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FUZZY LEFT AND RIGHT BI-QUASI IDEALS OF SEMIRINGS

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Abstract. In this paper,we introduce the notions of fuzzy left bi-quasi ideal , fuzzy right bi-quasi ideal and fuzzy bi-quasi ideal of semiring. We characterize the regular semiring in terms of fuzzy right (left) bi-quasi ideals of semiring.

1. Introduction

Semiring is an algebraic structure which is a common generalization of rings and distributive lattices, was first introduced by Vandiver [18] in 1934 but non-trivial examples of semirings had appeared in the studies on the theory of commutative ideals of rings by Dedekind in 19th century. Semiring is a universal algebra with two binary operations called addition and multiplication, where one of them distributive over the other. Bounded distributive lattices are commutative semirings which are both additively idempotent and multiplicatively idempotent. A natural example of semiring is the set of all natural numbers under usual addition and multiplication of numbers. In particular, if I is the unit interval on the real line, then (I, max, min) is a semiring in which 0 is the additive identity and 1 is the multiplicative identity. The theory of rings and the theory of semigroups have considerable impact on the development of the theory of semirings. In structure, semirings lie between semigroups and rings. Additive and multiplicative structures of a semiring play an important role in determining the structure of a semiring. Semiring as the basic algebraic structure was used in the areas of theoretical computer science as well as in the solutions of graph theory, optimization theory and in particular for studying automata , coding theory and formal languages. Semiring theory has many applications in other branches. The notion of ideals was introduced by Dedekind

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for the theory of algebraic numbers, was generalized by E. Noether for associative rings. The one and two sided ideals introduced by her, are still central concepts in ring theory. We know that the notion of an one sided ideal of any algebraic structure is a generalization of notion of an ideal. The quasi ideals are generalization of left and right ideals where as the bi-ideals are generalization of quasi ideals. The notion of bi-ideals in semigroups was introduced by Lajos [9]. Iseki [4, 5, 6] introduced the concept of quasi ideal for a semiring. Quasi ideals in $\Gamma-$semirings studied by Jagtap and Pawar [7]. M. Henriksen [3] studied ideals in semirings. As a further generalization of ideals, Steinfeld [16] first introduced the notion of quasi ideals for semigroups and then for rings.We know that the notion of the bi-ideal in semirings is a special case of (m, n) ideal introduced by S. Lajos. The concept of bi-ideals was first introduced by R.A. Good and D. R. Hughes [2] for a semigroup. Lajos and Szasz [10] introduced the concept of bi-ideals for rings.

The fuzzy set theory was developed by L. A. Zadeh [28] in 1965. Many papers on fuzzy sets appeared showing the importance of the concept and its applications to logic, set theory, group theory, ring theory, real analysis, topology, measure theory etc. The fuzzification of algebraic structure was introduced by A. Rosenfeld [24] and he introduced the notion of fuzzy subgroups in 1971. K. L. N. Swamy and U. M. Swamy [26] studied fuzzy prime ideals in rings in 1988. In 1982, W. J. Liu [11] defined and studied fuzzy subrings as well as fuzzy ideals in rings. D. Mandal [12] studied fuzzy ideals and fuzzy interior ideals in an ordered semiring. M. Murali Krishna Rao [13] studied fuzzy soft Γ-semiring and fuzzy soft $k-$ideal over $\Gamma-$semiring. N. Kuroki [8] studied fuzzy interior ideals in semigroups.Murali Krishna Rao [14] studied $T-$fuzzy ideals of ordered $\Gamma-$semirings. In 2017, Marapureddy Murali Krishna Rao [15, 16, 22, 23] introduced bi-quasi-ideals in semirings, bi-quasi-ideals and fuzzy bi-quasi–ideals in $\Gamma-$semigroups and T- fuzzy ideals in ordered $\Gamma-$semirings. In this paper we introduce the notion of right(left) bi-quasi ideal of semiring which is a generalization of bi-ideals of semiring and study the properties of right bi-quasi ideal. We characterize the right bi-quasi simple semiring and regular semiring. We also introduce the notion of fuzzy right (left) bi-quasi ideal of semiring and we characterize the regular semiring in terms of fuzzy right bi-quasi ideals of semiring.

2. Preliminaries

In this section we will recall some of the fundamental concepts and definitions, which are necessary for this paper.

Definition 2.1. A semigroup is an algebraic system $(S, .)$ consisting of a non-empty set S together with an associative binary operation $'\cdot'$.

Definition 2.2. A sub semigroup T of semigroup S is a non-empty subset T of S such that $TT \subseteq T$.

Definition 2.3. A non-empty subset T of semigroup S is called a left (right) ideal of S if $ST \subseteq T (TS \subseteq T)$.

DEFINITION 2.4. A non-empty subset T of semigroup S is called an ideal of S if it is both a left ideal and a right ideal of S.

DEFINITION 2.5. A non-empty Q of semigroup S is called a quasi ideal of S if $QS \cap SQ \subseteq Q$.

DEFINITION 2.6. A sub semigroup T of semigroup S is called a bi-ideal of S if $TST \subseteq T$.

