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ON M-BI IDEALS IN SEMIGROUPS

Mohammad Munir

Abstract. Bi ideals are the generalized form of the quasi ideals that are further a generalization of right and left ideals. In this article, we introduce the m-bi ideals as a generalization of bi ideals in semigroups. We present the basic and fundamental properties of the m-bi ideals in the semigroups from a pure algebraic point of view. The form of the m-bi ideals generated by an element, a subset and a subsemigroup of a semigroup is given. Their fundamental properties are also described.

1. Introduction

Theory of semigroups originated as a generalization of the group theory in the early twentieth century. Semigroups comparatively having simpler structures have uses and applications recognized in various fields of mathematics and science since their origin. They have wide spread applications in theoretical computer science because of having a natural association with the finite automata. Their uses in the theory of graphs, time-invariant processes and abstract evolution equations have been eminent. Ideals theory in semigroups, like all other algebraic structures, play an important role in studying them. Steinfeld gave the idea of quasi ideals in rings and semigroups respectively in his articles [**12**] and [**11**]. Iseki [**5**] developed this concept for semirings having no zero and studied important characterizations of semirings using quasi ideals.

Generalization of the ideals in algebraic structures have also been an interesting and a useable task for the mathematicians. Ideals were generalized to one-sided ideals; they were generalized to quasi ideals. The concept of bi ideals as generalized forms of quasi ideals were introduced by Lajos and Szasz [**7**] in associative rings. Later mathematicians introduced this concepts in different types of semigroups. S.Kar et al generalized bi ideals for the ternary semigroups [**6**]. Ayutthaya and

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Pibaljommee characterized the ordered semirings by the ordered quasi ideals [**10**]. Munir et al characterized regular and weakly regular semirings using their quasi and bi ideals[**8**]. In [**9**], the author of this paper generalized the bi ideals for semirings.

In this paper, we define the m-bi ideals as a generalization of bi ideals for semigroups. In Section 2, we present the idea of the m-bi ideal for the semigroups. The forms of the m-bi ideals generated by the nonempty sets, subemigroups and single elements of the semigroup are presented in Section 3. The conclusion of the article is given in Section 4.

2. m-Bi Ideals

In this section, before presenting the idea of the m-bi ideals in semigroups, we give a brief summary of the essential concepts and notions used in this field from the books [**3**] and [**4**] which will be used in the sequel.

A nonempty set $\mathcal{A}$ together with a binary operation $* : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is called a *semigroup* if $\mathcal{A}$ is closed under $\cdot$ i.e. $a * b \in \mathcal{A}$ for all $a, b \in \mathcal{A}$, and associative law holds in $\mathcal{A}$ i.e. $a * (b * c) = (a * b) * c$, for all $a, b, c \in \mathcal{A}$. A nonempty subset $\mathcal{K}$ of $\mathcal{A}$ is called its *subsemirgoup* if $\mathcal{K}$ itself is a semigroup under the operation $*$ of $\mathcal{A}$. A subsemigroup I of $\mathcal{A}$ is called a left (right) ideal of $\mathcal{A}$ if $\mathcal{A}I \subseteq I$ ($I\mathcal{A} \subseteq I$). If $\mathcal{A}$ is a left as well as a right ideal, then it is called an ideal (or a two-sided ideal) of $\mathcal{A}$. A *quasi ideal* $\mathcal{Q}$ of $\mathcal{A}$ is a subsemigroup $(\mathcal{Q}, +)$ of $\mathcal{A}$ satisfying the condition $\mathcal{A}\mathcal{Q} \cap \mathcal{Q}\mathcal{A} \subseteq \mathcal{Q}$. A subsemigroup $\mathcal{B}$ of $\mathcal{A}$ is called a *bi ideal* of $\mathcal{A}$ if $\mathcal{B}\mathcal{A}\mathcal{B} \subseteq \mathcal{B}$.

Now we define the notion of m-bi ideals and discuss their important properties.

Definition 2.1. Let $(\mathcal{A}, \cdot)$ be a semigroup. An *m-bi ideal* $\mathcal{B}$ of $\mathcal{A}$ is a subsemigroup of $\mathcal{A}$ such that $\mathcal{B}\mathcal{A}^m\mathcal{B} \subseteq \mathcal{B}$ where m is a positive integer, not necessarily 1, called the *bipotency* of the *bi ideal* $\mathcal{B}$.

