ON ZAGREB INDICES AND COINDICES
OF CLUSTER GRAPHS

Raju B. Jummannaver, Ivan Gutman,
and Ravikiran A. Mundewadi

Communicated by

Abstract. Graphs obtained by deleting a few edges from the complete graph are referred to as cluster graphs. We determine expressions for the first and second Zagreb indices, forgotten topological index, and hyper–Zagreb index of four classes of cluster graphs and their complements, as well as expressions for their coindices. In addition, we correct an error of one of the present authors, regrading an expression for the hyper–Zagreb coindex.

1. Introduction

Throughout this paper we only consider finite, connected graphs. Let $G$ be a graph with $n$ vertices and $m$ edges. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of $G$ and $E(G)$ be an edge set of $G$. The edge between the vertices $u$ and $v$ is denoted by $uv$. The degree of a vertex $v$ in $G$ is the number of edges incident to it and is denoted by $d(v)$. The complement of $G$, denoted by $\overline{G}$, is the graph having the same vertex set as $G$, in which two vertices are adjacent if and only if they are not adjacent in $G$.

A large number of vertex–based graph invariants is studied in the recent and current mathematical literature. The chief motivation for this is the fact that these invariants, usually called “topological indices” found numerous and important applications in chemistry [4, 5, 10, 19].

2010 Mathematics Subject Classification. 05C07; 05C90.

Key words and phrases. Hyper–Zagreb index, hyper–Zagreb coindex, Zagreb index, degree of vertex.
The oldest among these degree–based invariants are the first and second Zagreb indices:

\[ M_1(G) = \sum_{uv \in E(G)} [d(u) + d(v)] \quad \text{and} \quad M_2(G) = \sum_{uv \in E(G)} d(u) d(v) . \]

whose mathematical properties and chemical applications are studied in much detail [2, 12, 15].

Đošlić [6] defined the first and second Zagreb coindices as

\[ \overline{M}_1(G) = \sum_{uv \notin E(G)} [d(u) + d(v)] \quad \text{and} \quad \overline{M}_2(G) = \sum_{uv \notin E(G)} d(u) d(v) \]

respectively, where it is assumed that \( u \neq v \).

A so-called forgotten topological index, \( F(G) \), is defined as [8]

\[ F(G) = \sum_{u \in V(G)} d(u)^3 . \]

It is easy to show that [7, 8]

\[ F(G) = \sum_{uv \in E(G)} [d(u)^2 + d(v)^2] \]

from which the coindex of \( F \) is expressed as [3]

\[ \overline{F}(G) = \sum_{uv \notin E(G)} [d(u)^2 + d(v)^2] . \]

An extension of the Zagreb–index concept, called hyper–Zagreb index, was recently introduced by Shirdel et al. [18]:

\[ HZ(G) = \sum_{uv \in E(G)} [d(u) + d(v)]^2 . \]

The respective coindex is [20]

\[ \overline{HZ}(G) = \sum_{uv \notin E(G)} [d(u) + d(v)]^2 . \]

Recently, one of the present authors [11], determined the the basic relations between hyper–Zagreb index and its coindex for a graph \( G \) and of its complement \( \overline{G} \). In [11], the following relations have been obtained:

**Theorem 1.1.** [11] Let \( G \) be a graph with \( n \) vertices and \( m \) edges. Then

\[ \overline{HZ}(G) = 4m^2 + (n-2)M_1(G) - HZ(G) \]

\[ HZ(\overline{G}) = 2n(n-1)^3 - 12m(n-1)^2 + 4m^2 + (5n-6)M_1(G) - HZ(G) \]

\[ \overline{HZ}(\overline{G}) = 4m(n-1)^2 + 4(n-1)M_1(G) + HZ(G) . \]

The third equality in Theorem 1.1 contains an error. Its correct version should be:

\[ \overline{HZ}(\overline{G}) = 4m(n-1)^2 - 4(n-1)M_1(G) + HZ(G) . \]
2. Cluster graphs

From a chemical point of view, graphs with large number of edges may be considered as representations of inorganic clusters, so-called cluster graphs [14]. Bearing this in mind, we consider here the graphs obtained from the complete graph $K_n$ by removing some of its edges. These graphs were first studied in [13] in connection with graph energy. In the present paper we are concerned with the Zagreb, hyper-Zagreb, and forgotten indices of four classes of cluster graphs.

