*$-sg-open set containing S^c. Since S^c is $(1,2)^*$-$w\ddot{g}$-closed, we have $\tau_{1,2}\text{-}cl(\tau_{1,2}\text{-}int(S^c)) \subseteq G^c$. Therefore $G \subseteq \tau_{1,2}\text{-}int(\tau_{1,2}\text{-}cl(S))$.

Conversely, we suppose that $G \subseteq \tau_{1,2}\text{-}int(\tau_{1,2}\text{-}cl(S))$ whenever $G \subseteq S$ and G is $(1,2)^*$-sg-closed. Then G^c is a $(1,2)^*$-sg-open set containing S^c and $G^c \supseteq (\tau_{1,2}\text{-}int(\tau_{1,2}\text{-}cl(S)))^c$. It follows that $G^c \supseteq \tau_{1,2}\text{-}cl(\tau_{1,2}\text{-}int(S^c))$. Hence S^c is $(1,2)^*$-$w\ddot{g}$-closed and so S is $(1,2)^*$-$w\ddot{g}$-open. □

DEFINITION 3.3. Let $S \subseteq X$. The $(1,2)^*$-kernel of S is defined as the intersection of all $\tau_{1,2}$-open supersets of the set S and is denoted by $(1,2)^*$-$ker(S)$.

LEMMA 3.1. *The following properties hold for subsets* P, Q *of a space* X*:*

(1) $x \in (1,2)^*\text{-}ker(P)$ *if and only if* $P \cap F \neq \phi$ *for any* $\tau_{1,2}$-*closed set* F *containing* x.

(2) $P \subseteq (1,2)^*\text{-}ker(P)$ *and* $P = (1,2)^*\text{-}ker(P)$ *if* P *is* $\tau_{1,2}$-*open in* X.

(3) *If* $P \subseteq Q$, *then* $(1,2)^*\text{-}ker(P) \subseteq (1,2)^*\text{-}ker(Q)$.

THEOREM 3.11. *The following are equivalent for a function* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$.

(1) f *is contra* $(1,2)^*$-$\ddot{g}$-*continuous,*
(2) *the inverse image of every* $\sigma_{1,2}$-*closed set of* Y *is* $(1,2)^*$-$\ddot{g}$-*open.*

PROOF. Let P be any $\sigma_{1,2}$-closed set of Y. Since $Y \backslash P$ is $\sigma_{1,2}$-open, then by (1), it follows that $f^{-1}(Y \backslash P) = X \backslash f^{-1}(P)$ is $(1,2)^*$-$\ddot{g}$-closed. This shows that $f^{-1}(P)$ is $(1,2)^*$-$\ddot{g}$-open in X.

Converse is similar. □

THEOREM 3.12. *Suppose that* $(1,2)^*$-$\ddot{G}C(X)$ *is closed under arbitrary intersections. Then the following are equivalent for a function* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$.

(1) f *is contra* $(1,2)^*$-$\ddot{g}$-*continuous,*
(2) *the inverse image of every* $\sigma_{1,2}$-*closed set of* Y *is* $(1,2)^*$-$\ddot{g}$-*open in* X,
(3) *for each* $x \in X$ *and each* $\sigma_{1,2}$-*closed set* Q *in* Y *with* $f(x) \in Q$, *there exists a* $(1,2)^*$-$\ddot{g}$-*open set* P *in* X *such that* $x \in P$ *and* $f(P) \subseteq Q$,
(4) $f((1,2)^*\text{-}\ddot{g}\text{-}cl(P)) \subseteq (1,2)^*\text{-}ker(f(P))$ *for every subset* P *of* X,
(5) $(1,2)^*\text{-}\ddot{g}\text{-}cl(f^{-1}(Q)) \subseteq f^{-1}((1,2)^*\text{-}ker(Q))$ *for every subset* Q *of* Y.

PROOF. (1) $\Rightarrow$ (3). Let $x \in X$ and Q be a $\sigma_{1,2}$-closed set in Y with $f(x) \in Q$. By (1), it follows that $f^{-1}(Y \backslash Q) = X \backslash f^{-1}(Q)$ is $(1,2)^*$-$\ddot{g}$-closed and so $f^{-1}(Q)$ is $(1,2)^*$-$\ddot{g}$-open. Take $P = f^{-1}(Q)$. We obtain that $x \in P$ and $f(P) \subseteq Q$.