$\mathcal{B}\mathcal{A}^m\mathcal{B} \subseteq \mathcal{B}$ is called the *bipotency condition.* This is to be noted that every bi ideal $\mathcal{B}$ of $\mathcal{A}$ is a 1-bi ideal of $\mathcal{A}$ (bi ideal with bipotency 1). All the so-called 1-bi ideals are simply the bi ideals, whereas those with bipotency $m > 1$ are to be indicated with the value of m.

Proposition 2.1. *In a semigroup $\mathcal{A}$, every bi ideal is m-bi ideal for any $m \geqslant 1$.*

Proof. Since $\mathcal{B}$ is a bi ideal of $\mathcal{A}$ [**8**], then we can write $\mathcal{B}\mathcal{A}\mathcal{B} \subseteq \mathcal{B}$ because $\mathcal{B}\mathcal{A}^1\mathcal{B} \subseteq \mathcal{B}$ which makes $\mathcal{B}$ an m-bi ideal of A. □

The converse of the above proposition is not true. This follows from the examples.

Example 2.1. If $\mathcal{A} =$

$$\left\{ \begin{pmatrix} 0 & s & t & u \\ 0 & 0 & v & w \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 \end{pmatrix} : s, t, u, v, w, x \text{ are any positive real numbers} \right\},$$

then $(\mathcal{A}, \cdot)$ is a semigroup under the usual operations of multiplication $' \cdot '$ of matrices. If

$$\mathcal{B} = \left\{ \begin{pmatrix} 0 & s & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 \end{pmatrix} : s, x \text{ are any positive real numbers} \right\},$$

then $\mathcal{B}$ is 2-bi ideal of $\mathcal{A}$ as $\mathcal{B}\mathcal{A}^2\mathcal{B} \subseteq \mathcal{B}$, and $\mathcal{B}\mathcal{A}\mathcal{B} \not\subset \mathcal{B}$.

The following examples characterize the m-bi ideals for the categories of *idempotent* and *nilpotent* matrices.

EXAMPLE 2.2. Let $\mathcal{A}$ be the set of all idempotent matrices of idempotency m each and commuting with each other, then $\mathcal{A}$ forms a semigroup under the ordinary multiplication of matrices. In this case, for any two matrices $\mathcal{M}$ and $\mathcal{N}$ belonging to $\mathcal{A}$, we have $(\mathcal{M}\mathcal{N})^m = \mathcal{N}^m\mathcal{M}^m = \mathcal{N}\mathcal{M} = \mathcal{M}\mathcal{N}$, therefore $\mathcal{M}\mathcal{N}$ is idempotent with idempotency m. Moreover, since the multiplication of matrices is associative , so associative law holds in $\mathcal{A}$. Therefore, $\mathcal{A}$ is a semigroup. Next, let $\mathcal{B}$ be any bi ideal of $\mathcal{A}$, then $\mathcal{B}\mathcal{A}^m\mathcal{B} = \mathcal{B}^m\mathcal{A}^m\mathcal{B}^m = (\mathcal{B}\mathcal{A}\mathcal{B})^m = \mathcal{B}^m = \mathcal{B}$. Therefore, $\mathcal{B}\mathcal{A}^m\mathcal{B} \subseteq \mathcal{B}$. Thus $\mathcal{B}$ is a bi ideal of $\mathcal{A}$ with bipotency m.

EXAMPLE 2.3. Let $\mathcal{A}$ be the set of all nilpotent matrices of nilpotency m, then $\mathcal{A}$ also forms a semigroup under the usual multiplication of matrices. In this case, every subsemigroup $\mathcal{B}$ of $\mathcal{A}$ forms its m-bi ideal; as $\mathcal{B}\mathcal{A}^m\mathcal{B} = \mathcal{B}0\mathcal{B} = 0 \subseteq \mathcal{B}$ implies $\mathcal{B}\mathcal{A}^m\mathcal{B} \subseteq \mathcal{B}$,0 is the null matrix.

The left ideal $\mathcal{L}$ and the right ideal $\mathcal{R}$ of the semigroup $\mathcal{A}$ are the bi ideals or the 1-bi ideals. Every ideal of $\mathcal{A}$ is a 1-bi ideal of $\mathcal{A}$.