**Definition 2.1.** [13] Let $e_i$, $i = 1, 2, \ldots, k$, $1 \leq k \leq n - 2$, be distinct edges of the complete graph $K_n$, $n \geq 3$, all being incident to a single vertex. The graph $Ka_n(k)$ is obtained by deleting $e_i$, $i = 1, 2, \ldots, k$ from $K_n$. In addition, $Ka_n(0) \cong K_n$.

**Definition 2.2.** [13] Let $f_i$, $i = 1, 2, \ldots, k$, $1 \leq k \leq \lfloor n/2 \rfloor$ be independent edges of the complete graph $K_n$, $n \geq 3$. The graph $Kb_n(k)$ is obtained by deleting $f_i$, $i = 1, 2, \ldots, k$ from $K_n$. In addition, $Kb_n(0) \cong K_n$.

**Definition 2.3.** [13] Let $V_k$ be a $k$-element subset of the vertex set of the complete graph $K_n$, $2 \leq k \leq n - 1$, $n \geq 3$. The graph $Kc_n(k)$ is obtained by deleting from $K_n$ all the edges connecting pairs of vertices from $V_k$. In addition, $Kc_n(0) \cong Kc_n(1) \cong K_n$.

**Definition 2.4.** [13] Let $3 \leq k \leq n$, $n \geq 3$. The graph $Kd_n(k)$ is obtained by deleting from $K_n$ the edges belonging to a $k$-membered cycle.

3. Results

**Theorem 3.1.** For $n \geq 3$ and $1 \leq k \leq n - 2$,

$$HZ(Ka_n(k)) = 2n^4 - 6n^3 + 6n^2 - 12n^2k + 5nk^2 + 29nk - 2n - k^3 - 4k^2 - 19k.$$  

**Proof.** The graph $Ka_n(k)$ has $n$ vertices and $n(n - 1)/2 - k$ edges. Its edge set can be partitioned into four sets $E_1$, $E_2$, $E_3$, and $E_4$, such that

- $E_1 = \{uv \mid d(u) = n - 1 - k \& d(v) = n - 1\}$
- $E_2 = \{uv \mid d(u) = n - 2 \& d(v) = n - 2\}$
- $E_3 = \{uv \mid d(u) = n - 2 \& d(v) = n - 1\}$
- $E_4 = \{uv \mid d(u) = n - 1 \& d(v) = n - 1\}$.

It is easy to check that $|E_1| = n - k - 1$, $|E_2| = k(k - 1)/2$, $|E_3| = (n - k - 1)k$, and $|E_4| = (n - k - 1)(n - k - 2)/2$. Therefore

$$HZ(Ka_n(k)) = \sum_{u \in E_1} [d(u) + d(v)]^2 + \sum_{u \in E_2} [d(u) + d(v)]^2 + \sum_{u \in E_3} [d(u) + d(v)]^2 + \sum_{u \in E_4} [d(u) + d(v)]^2$$

$$+ \sum_{u \in E_4} [d(u) + d(v)]^2 = (n - k - 1) [(n - 1 - k) + (n - 1)]^2$$

$$+ \frac{1}{2} k(k - 1) [(n - 2) + (n - 2)]^2 + (nk - k^2 - k) [(n - 2) + (n - 1)]^2$$
and the expression given in Theorem 3.1 follows. \[\square\]

**Theorem 3.2.** For \(n \geq 3\) and \(1 \leq k \leq \lfloor n/2 \rfloor\),
\[
HZ(Kb_n(k)) = 2n^4 - 12kn^2 - 6n^3 + 4k^2 + 34kn + 6n^2 - 28k - 2n.
\]

**Proof.** The graph \(Kb_n(k)\) has \(n\) vertices and \(n(n-1)/2 - k\) edges. Its edge set can be partitioned into three sets \(E_1, E_2,\) and \(E_3,\) such that
\[
E_1 = \{uv \mid d(u) = n - 2 \land d(v) = n - 1\},
E_2 = \{uv \mid d(u) = n - 1 \land d(v) = n - 1\},
E_3 = \{uv \mid d(u) = n - 2 \land d(v) = n - 2\}.
\]

Then \(|E_1| = 2k(n-2k), |E_2| = (n-2k)(n-2k-1)/2,\) and \(|E_3| = 2k(2k-1)/2 - k.\) Thus
\[
HZ(Kb_n(k)) = \sum_{uv \in E_1} [d(u) + d(v)]^2 + \sum_{uv \in E_2} [d(u) + d(v)]^2 + \sum_{uv \in E_3} [d(u) + d(v)]^2
= (2nk - 4k^2) [(n-2) + (n-1)]^2 + \frac{1}{2} (n-2k)(n-2k-1) [(n-1) + (n-1)]^2
+ \left[\frac{(2k(2k-1))}{2} - k\right] [(n-2) + (n-2)]^2
\]
and the expression given in Theorem 3.2 follows. \[\square\]