$(3) \Rightarrow (2)$. Let Q be $\sigma_{1,2}$-closed set in Y with $x \in f^{-1}(Q)$. Since $f(x) \in Q$, by (3) there exists a $(1,2)^*$-$\ddot{g}$-open set P in X containing x such that $f(P) \subseteq Q$. It follows that $x \in P \subseteq f^{-1}(Q)$. Hence $f^{-1}(Q)$ is $(1,2)^*$-$\ddot{g}$-open.

$(2) \Rightarrow (1)$. Follows from the previous Theorem.

$(2) \Rightarrow (4)$. Let P be any subset of X. Let $y \notin (1,2)^*$-$ker(f(P))$. Then there exists a $\sigma_{1,2}$-closed set F containing y such that $f(P) \cap F = \phi$. Hence, we have $P \cap f^{-1}(F) = \phi$ and $(1,2)^*$-$\ddot{g}$-$cl(P) \cap f^{-1}(F) = \phi$. Hence, we obtain $f((1,2)^*$-$\ddot{g}$-$cl(P)) \cap F = \phi$ and $y \notin f((1,2)^*$-$\ddot{g}$-$cl(P))$. Thus, $f((1,2)^*$-$\ddot{g}$-$cl(P)) \subseteq (1,2)^*$-$ker(f(P))$.

$(4) \Rightarrow (5)$. Let Q be any subset of Y. By (4), $f((1,2)^*$-$\ddot{g}$-$cl(f^{-1}(Q))) \subseteq (1,2)^*$-$ker(Q)$ and $(1,2)^*$-$\ddot{g}$-$cl(f^{-1}(Q)) \subseteq f^{-1}((1,2)^*$-$ker(Q))$.

$(5) \Rightarrow (1)$. Let Q be any $\sigma_{1,2}$-open set of Y. By (5), $(1,2)^*$-$\ddot{g}$-$cl(f^{-1}(Q)) \subseteq f^{-1}((1,2)^*$-$ker(Q)) = f^{-1}(Q)$ and $(1,2)^*$-$\ddot{g}$-$cl(f^{-1}(Q)) = f^{-1}(Q)$. We obtain that $f^{-1}(Q)$ is $(1,2)^*$-$\ddot{g}$-closed in X. □

4. WEAKLY $(1,2)^*$-$\ddot{g}$-OPEN AND WEAKLY $(1,2)^*$-$\ddot{g}$-CLOSED FUNCTIONS

DEFINITION 4.1. Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be bitopological spaces. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called weakly $(1,2)^*$-$\ddot{g}$-open (briefly, $(1,2)^*$-$w\ddot{g}$-open) if $f(V)$ is a $(1,2)^*$-$w\ddot{g}$-open set in Y for each $\tau_{1,2}$-open set V of X.

REMARK 4.1. Every $(1,2)^*$-$\ddot{g}$-open function is $(1,2)^*$-$w\ddot{g}$-open but not conversely.

EXAMPLE 4.1. Let $X = Y = \{a_1, a_2, a_3, a_4\}$, $\tau_1 = \{\phi, X, \{a_1\}\}$ and $\tau_2 = \{\phi, X, \{a_1, a_2, a_4\}\}$. Then the sets in $\{\phi, X, \{a_1\}, \{a_1, a_2, a_4\}\}$ are called $\tau_{1,2}$-open and the sets in $\{\phi, X, \{a_3\}, \{a_2, a_3, a_4\}\}$ are called $\tau_{1,2}$-closed. Let $\sigma_1 = \{\phi, Y, \{a_1\}\}$ and $\sigma_2 = \{\phi, Y, \{a_2, a_3\}\}$. Then the sets in $\{\phi, Y, \{a_1\}, \{a_2, a_3\}, \{a_1, a_2, a_3\}\}$ are called $\sigma_{1,2}$-open and the sets in $\{\phi, Y, \{a_4\}, \{a_1, a_4\}, \{a_2, a_3, a_4\}\}$ are called $\sigma_{1,2}$-closed. Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be the identity function. Then f is $(1,2)^*$-$w\ddot{g}$-open but not $(1,2)^*$-$\ddot{g}$-open.