PROPOSITION 2.2. *The product of any m_1-bi ideals and m_2-bi-ideal of a semigroup $\mathcal{A}$, with identity e, is max(m_1, m_2)-bi ideal of $\mathcal{A}$.*

PROOF. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two bi ideals of a semigroup $\mathcal{A}$ with bipotencies m_1 and m_2 respectively i.e., $\mathcal{B}_1\mathcal{A}^{m_1}\mathcal{B}_1 \subseteq \mathcal{B}_1$ and $\mathcal{B}_2\mathcal{A}^{m_1}\mathcal{B}_2 \subseteq \mathcal{B}_2$, m_1 and m_2 are any positive integers. We have, $(\mathcal{B}_1\mathcal{B}_2)^2 = (\mathcal{B}_1\mathcal{B}_2)(\mathcal{B}_1\mathcal{B}_2) = (\mathcal{B}_1\mathcal{A}\mathcal{B}_1)\mathcal{B}_2 = (\mathcal{B}_1\mathcal{A}e...e\mathcal{B}_1)\mathcal{B}_2 \subseteq (\mathcal{B}_1\mathcal{A}\mathcal{A}...\mathcal{A}\mathcal{B}_1)\mathcal{B}_2 \subseteq (\mathcal{B}_1\mathcal{A}^m\mathcal{B}_1)\mathcal{B}_2 \subseteq \mathcal{B}_1\mathcal{B}_2$, i.e., $(\mathcal{B}_1\mathcal{B}_2)^2 \subseteq \mathcal{B}_1\mathcal{B}_2$. So, $\mathcal{B}_1\mathcal{B}_2$ is closed under multiplication and $\mathcal{B}_1\mathcal{B}_2$ is a subsemigroup of $\mathcal{A}$. Moreover, $\mathcal{B}_1\mathcal{B}_2(\mathcal{A}^{\max(m_1,m_2)})\mathcal{B}_1\mathcal{B}_2 \subseteq \mathcal{B}_1\mathcal{A}\mathcal{A}^{\max(m_1,m_2)}\mathcal{B}_1\mathcal{B}_2 = \mathcal{B}_1\mathcal{A}^{1+\max(m_1,m_2)}\mathcal{B}_1\mathcal{B}_2 \subseteq \mathcal{B}_1\mathcal{A}^{m_1}\mathcal{B}_1\mathcal{B}_2 \subseteq \mathcal{B}_1\mathcal{B}_2$. We have used the result $\mathcal{A}^{1+\max(m_1,m_2)} \subseteq \mathcal{A}^{m_1}$ (See [**8**]). So, $\mathcal{B}_1\mathcal{B}_2(\mathcal{A}^{\max(m_1,m_2)})\mathcal{B}_1\mathcal{B}_2 \subseteq \mathcal{B}_1\mathcal{B}_2$. Thus, $\mathcal{B}_1\mathcal{B}_2$ is an max(m_1, m_2)-bi ideal of $\mathcal{A}$. □

PROPOSITION 2.3. *Let $\mathcal{A}$ be a semigroup with identity e, $\mathcal{T}$ be its arbitrary subset and $\mathcal{B}$ be its m-bi ideal; m not necessarily 1, then $\mathcal{B}\mathcal{T}$ is its m-bi ideal.*

PROOF. We have $(\mathcal{B}\mathcal{T})^2 = (\mathcal{B}\mathcal{T})(\mathcal{B}\mathcal{T}) = (\mathcal{B}\mathcal{T}\mathcal{B})\mathcal{T} \subseteq (\mathcal{B}\mathcal{A}\mathcal{B}) \subseteq \mathcal{B}\mathcal{A}e...e\mathcal{B} \subseteq \mathcal{B}\mathcal{A}\mathcal{A}...\mathcal{A}\mathcal{B} \subseteq (\mathcal{B}\mathcal{A}^m\mathcal{B})\mathcal{T} \subseteq \mathcal{B}\mathcal{T}$. So, $\mathcal{B}\mathcal{T}^2 \subseteq \mathcal{B}\mathcal{T}$ making it a subsemigroup of $\mathcal{A}$. Moreover, $\mathcal{B}\mathcal{T}(\mathcal{A}^m)\mathcal{B}\mathcal{T} \subseteq \mathcal{B}\mathcal{A}\mathcal{A}^m\mathcal{B}\mathcal{T} \subseteq \mathcal{B}\mathcal{A}^{1+m}\mathcal{B}\mathcal{T} \subseteq \mathcal{B}\mathcal{A}^m\mathcal{B}\mathcal{T} \subseteq \mathcal{B}\mathcal{T}$. Therefore, $\mathcal{B}\mathcal{T}$ is m-bi ideal of $\mathcal{A}$. □

In a similar way, we can also prove $\mathcal{T}\mathcal{B}$ an m-bi ideal of $\mathcal{A}$.