**Theorem 3.3.** For \(n \geq 3\) and \(2 \leq k \leq n-1,\)
\[
HZ(Kc_n(k)) = (-k^2 + nk)(2n-k-1)^2 + \frac{1}{2} (n-k)(n-k-1)(2n-2)^2.
\]

**Proof.** The graph \(Kc_n(k)\) has \(n\) vertices and \(\frac{1}{4}(n-k)(n+k+1)\) edges. The edge set \(E(Kc_n(k))\) can be partitioned into two sets \(E_1\) and \(E_2,\) where
\[
E_1 = \{uv \mid d(u) = n-k \land d(v) = n-1\},
E_2 = \{uv \mid d(u) = n-1 \land d(v) = n-1\}.
\]

We have that \(|E_1| = (n-k)k\) and \(|E_2| = (n-k)(n-k-1)/2.\) Therefore
\[
HZ(Kc_n(k)) = \sum_{uv \in E_1} [d(u) + d(v)]^2 + \sum_{uv \in E_2} [d(u) + d(v)]^2
= (nk - k^2) [(n-k) + (n-1)]^2 + \frac{1}{2} (n-k)(n-k-1) [(n-1) + (n-1)]^2
\]
implying the expression given in Theorem 3.3. \[\square\]

**Theorem 3.4.** For \(3 \leq k \leq n\) and \(n \geq 3,\)
\[
HZ(Kd_n(k)) = 2n^4 - 12kn^2 - 6n^3 + 4k^2 + 44kn + 6n^2 - 52k - 2n.
\]
Proof. The graph $Kd_n(k)$ has $n$ vertices and $n(n - 1)/2 - k$ edges. The edge set $E(Kd_n(k))$ can be partitioned into three sets $E_1$, $E_2$, and $E_3$, such that

$$E_1 = \{uv \mid d(u) = n - 3 \& d(v) = n - 3\}$$

$$E_2 = \{uv \mid d(u) = n - 3 \& d(v) = n - 1\}$$

$$E_3 = \{uv \mid d(u) = n - 1 \& d(v) = n - 1\}.$$  

It is easy to check that $|E_1| = k(k - 1)/2 - k$, $|E_2| = (n - k)k$, and $|E_3| = (n - k)(n - k - 1)/2$. Therefore

$$HZ(Kd_n(k)) = \sum_{uv \in E_1} [d(u) + d(v)]^2 + \sum_{uv \in E_2} [d(u) + d(v)]^2 + \sum_{uv \in E_3} [d(u) + d(v)]^2$$

$$= \left(\frac{k^2 - k}{2} - k\right)\left[(n - 3) + (n - 3)\right]^2 + (nk - k^2)\left[(n - 3) + (n - 1)\right]^2$$

$$+ \frac{1}{2}(n - k)(n - k - 1)\left[(n - 1) + (n - 1)\right]^2$$

resulting in the expression given in Theorem 3.4. \qed

Recently, the following formulas for the first and second Zagreb indices and forgotten index of the graphs $Ka_n(k)$, $Kb_n(k)$, $Kc_n(k)$, and $Kd_n(k)$ were obtained by Ramane et al. [16, 17]:

**Theorem 3.5.** For $n \geq 3$ and $2 \leq k \leq n - 1$,

$$M_1(Ka_n(k)) = (n - k - 1)(n^2 - n - 3k + nk) + k(k - 1)(n - 2)$$

$$M_2(Ka_n(k)) = \frac{1}{2} \left(n^4 - k^2 - n\right) - \frac{3}{2} \left(n^3 - n^2\right) - \frac{9}{2} k + k^2 n - 3k n^2 + 7kn$$

$$F(Ka_n(k)) = n^4 + 3n^2 + 3n k^2 + 15nk - 3n^3 - 6n^2 k - n - k^3 - 3k^2 - 10k.$$