DEFINITION 4.2. Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be bitopological spaces. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called weakly $(1,2)^*$-$\ddot{g}$-closed (briefly, $(1,2)^*$-$w\ddot{g}$-closed) if $f(V)$ is a $(1,2)^*$-$w\ddot{g}$-closed set in Y for each $\tau_{1,2}$-closed set V of X.

It is clear that an $(1,2)^*$-open function is $(1,2)^*$-$w\ddot{g}$-open and a $(1,2)^*$-closed function is $(1,2)^*$-$w\ddot{g}$-closed.

THEOREM 4.1. *Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be bitopological spaces. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is $(1,2)^*$-$w\ddot{g}$-closed if and only if for each subset Q of Y and for each $\tau_{1,2}$-open set G containing $f^{-1}(Q)$ there exists a $(1,2)^*$-$w\ddot{g}$-open set F of Y such that $Q \subseteq F$ and $f^{-1}(F) \subseteq G$.*

PROOF. Let Q be any subset of Y and let G be an $\tau_{1,2}$-open subset of X such that $f^{-1}(Q) \subseteq G$. Then $F = Y \backslash f(X \backslash G)$ is $(1,2)^*$-$w\ddot{g}$-open set containing Q and $f^{-1}(F) \subseteq G$.

Conversely, let U be any $\tau_{1,2}$-closed subset of X. Then $f^{-1}(Y \backslash f(U)) \subseteq X \backslash U$ and $X \backslash U$ is $\tau_{1,2}$-open. According to the assumption, there exists a $(1,2)^*$-$w\ddot{g}$-open set F of Y such that $Y\backslash f(U) \subseteq F$ and $f^{-1}(F) \subseteq X\backslash U$. Then $U \subseteq X\backslash f^{-1}(F)$. From $Y\backslash F \subseteq f(U) \subseteq f(X\backslash f^{-1}(F)) \subseteq Y\backslash F$, it follows that $f(U) = Y\backslash F$, so $f(U)$ is $(1,2)^*$-$w\ddot{g}$-closed in Y. Therefore f is a $(1,2)^*$-$w\ddot{g}$-closed function. □

REMARK 4.2. The composition of two $(1,2)^*$-$w\ddot{g}$-closed functions need not be a $(1,2)^*$-$w\ddot{g}$-closed as we can see from the following example.

EXAMPLE 4.2. Let $X = Y = Z = \{a_1, a_2, a_3\}$, $\tau_1 = \{\phi, X, \{a_1\}\}$ and $\tau_2 = \{\phi, X, \{a_1, a_2\}\}$. Then the sets in $\{\phi, X, \{a_1\}, \{a_1, a_2\}\}$ are called $\tau_{1,2}$-open and the sets in $\{\phi, X, \{a_3\}, \{a_2, a_3\}\}$ are called $\tau_{1,2}$-closed. Let $\sigma_1 = \{\phi, Y, \{a_1\}\}$ and $\sigma_2 = \{\phi, Y, \{a_2, a_3\}\}$. Then the sets in $\{\phi, Y, \{a_1\}, \{a_2, a_3\}\}$ are called $\sigma_{1,2}$-open and the sets in $\{\phi, Y, \{a_1\}, \{a_2, a_3\}\}$ are called $\sigma_{1,2}$-closed. Let $\eta_1 = \{\phi, Z, \{a_1, a_2\}\}$ and $\eta_2 = \{\phi, Z\}$. Then the sets in $\{\phi, Z, \{a_1, a_2\}\}$ are called $\eta_{1,2}$-open and the sets in $\{\phi, Z, \{a_3\}\}$ are called $\sigma_{1,2}$-closed. We define $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ by $f(a_1) = a_3$, $f(a_2) = a_2$ and $f(a_3) = a_1$ and let $g : (Y, \sigma_1, \sigma_2) \to (Z, \eta_1, \eta_2)$ be the identity function. Hence both f and g are $(1,2)^*$-$w\ddot{g}$-closed functions. For a $\tau_{1,2}$-closed set $U = \{a_2, a_3\}$, $(g \circ f)(U) = g(f(U)) = g(\{a_1, a_2\}) = \{a_1, a_2\}$ which is not $(1,2)^*$-$w\ddot{g}$-closed in Z. Hence the composition of two $(1,2)^*$-$w\ddot{g}$-closed functions need not be a $(1,2)^*$-$w\ddot{g}$-closed.