PROPOSITION 2.4. *The intersection of a family of bi ideals of a semigroup $\mathcal{A}$ with bipotencies $m_1, m_2, ...,$ is also a bi ideal with bipotency max$\{m_1, m_2, ...\}$.*

PROOF. Let $\{\mathcal{B}_\lambda : \lambda \in \wedge\}$ be a family of m-bi ideals of semigroup $\mathcal{A}$, then $\mathcal{B} = \bigcap_{\lambda\in\wedge} \mathcal{B}_\lambda$, being the intersection of subsemigroups of $\mathcal{A}$ is a subsemigroup of $\mathcal{A}$. Since $\mathcal{B}_\lambda\mathcal{A}^{m_\lambda}\mathcal{B}_\lambda \subseteq \mathcal{B}_\lambda \quad \forall \quad \lambda \in \wedge$, and $\mathcal{B} \subseteq \mathcal{B}_\lambda \quad \forall \quad \lambda \in \wedge$, therefore $\mathcal{B}\mathcal{A}^{\max\{m_\lambda:\lambda\in\wedge\}}\mathcal{B} \subseteq \mathcal{B}_\lambda\mathcal{A}^{m_\lambda}\mathcal{B}_\lambda \subseteq \mathcal{B}_\lambda \quad \forall \quad \lambda \in \wedge$. This implies that $\mathcal{B}\mathcal{A}^{\max\{m_\lambda:\lambda\in\wedge\}}\mathcal{B} \subseteq \mathcal{B}_\lambda \quad \forall \quad \lambda \in \wedge$. This gives $\mathcal{B}\mathcal{A}^{\max\{m_\lambda:\lambda\in\wedge\}}\mathcal{B} \subseteq \bigcap_{\lambda\in\wedge} \mathcal{B}_\lambda = \mathcal{B}$. Therefore, $\mathcal{B}\mathcal{A}^{\max\{m_\lambda:\lambda\in\wedge\}}\mathcal{B} \subseteq \mathcal{B}$. Thus $\mathcal{B}$ is an m-bi ideal with *bipotency* $\max\{m_1, m_2, ...\}$. □

Sum of two m-bi ideals of a semigroup is not an m-bi ideals.

EXAMPLE 2.4. If

$$\mathcal{A} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \text{a, b, c, d are non-negative integers} \right\},$$

then $\mathcal{A}$ is a semigroup under multiplication of matrices. If

$$\mathcal{B}_1 = \left\{ \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} : x \text{ is a non-negative integers} \right\}$$

and

$$\mathcal{B}_2 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix} : y \text{ is a non-negative integers} \right\},$$

then $\mathcal{B}_1$ and $\mathcal{B}_2$ are 1-bi ideals of $\mathcal{A}$. But $\mathcal{B} = \mathcal{B}_1 + \mathcal{B}_2$, is not a bi ideal of $\mathcal{A}$ because, in this case, we have

$$\mathcal{B} = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} : x \text{ and } y \text{ are non-negative integers} \right\}.$$

So, $\mathcal{B}\mathcal{A}\mathcal{B} \not\subseteq \mathcal{B}$.

PROPOSITION 2.5. *Every (m,m)-quasi ideal $\mathcal{Q}$ of a semigroup $\mathcal{A}$ is its m-bi ideal.*

PROOF. Consider $\mathcal{Q}\mathcal{A}^m\mathcal{Q} \subseteq \mathcal{Q}\mathcal{A}^m\mathcal{A} = \mathcal{Q}\mathcal{A}^{m+1} \subseteq \mathcal{Q}\mathcal{A}^m$, and so $\mathcal{Q}\mathcal{A}^m\mathcal{Q} \subseteq \mathcal{Q}\mathcal{A}^m$. Similarly, $\mathcal{Q}\mathcal{A}^m\mathcal{Q} \subseteq \mathcal{A}^m\mathcal{Q}$. Combining these two, we $\mathcal{Q}\mathcal{A}^m\mathcal{Q} \subseteq \mathcal{Q}\mathcal{A}^m \cap \mathcal{A}^m\mathcal{Q} \subseteq \mathcal{Q}$. Thus $\mathcal{Q}\mathcal{A}^m\mathcal{Q} \subseteq \mathcal{Q}$. That is, $\mathcal{Q}$ is bi ideal with bipotency m. □