**Theorem 3.6.** For $n \geq 3$ and $1 \leq k \leq \lfloor n/2 \rfloor$,

$$M_1(Kb_n(k)) = (n - 2k)(n^2 + 2nk - 4k - 2n + 1) + 4k(k - 1)(n - 2)$$

$$M_2(Kb_n(k)) = 2k(n - 2k)(n - 2)(n - 1) + \frac{1}{2}(n - 2k)(n - 2k - 1)(n - 1)^2$$

$$+ 2k(k - 1)(n - 2)^2$$

$$F(Kb_n(k)) = 18nk - 6n^2 k - 14k + n^4 - 3n^3 + 3n^2 - n.$$  

**Theorem 3.7.** For $n \geq 3$ and $2 \leq k \leq n - 1$,

$$M_1(Kc_n(k)) = (n - k)k(2n - k - 1) + (n - 1)(n - k)(n - k - 1)$$

$$M_2(Kc_n(k)) = k(n - 1)(n - k)^2 + \frac{1}{2}(n - k)(n - k - 1)(n - 1)^2$$

$$F(Kc_n(k)) = n^4 - 3n^3 + 3n^2 - 3n^2 k^2 + 3n^2 k + 3nk^3 - 3nk - n - k^4 + k.$$  

**Theorem 3.8.** For $3 \leq k \leq n$ and $n \geq 3$,

$$M_1(Kd_n(k)) = k(k - 3)(n - 3) + k(n - k)(2n - 4) + (n - k)(n - k - 1)(n - 1)$$

$$M_2(Kd_n(k)) = \frac{1}{2} k(k - 3)(n - 3)^2 + k(n - k)(n - 3)(n - 1)$$
\[ F(Kd_n(k)) = n^4 - 3n^3 + 3n^2 - n - 6n^2k + 26k + 24nk. \]

4. Coindices of cluster graphs and their complements

The following earlier established results will be needed for the present considerations.

**Theorem 4.1.** [1, 12] Let \( G \) be a simple graph with \( n \) vertices and \( m \) edges. Then

\[
\begin{align*}
M_1(G) &= n(n-1)^2 - 4m(n-1) + M_1(G) \\
M_1(G) &= 2m(n-1) - M_1(G) \\
M_2(G) &= \frac{1}{2} n(n-1)^3 - 3m(n-1)^2 + 2m^2 + \frac{1}{2} (2n-3)M_1(G) - M_2(G) \\
M_2(G) &= 2m^2 - M_2(G) - \frac{1}{2} M_1(G) \\
M_2(G) &= m(n-1)^2 - (n-1)M_1(G) + M_2(G).
\end{align*}
\]

**Theorem 4.2.** [9] Let \( G \) be a simple graph with \( n \) vertices and \( m \) edges. Then

\[
\begin{align*}
F(G) &= n(n-1)^3 - 6m(n-1)^2 + 3(n-1)M_1(G) - F(G) \\
F(G) &= (n-1)M_1(G) - F(G) \\
F(G) &= 2m(n-1)^2 - 2(n-1)M_1(G) + F(G).
\end{align*}
\]

The next lemma follows from Theorems 3.5–3.8 and Eqs. (4.2)–(4.4) by taking into account that \( Ka_n(k), Kb_n(k), Kd_n(k) \) have \( m - k \) edges whereas \( Kc_n(k) \) has \( m - k(k - 1)/2 \) edges:

**Lemma 4.1.** Let \( Ka_n(k), Kb_n(k), Kc_n(k) \) and \( Kd_n(k) \) be the graphs defined in Section 2. Then

\[
\begin{align*}
M_1(Ka_n(k)) &= k^2 + k \\
M_1(Kb_n(k)) &= 2k \\
M_1(Kc_n(k)) &= k^3 - 2k^2 + k \\
M_1(Kd_n(k)) &= 4k \\
\overline{M}_1(Ka_n(k)) &= 2nk - k^2 - 3k \\
\overline{M}_1(Kb_n(k)) &= 2nk - 4k \\
\overline{M}_1(Kc_n(k)) &= k^2n - kn - k^3 + k^2 \\
\overline{M}_1(Kd_n(k)) &= 2nk - 6k \\
\overline{M}_1(Ka_n(k)) &= 2nk - k^2 - 3k
\end{align*}
\]
\[ M_1(Kb_n(k)) = 2nk - 4k \]
\[ M_1(Kc_n(k)) = k^2n - kn - k^3 + k^2 \]
\[ M_1(Kd_n(k)) = 2nk - 6k \]

From Theorems 3.5–3.8 and Eqs. (4.4)–(4.6), we obtain:

**Lemma 4.2.** Using the same notation as in Lemma 4.1,

\[
\begin{align*}
M_2(Ka_n(k)) &= k^2 \\
M_2(Kb_n(k)) &= k \\
M_2(Kc_n(k)) &= \frac{1}{2}k^4 - \frac{3}{2}k^3 + \frac{3}{2}k^2 - \frac{1}{2}k \\
M_2(Kd_n(k)) &= 4k \\
M_2(Ka_n(k)) &= 2k^2 - nk^2 + n^2k - 3nk + 2k \\
M_2(Kb_n(k)) &= n^2k - 4nk + 4k \\
M_2(Kc_n(k)) &= \frac{1}{2}k^4 - k^3n + \frac{1}{2}k^2n^2 - \frac{1}{2}k^3 + k^2n - \frac{1}{2}kn^2 \\
M_2(Kd_n(k)) &= n^2k - 6nk + 9k \\
M_2(Ka_n(k)) &= \frac{1}{2}k^2 - \frac{1}{2}k \\
M_2(Kb_n(k)) &= 2k^2 - 2k \\
M_2(Kc_n(k)) &= 0 \\
M_2(Kd_n(k)) &= 2k^2 - 6k \\
\end{align*}
\]

From Theorems 3.5–3.8 and Eqs. (4.7)–(4.9), we obtain:

**Lemma 4.3.** Using the same notation as in Lemma 4.1,

\[
\begin{align*}
F(Ka_n(k)) &= k^3 + k \\
F(Kb_n(k)) &= 2k \\
F(Kc_n(k)) &= k^4 - 3k^3 + 3k^2 - k \\
F(Kd_n(k)) &= 8k \\
\overline{F}(Ka_n(k)) &= 2n^2k - 2n^2k - 6nk + k^3 + 2k^2 + 5k \\
\overline{F}(Kb_n(k)) &= 2n^2k - 8nk + 8k \\
\overline{F}(Kc_n(k)) &= n^2k^2 - n^2k - 2nk^3 + 2nk^2 + k^4 - k^3 \\
\overline{F}(Kd_n(k)) &= 2n^2k - 12nk + 18k \\
\end{align*}
\]
\[
F(\overline{K_a}(k)) = nk + nk^2 - k^3 - k^2 - 2k \\
F(\overline{K_b}(k)) = 2kn - 4k \\
F(\overline{K_c}(k)) = -2nk^2 + nk + n k^3 - k^4 + 2k^3 - k^2 \\
F(\overline{K_d}(k)) = 4kn - 12k.
\]

From Theorem 3.1–3.4 and the corrected Theorem 1.1, we obtain:

**Lemma 4.4.** Using the same notation as in Lemma 4.1,
\[
HZ(\overline{K_a}(k)) = k^3 + 2k^2 + k \\
HZ(\overline{K_b}(k)) = 4k \\
HZ(\overline{K_c}(k)) = 2k^4 - 6k^3 + 6k^2 - 2k \\
HZ(\overline{K_d}(k)) = 16k \\
HZ(K_a(k)) = 4n^2 k - 4nk^2 - 12nk + k^3 + 6k^2 + 9k \\
HZ(K_b(k)) = 4k n^2 - 16kn + 16k \\
HZ(K_c(k)) = 2k^2 n^2 - 2kn^2 - 4k^3 n + 4k^2 n + 2k^4 - 2k^3 \\
HZ(K_d(k)) = 2n^4 - 6n^3 - 8n^2 k + 6n^2 + 20nk - 2n + 4k^2 - 32k \\
HZ(K_a(k)) = nk + nk^2 - k^3 - 3k \\
HZ(K_b(k)) = 4k^2 + 2kn - 8k \\
HZ(K_c(k)) = k^3 n - 2k^2 n + kn - k^4 + 2k^3 - k^2 \\
HZ(K_d(k)) = 4k^2 + 4kn - 24k.
\]

**References**


Received by editors 13.07.2018; Available online 01.10.2018.

DEPARTMENT OF MATHEMATICS, BEARYS INSTITUTE OF TECHNOLOGY, MANGALORE – 574153, KARNATAKA, INDIA
E-mail address: rajesh.rbj065@gmail.com

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KRAGUJEVAC, Kragujevac, Serbia
E-mail address: gutman@kg.ac.rs

DEPARTMENT OF MATHEMATICS, P. A. COLLEGE OF ENGINEERING, MANGALORE – 574153, KARNATAKA, INDIA
E-mail address: rkmundevadi@gmail.com