THEOREM 4.2. *Let (X, τ_1, τ_2), (Y, σ_1, σ_2) and (Z, η_1, η_2) be bitopological spaces. If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a $(1,2)^*$-closed function and $g : (Y, \sigma_1, \sigma_2) \to (Z, \eta_1, \eta_2)$ is a $(1,2)^*$-$w\ddot{g}$-closed function, then $g \circ f : (X, \tau_1, \tau_2) \to (Z, \eta_1, \eta_2)$ is a $(1,2)^*$-$w\ddot{g}$-closed function.*

DEFINITION 4.3. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called a weakly $(1,2)^*$-$\ddot{g}$-irresolute (briefly, $(1,2)^*$-$w\ddot{g}$-irresolute) function if $f^{-1}(Q)$ is a $(1,2)^*$-$w\ddot{g}$-open set in X for each $(1,2)^*$-$w\ddot{g}$-open set Q of Y.

EXAMPLE 4.3. Let $X = Y = \{a_1, a_2, a_3\}$, $\tau_1 = \{\phi, X, \{a_2\}\}$ and $\tau_2 = \{\phi, X, \{a_1, a_3\}\}$. Then the sets in $\{\phi, X, \{a_2\}, \{a_1, a_3\}\}$ are called $\tau_{1,2}$-open and the sets in $\{\phi, X, \{a_2\}, \{a_1, a_3\}\}$ are called $\tau_{1,2}$-closed. Let $\sigma_1 = \{\phi, Y, \{a_2\}\}$ and $\sigma_2 = \{\phi, Y\}$. Then the sets in $\{\phi, Y, \{a_2\}\}$ are called $\sigma_{1,2}$-open and the sets in $\{\phi, Y, \{a_1, a_3\}\}$ are called $\sigma_{1,2}$-closed. Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be the identity function. Then f is $(1,2)^*$-$w\ddot{g}$-irresolute.

REMARK 4.3. The following examples show that $(1,2)^*$-sg-irresoluteness and $(1,2)^*$-$w\ddot{g}$-irresoluteness are independent of each other.

EXAMPLE 4.4. Let $X = Y = \{a_1, a_2, a_3\}$, $\tau_1 = \{\phi, X, \{a_1, a_2\}\}$ and $\tau_2 = \{\phi, X\}$. Then the sets in $\{\phi, X, \{a_1, a_2\}\}$ are called $\tau_{1,2}$-open and the sets in $\{\phi, X, \{a_3\}\}$ are called $\tau_{1,2}$-closed. Let $\sigma_1 = \{\phi, Y, \{a_1\}\}$ and $\sigma_2 = \{\phi, Y\}$. Then the sets in $\{\phi, Y, \{a_1\}\}$ are called $\sigma_{1,2}$-open and the sets in $\{\phi, Y, \{a_2, a_3\}\}$ are called $\sigma_{1,2}$-closed. Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be the identity function. Then f is $(1,2)^*$-$w\ddot{g}$-irresolute but not $(1,2)^*$-sg-irresolute.

EXAMPLE 4.5. Let $X = Y = \{a_1, a_2, a_3\}$, $\tau_1 = \{\phi, X, \{a_1\}\}$ and $\tau_2 = \{\phi, X, \{a_2\}\}$. Then the sets in $\{\phi, X, \{a_1\}, \{a_2\}, \{a_1, a_2\}\}$ are called $\tau_{1,2}$-open and the sets in $\{\phi, X, \{a_3\}, \{a_1, a_3\}, \{a_2, a_3\}\}$ are called $\tau_{1,2}$-closed. Let $\sigma_1 = \{\phi, Y, \{a_1, a_2\}\}$

and $\sigma_2 = \{\phi, Y\}$. Then the sets in $\{\phi, Y, \{a_1, a_2\}\}$ are called $\sigma_{1,2}$-open and the sets in $\{\phi, Y, \{a_3\}\}$ are called $\sigma_{1,2}$-closed. Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be the identity function. Then f is $(1,2)^*$-sg-irresolute but not $(1,2)^*$-$w\ddot{g}$-irresolute.