PROPOSITION 2.6. *The product of any (m_1, m_2)-quasi ideal and (n_1, n_2)-quasi-ideal of a semigroup $\mathcal{A}$, with identity e, is max$\{m_1, m_2, n_1, n_2\}$-bi ideal of $\mathcal{A}$.*

PROOF. Since $(\mathcal{Q}_1\mathcal{Q}_2)(\mathcal{Q}_1\mathcal{Q}_2) \subseteq \mathcal{Q}_1(\mathcal{Q}_2\mathcal{A}\mathcal{Q}_2) \subseteq \mathcal{Q}_1\mathcal{Q}_2$, i.e., $(\mathcal{Q}_1\mathcal{Q}_2)^2 \subseteq \mathcal{Q}_1\mathcal{Q}_2$, therefore $\mathcal{Q}_1\mathcal{Q}_2$ is closed under multiplication. $(\mathcal{Q}_1\mathcal{Q}_2)\mathcal{A}^{\max\{m_1,m_2,n_1,n_2\}}(\mathcal{Q}_1\mathcal{Q}_2) \subseteq \mathcal{Q}_1\mathcal{Q}_2\mathcal{A}^{\max\{m_1,m_2,n_1,n_2\}}(\mathcal{A}\mathcal{Q}_2) \subseteq \mathcal{Q}_1(\mathcal{Q}_2\mathcal{A}^{\max\{m_1,m_2,n_1,n_2\}+1}\mathcal{Q}_2) \subseteq \mathcal{Q}_1\mathcal{Q}_2$. Thus, $\mathcal{Q}_1\mathcal{Q}_2$ is a bi ideal of $\mathcal{A}$ with bipotency $\max\{m_1, m_2, n_1, n_2\}$. □

DEFINITION 2.2. A subsemigroup $\mathcal{L}$ of a semigroup $\mathcal{A}$ is said to be its m-left ideal if $\mathcal{A}^m\mathcal{L} \subseteq \mathcal{L}$ for a positive integer m.

The n-right ideal R of $\mathcal{A}$ is defined similarly, where n is a positive integer ([**1**],[**2**]).

PROPOSITION 2.7. *An m-left ideal of a semigroup $\mathcal{A}$ is its m-bi ideal.*

PROOF. Let $\mathcal{L}$ be the m-left ideal of $\mathcal{A}$, then $\mathcal{L}\mathcal{A}^m\mathcal{L} \subseteq \mathcal{L}\mathcal{L} \subseteq \mathcal{L}$. This implies that L is m-bi ideal of $\mathcal{A}$. □

COROLLARY 2.1. *An n-right ideal of $\mathcal{A}$ is its m-bi ideal.*

PROOF. As above. □

THEOREM 2.1. *If $\mathcal{L}_i$ ($\mathcal{R}_i$) is an m-left (n-right) ideal of a semigroup $\mathcal{A}$ for $i \in I$, then $\bigcap_{i\in I} \mathcal{L}_i$ ($\bigcap_{i\in I} \mathcal{R}_i$) is also m-left (n-right) ideal of $\mathcal{A}$.*

PROOF. As Proposition 2.4. □

THEOREM 2.2. *For an m-left $\mathcal{L}$ and n-right $\mathcal{R}$ of a semigroup $\mathcal{A}$, their intersection $\mathcal{L} \cap \mathcal{R}$ is its t-bi ideal, where $t = max(m, n)$.*

PROOF. Since $0 \in \mathcal{L} \cap \mathcal{R}$, therefore by the above Lemma 3.1, we see that $\mathcal{L} \cap \mathcal{R}$ is a subsemigroup of $\mathcal{A}$. Now, since $\mathcal{L}$ and $\mathcal{R}$ are also m-bi and n-bi ideals of $\mathcal{A}$, their intersection turns out to be $max(m, n)$-bi ideals from the above result 2.4. Alternatively, $\mathcal{L} \cap \mathcal{R}(\mathcal{A}^{max\{m,n\}})\mathcal{L} \cap \mathcal{R} \subseteq \mathcal{L}\mathcal{A}^{max\{m,n\}}\mathcal{L} \subseteq \mathcal{A}^{max\{m,n\}+1}\mathcal{L} \subseteq \mathcal{A}^m\mathcal{L} \subseteq \mathcal{L}$.