Remark 4.4. Every $(1,2)^*$-$\ddot{g}$-irresolute function is $(1,2)^*$-$w\ddot{g}$-continuous but not conversely. Also, the concepts of $(1,2)^*$-$\ddot{g}$-irresoluteness and $(1,2)^*$-$w\ddot{g}$-irresoluteness are independent of each other.

Example 4.6. Let $X = Y = \{a_1, a_2, a_3, a_4\}$, $\tau_1 = \{\phi, X, \{a_1\}\}$ and $\sigma_2 = \{\phi, X, \{a_2, a_3\}\}$. Then the sets in $\{\phi, X, \{a_1\}, \{a_2, a_3\}, \{a_1, a_2, a_3\}\}$ are called $\tau_{1,2}$-open and the sets in $\{\phi, X, \{a_4\}, \{a_1, a_4\}, \{a_2, a_3, a_4\}\}$ are called $\tau_{1,2}$-closed. Let $\sigma_1 = \{\phi, Y, \{a_1\}\}$ and $\sigma_2 = \{\phi, Y, \{a_1, a_2, a_4\}\}$. Then the sets in $\{\phi, Y, \{a_1\}, \{a_1, a_2, a_4\}\}$ are called $\sigma_{1,2}$-open and the sets in $\{\phi, Y, \{a_3\}, \{a_2, a_3, a_4\}\}$ are called $\sigma_{1,2}$-closed. Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be the identity function. Then f is $(1,2)^*$-$w\ddot{g}$-continuous but not $(1,2)^*$-$\ddot{g}$-irresolute.

Example 4.7. Let $X = Y = \{a_1, a_2, a_3\}$, $\tau_1 = \{\phi, X, \{a_1\}\}$ and $\tau_2 = \{\phi, X, \{a_2, a_3\}\}$. Then the sets in $\{\phi, X, \{a_1\}, \{a_2, a_3\}\}$ are called $\tau_{1,2}$-open and the sets in $\{\phi, X, \{a_1\}, \{a_2, a_3\}\}$ are called $\tau_{1,2}$-closed. Let $\sigma_1 = \{\phi, Y, \{a_1\}\}$ and $\sigma_2 = \{\phi, Y, \{a_1, a_2\}\}$. Then the sets in $\{\phi, Y, \{a_1\}, \{a_1, a_2\}\}$ are called $\sigma_{1,2}$-open and the sets in $\{\phi, Y, \{a_3\}, \{a_2, a_3\}\}$ are called $\sigma_{1,2}$-closed. Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be the identity function. Then f is $(1,2)^*$-$w\ddot{g}$-irresolute but not $(1,2)^*$-$\ddot{g}$-irresolute.

Example 4.8. In Example 4.5, then f is $(1,2)^*$-$\ddot{g}$-irresolute but not $(1,2)^*$-$w\ddot{g}$-irresolute.

Theorem 4.3. *The composition of two $(1,2)^*$-$w\ddot{g}$-irresolute functions is also $(1,2)^*$-$w\ddot{g}$-irresolute.*

Theorem 4.4. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ and $g : (Y, \sigma_1, \sigma_2) \to (Z, \eta_1, \eta_2)$ be functions such that $g \circ f : (X, \tau_1, \tau_2) \to (Z, \eta_1, \eta_2)$ is $(1,2)^*$-$w\ddot{g}$-closed function. Then the following statements hold:*

(1) *if f is $(1,2)^*$-continuous and injective, then g is $(1,2)^*$-$w\ddot{g}$-closed.*
(2) *if g is $(1,2)^*$-$w\ddot{g}$-irresolute and surjective, then f is $(1,2)^*$-$w\ddot{g}$-closed.*

Proof. (1) Let F be a $\sigma_{1,2}$-closed set of Y. Since $f^{-1}(F)$ is $\tau_{1,2}$-closed in X, we can conclude that $(g \circ f)(f^{-1}(F))$ is $(1,2)^*$-$w\ddot{g}$-closed in Z. Hence $g(F)$ is $(1,2)^*$-$w\ddot{g}$-closed in Z. Thus g is a $(1,2)^*$-$w\ddot{g}$-closed function.