Similarly, we can show that $\mathcal{L} \cap \mathcal{R}(\mathcal{A}^{max\{m,n\}})\mathcal{L} \cap \mathcal{R} \subseteq \mathcal{R}$. Consequently, $\mathcal{L} \cap \mathcal{R}\mathcal{A}^{max\{m,n\}}\mathcal{L} \cap \mathcal{R} \subseteq \mathcal{L} \cap \mathcal{R}$ □

3. Finitely Generated m-bi Ideals

DEFINITION 3.1. For a semigroup $\mathcal{A}$, let τ be the collection of m-bi ideals $\mathcal{B}$ of $\mathcal{A}$ containing $\mathcal{G}$, where $\mathcal{G}$ is a subset of $\mathcal{A}$. Then, we can write $\tau = \{\mathcal{B} :$ $\mathcal{B}$ is m-bi ideal of $\mathcal{A}$ such that it contains $\mathcal{G}$ $\}$. τ is nonempty as $\mathcal{A} \in \tau$. Let $< \mathcal{G} >_{m-b}= \bigcap_{\mathcal{B}\in\tau} \mathcal{B}$. Clearly, $< \mathcal{G} >_{m-b}$ is nonempty because $0 \in< \mathcal{G} >_{m-b}$. Since the intersection of m-bi ideals is m-bi ideal, so is $< \mathcal{G} >_{m-b}$ of $\mathcal{A}$. Next, $< \mathcal{G} >_{m-b}$ being the intersection of all m-bi ideals of $\mathcal{A}$ containing $\mathcal{G}$ is its smallest m-bi ideal which contains $\mathcal{G}$. $< \mathcal{G} >_{m-b}$ is said to be the m-bi ideal of $\mathcal{A}$ generated by $\mathcal{G}$.

It is clear that $< \phi >_{m-b}=< 0 >_{m-b}= \{0\}$.

THEOREM 3.1. *Let $\mathcal{G}$ be a non-void subset of $\mathcal{A}$, then the m-bi ideal generated by $\mathcal{G}$ is $< \mathcal{G} >_{m-b}= \mathcal{G} \cup \mathcal{G}^2 \cup \mathcal{G}\mathcal{A}^m\mathcal{G}$.*

PROOF. We need to show that $< \mathcal{G} >_{m-b}= \mathcal{G} \cup \mathcal{G}^2 \cup \mathcal{G}\mathcal{A}^m\mathcal{G}$ is the smallest m-bi ideal of $\mathcal{A}$ which contains $\mathcal{G}$. $< \mathcal{G} >_{m-b}$ is nonempty. Let $a \in< \mathcal{G} >_{m-b}$. Then $a \in \bigcap_{\mathcal{G}\subseteq\mathcal{B}\in\tau} \mathcal{B}$. This implies $a \in \mathcal{B}$, for all $\mathcal{B} \in \tau$, such that $\mathcal{B} \supseteq \mathcal{G}$. Here, we claim that $a \in \mathcal{G}\mathcal{A}^m\mathcal{G}$; for if $a \notin \mathcal{G}\mathcal{A}^m\mathcal{G}$, then since $a \notin \mathcal{G}\mathcal{A}^m\mathcal{G} \subseteq \mathcal{B}\mathcal{A}^2\mathcal{B}$, therefore, $a \notin \mathcal{B}\mathcal{A}^m\mathcal{B}$ for some $\mathcal{B} \in \tau$. This is a contradiction to our hypothesis that $a \in \bigcap\limits_{\mathcal{G}\subseteq\mathcal{B}\in\tau} \mathcal{B}$. So, $a \in \mathcal{G}\mathcal{A}^m\mathcal{G}$, and thus, $a \in \mathcal{G} \cup \mathcal{G}^2 \cup \mathcal{G}\mathcal{A}^m\mathcal{G}$. Thus, $\mathcal{G}$ is $< \mathcal{G} >_{m-b} \subseteq \mathcal{G} \cup \mathcal{G}^2 \cup \mathcal{G}\mathcal{A}^m\mathcal{G}$.