(2) It can be proved in a similar manner as (1). □

Theorem 4.5. *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is an $(1,2)^*$-$w\ddot{g}$-irresolute function, then it is $(1,2)^*$-$w\ddot{g}$-continuous.*

Remark 4.5. The converse of the above theorem need not be true in general. The function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ in the Example 4.5 is $(1,2)^*$-$w\ddot{g}$-continuous but not $(1,2)^*$-$w\ddot{g}$-irresolute.

Theorem 4.6. *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is surjective $(1,2)^*$-$w\ddot{g}$-irresolute function and X is $(1,2)^*$-$w\ddot{g}$-compact, then Y is $(1,2)^*$-$w\ddot{g}$-compact.*

Theorem 4.7. *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is surjective $(1,2)^*$-$w\ddot{g}$-irresolute function and X is $(1,2)^*$-$w\ddot{g}$-connected, then Y is $(1,2)^*$-$w\ddot{g}$-connected.*

References

[1] P. Bhattacharya and B. K. Lahiri. Semi-generalized closed sets in topology. *Indian J. Math.*, **29**(3)(1987), 375–382.

[2] E. Ekici. On a weaker form of RC-continuity. *An. Uni. Vest Din Timisoara, Ser. Math. Inf.*, **XLII** (fasc.1) (2004), 79–91.

[3] M. Kamaraj, K. Kumaresan, O. Ravi and A. Pandi. $(1,2)^*$-$\ddot{g}$-closed sets in bitopological spaces. *Int. J. Adv. Pure Appl. Math.*, **1**(3)(2011), 98–111.

[4] M. Kamaraj, K. Kumaresan, O. Ravi and A. Pandi. Decompostion of $(1,2)^*$-$\ddot{g}$-continuity in bitopological spaces. (Tp appear).

[5] M. Kamaraj, K. Kumaresan, O. Ravi and A. Pandi. $(1,2)^*$-$\ddot{g}$-closed and $(1,2)^*$-$\ddot{g}$-open maps in bitopological spaces. (To appear).

[6] N. Levine. Generalized closed sets in topology. *Rend. Circ. Math. Palermo*, **19**(1)(1970), 89–96.

[7] O. Ravi and S. Ganesan. $\ddot{g}$-closed sets in topology. *Int. J. Comp. Sci. and Emerging Tech.*, **2**(3)(2011), 330–337.

[8] O. Ravi and M. L. Thivagar. On stronger forms of $(1,2)^*$-quotient mappings in bitopological spaces., *Int. J. Math. Game Theory and Algebra*, **14**(6)(2004), 481–492.

[9] o. Ravi and m. l. Thivagar. A bitopological $(1,2)^*$-semi-generalized continuous maps. *Bull. Malaysian Math. Sci. Soc.*, **2**(29)(1)(2006), 76–88.

[10] O. Ravi, M. L. Thivagar and E. Ekici. Decomposition of $(1,2)^*$-continuity and complete $(1,2)^*$-continuity in bitopological spaces. *An. Uni. Din Oradea. Fasc. Mat.*, Tom **XV**(2008), 29–37.

[11] O. Ravi, M. L. Thivagar and X. Jinjinli. Remarks on extensions of $(1,2)^*$-g-closed mappings in bitopology. *Archimedes J. Math.*, **1**(2)(2011), 177–187.

[12] O. Ravi. S. P. Missier and T. S. Parkunan. On bitopological $(1,2)^*$-generalized homeomorphisms. *Int. J. Contemp. Math. Sci.*, **5**(11)(2010), 543–557.

[13] O. Ravi, A. Pandi, S. P. Missier and T. S. Parkunan. Remarks on bitopological $(1,2)^*$-$r\omega$-homeomorphism. *Int.J. Math. Archive*, **2**(4)(2011), 465–475.

[14] O. Ravi, M. L. Thivagar, K. Kayathri and M. J. Isreal. Mildly $(1,2)^*$-normal spaces and some bitopological functions. *Math. Bohemica*, **135**(1)(2010),1–13.

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