Conversely, let $a \in \mathcal{G}\cup\mathcal{G}^2\cup\mathcal{G}\mathcal{A}^m\mathcal{G}$. Then, $a \in \mathcal{G}$ or $a \in \mathcal{G}^2$ or $a \in \mathcal{G}\mathcal{A}^m\mathcal{G}$. When $a \in \mathcal{G}$, then $a \in < \mathcal{G} >_{m-b}$. When $a \in \mathcal{G}^2$, then $a \in \mathcal{G}\mathcal{G} \subseteq < \mathcal{G} >_{m-b} < \mathcal{G} >_{m-b} \subseteq (< \mathcal{G} >_{m-b})^2 \subseteq < \mathcal{G} >_{m-b}$. That is, $a \in < \mathcal{G} >_{m-b}$. Lastly, when $a \in \mathcal{G}\mathcal{A}^m\mathcal{G}$, then $a \in \mathcal{B}\mathcal{A}^m\mathcal{B}$, for all $\mathcal{B} \in \tau$ containing $\mathcal{G}$. So, $a \in \bigcap\limits_{\mathcal{G}\subseteq\mathcal{B}\in\tau} \mathcal{B} = < \mathcal{G} >_{m-b}$. So, $\mathcal{G} \cup \mathcal{G}^2 \cup \mathcal{G}\mathcal{A}^m\mathcal{G} \subseteq < \mathcal{G} >_{m-b}$. Consequently, $< \mathcal{G} >_{m-b} = \mathcal{G} \cup \mathcal{G}^2 \cup \mathcal{G}\mathcal{A}^m\mathcal{G}$.

Next, we need to show that $< \mathcal{G} >_{m-b} \mathcal{A}^m < \mathcal{G} >_{m-b} \subseteq < \mathcal{G} >_{m-b}$. Consider $< \mathcal{G} >_{m-b} \mathcal{A}^m < \mathcal{G} >_{m-b} = \big(\mathcal{G} \cup \mathcal{G}^2 \cup \mathcal{G}\mathcal{A}^m\mathcal{G}\big)\mathcal{A}^m\big(\mathcal{G} \cup \mathcal{G}^2 \cup \mathcal{G}\mathcal{A}^m\mathcal{G}\big) \subseteq \big(\mathcal{G} \cup \mathcal{G}^2 \cup \mathcal{G}\mathcal{A}^m\mathcal{A}\big)\mathcal{A}^m\big(\mathcal{A} \cup \mathcal{A}^2 \cup \mathcal{A}\mathcal{A}^m\mathcal{G}\big) = \big(\mathcal{G} \cup \mathcal{G}^2 \cup \mathcal{G}\mathcal{A}^{m+1}\big)\mathcal{A}^m\big(\mathcal{A} \cup \mathcal{A}^{m+1}\mathcal{G}\big) = \big(\mathcal{G} \cup \mathcal{G}^2 \cup \mathcal{G}\mathcal{A}^{m+1}\big)\mathcal{A}^m\big((\mathcal{A}\cup\mathcal{A}^{m+1})\mathcal{G}\big) \subseteq \big(\mathcal{G}\cup\mathcal{G}^2\cup\mathcal{G}\mathcal{A}^{m+1}\big)\mathcal{A}^m\big(\mathcal{A}\mathcal{G}\big) \subseteq \big(\mathcal{G}\cup\mathcal{G}^2\cup\mathcal{G}\mathcal{A}^{2(m+1)}\mathcal{G}\big) \subseteq \big(\mathcal{G} \cup \mathcal{G}^2 \cup \mathcal{G}\mathcal{A}^m\mathcal{G}\big)$. Therefore, $< \mathcal{G} >_{m-b} \mathcal{A}^m < \mathcal{G} >_{m-b} \subseteq < \mathcal{G} >_{m-b}$. That is, $< \mathcal{G} >_{m-b} \mathcal{A}^m < \mathcal{G} >_{m-b}$ is an m-bi ideal containing $\mathcal{G}$. To show that $< \mathcal{G} >_{m-b}$ is the smallest m-bi ideal of $\mathcal{A}$ that contains $\mathcal{G}$, take $\mathcal{B}'$ to be any other m-bi ideal of $\mathcal{A}$ containing $\mathcal{G}$, then, $\mathcal{G}\mathcal{G} \subseteq \mathcal{B}'\mathcal{B}' \subseteq (\mathcal{B}')^2 \subseteq \mathcal{B}'$. That is, $\mathcal{G}^2 \subseteq \mathcal{B}'$. Now $\mathcal{G}\mathcal{A}^m\mathcal{G} \subseteq \mathcal{B}'\mathcal{A}^m\mathcal{B}' \subseteq \mathcal{B}'$. Therefore, $< \mathcal{G} >_{m-b} = \mathcal{G} \cup \mathcal{G}^2 \cup \mathcal{G}\mathcal{A}^m\mathcal{G} \subseteq B'$, that is, $< \mathcal{G} >_{m-b}$ is the smallest m-bi ideal of $\mathcal{A}$ containing $\mathcal{G}$. □

Corollary 3.1. *If $a \in \mathcal{A}$, the m-bi ideal generated by a is $< a >_{m-b} = \{a\} \cup \{a^2\} \cup a\mathcal{A}^m a$.*

Corollary 3.2. *If $\mathcal{G}$ is a subsemigroup of $\mathcal{A}$, then the m-bi ideal generated by $\mathcal{G}$ is $< \mathcal{G} >_{m-b} = \mathcal{G} \cup \mathcal{G}\mathcal{A}^m\mathcal{G}$.*

Proof. Since $\mathcal{G}$ is subsemigroup of $\mathcal{A}$, $\mathcal{G}^2 = \mathcal{G}\mathcal{G} \subseteq \mathcal{G}$. Therefore, the m-bi ideal generated by $\mathcal{G}$ is $< \mathcal{G} >_{m-b} = \mathcal{G} \cup \mathcal{G}\mathcal{A}^m\mathcal{G}$. □

Corollary 3.3. *If the semigroup $(\mathcal{A}, \cdot)$ contains the multiplicative identity e, then the m-bi ideal generated by the nonempty set $\mathcal{G}$ is $< \mathcal{G} >_{m-b} = \mathcal{G}\mathcal{A}^m\mathcal{G}$.*

Proof. Since $\mathcal{A}$ contains the multiplicative identity e, $\mathcal{G} \subseteq \mathcal{G}\mathcal{A}^m\mathcal{G}$. Therefore, $< \mathcal{G} >_{m-b} = \mathcal{G}\mathcal{A}^m\mathcal{G}$. □

Corollary 3.4. *If a is an element of a semigroup $\mathcal{A}$ with identity e, then the m-bi ideal generated by a is $< a >_{m-b} = a\mathcal{A}^m a$.*

Remark 3.1. For $m = 1$, the bi ideal generated by a nonempty set $\mathcal{G}$ is $< \mathcal{G} >_b = \mathcal{G} \cup \mathcal{G}^2 \cup \mathcal{G}\mathcal{A}\mathcal{G}$. If $\mathcal{G}$ is a subsemigroup of $\mathcal{A}$, then the bi ideal generated by $\mathcal{G}$ is $< \mathcal{G} >_b = \mathcal{G} \cup \mathcal{G}\mathcal{A}\mathcal{G}$. If $\mathcal{A}$ possesses the identity element, then the bi ideal generated by $\mathcal{G}$ is $< \mathcal{G} >_b = \mathcal{G}\mathcal{A}\mathcal{G}$.

Definition 3.2. An m-bi ideal is called principal m-bi ideal if it is generated by a single element.

Theorem 3.2. *In a semigroup $\mathcal{A}$ the following hold*

1. *for a nonempty subset $\mathcal{G}$ of $\mathcal{A}$, $< \mathcal{G} >_{m-b} \subseteq < \mathcal{G} >_b$,*
2. *for a semigroup $\mathcal{G}$ of $\mathcal{A}$, $< \mathcal{G} >_{m-b} \subseteq < \mathcal{G} >_b$,*
3. *for an element a of $\mathcal{A}$, $< a >_{m-b} \subseteq < a >_b$.*

PROOF. (1) Since for any positive integer m, $\mathcal{G} \cup \mathcal{G}^2 \cup \mathcal{G}\mathcal{A}^m\mathcal{G} \subseteq \mathcal{G} \cup \mathcal{G}^2 \cup \mathcal{G}\mathcal{A}\mathcal{G}$. Therefore, $< \mathcal{G} >_{m-b} \subseteq < \mathcal{G} >_b$.

(2) Analogously.

(3) Analogously. □

4. Conclusions

We have introduced the notion of m-bi ideal in semigroups as a generalization of their bi ideals for a positive integer m, then studied their basic properties. We have also presented the forms of the m-bi ideals of a semigroup generated by a nonempty subset, a subsemigroup and a single element of the semigroup. The idea of m-bi ideals will be useful to characterize some more classes of the semigroups like regular semigroups, intra-regular and weakly-regular semigroups.

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MATHEMATICS DEPARTMENT, GOVERNMENT POSTGRADUATE COLLEGE NO.1, ABBOTTABAD, PAKISTAN.

E-mail address: dr.mohammadmunir@gpgc1-atd.edu.pk dr.mohammadmunir@gmail.com