DEFINITION 2.7. A sub semigroup T of semigroup S is called an interior ideal of S if $STS \subseteq T$.

DEFINITION 2.8. An element a of a semigroup S is called a regular element if there exists an element b of S such that $a = aba$.

DEFINITION 2.9. A semigroup S is called a regular semigroup if every element of S is a regular element.

DEFINITION 2.10. A set S together with two associative binary operations called addition and multiplication (denoted by $+$ and $\cdot$ respectively) will be called a semiring provided

(i) addition is a commutative operation.

(ii) multiplication distributes over addition both from the left and from the right.

(iii) there exists $0 \in S$ such that $x + 0 = x$ and $x \cdot 0 = 0 \cdot x = 0$ for each $x \in S$.

DEFINITION 2.11. A non-empty subset A of semiring M is called

(i) a subsemiring of M if $(A, +)$ is a subsemigroup of $(M, +)$ and $AA \subseteq A$.

(ii) a quasi ideal of M if A is a subsemiring of M and $AM \cap M \subseteq A$.

(iii) a bi-ideal of M if A is a subsemiring of M and $AMA \subseteq A$.

(iv) an interior ideal of M if A is a subsemiring of M and $MAM \subseteq A$.

(v) a left (right) ideal of M if A is a subsemiring of M and
$$MA \subseteq A(AM \subseteq A).$$

(vi) an ideal if A is a subsemiring of $M, A \subseteq A$ and $MA \subseteq A$.

(vii) a $k-$ideal if A is a subsemiring of M if holds
$$AM \subseteq A, MA \subseteq A \text{ and } x \in M,\ x + y \in A, y \in A \text{ then } x \in A.$$

DEFINITION 2.12. Let M be a non-empty set. A mapping $\mu : M \to [0, 1]$ is called a fuzzy subset of M.

DEFINITION 2.13. If μ is a fuzzy subset of M, for $t \in [0, 1]$ then the set $\mu_t = \{x \in M \mid \mu(x) \geqslant t\}$ is called a level subset of M with respect to a fuzzy subset μ.

DEFINITION 2.14. A fuzzy subset $\mu : M \to [0, 1]$ is a non-empty fuzzy subset if μ is not a constant function.

DEFINITION 2.15. For any two fuzzy subsets λ and μ of M, $\lambda \subseteq \mu$ means $\lambda(x) \leqslant \mu(x)$ for all $x \in M$.

Definition 2.16. Let A be a non-empty subset of M. The characteristic function of A is a fuzzy subset of M is defined by $\chi_{_A}(x) = \begin{cases} 1, & \text{if } x \in A; \\ 0, & \text{if } x \notin A. \end{cases}$

Definition 2.17. A fuzzy subset μ of semiring M is called

(i) a fuzzy subsemiring of M if it satisfies the following conditions
(a) $\mu(x + y) \geqslant min\ \{\mu(x), \mu(y)\}$
(b) $\mu(xy) \geqslant min\ \{\mu(x), \mu(y)\}$, for all $x, y \in M$.

(ii) a fuzzy left (right) ideal of M if it satisfies the following conditions
(a) $\mu(x + y) \geqslant min\{\mu(x), \mu(y)\}$
(b) $\mu(xy) \geqslant \mu(y)\ (\mu(x))$, for all $x, y \in M$.

(iii) a fuzzy ideal of M if it satisfies the following conditions
(a) $\mu(x + y) \geqslant min\{\mu(x), \mu(y)\}$
(b) $\mu(xy) \geqslant max\ \{\mu(x), \mu(y)\}$, for all $x, y \in M$.

(iv) a fuzzy bi-ideal of M if it satisfies the following conditions
(a) $\mu(x + y) \geqslant \min\{\mu(x), \mu(y)\}$ for all $x, y \in M$
(b) $\mu \circ \chi_M \circ \mu \subseteq \mu$.

(v) a fuzzy quasi -ideal ofM if it satisfies the following conditions
(a) $\mu(x + y) \geqslant \min\{\mu(x), \mu(y)\}$ for all $x, y \in M$
(b)$\mu \circ \chi_M \cap \chi_M \circ \mu \subseteq \mu$.

Definition 2.18. Let M be a semiring. A non-empty subset L of M is said to be left bi-quasi ideal (right bi-quasi ideal) of M if L is a subsemigroup of $(M, +)$ and $ML \cap LML \subseteq L$ $(LM \cap LML \subseteq L)$.

Definition 2.19. Let M be a semiring. L is said to be bi-quasi ideal of M if it is both a left bi-quasi and a right bi-quasi ideal of M.

Definition 2.20. A semiring M is called a right bi-quasi simple semiring if M has no right bi-quasi ideal other than M itself.

Example 2.1. (i) Let Q be the set of all rational numbers,

$$M = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in Q \right\}$$

be the additive semigroup of M matrices. Binary operation AB is defined as usual matrix multiplication of A, B, for all $A, B \in M$.

(a) If $R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid 0 \neq a, 0 \neq b \in Q \right\}$ then R is a quasi ideal of semiring M and R is neither a left ideal nor a right ideal.

(b) If $S = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid 0 \neq a, 0 \neq b \in Q \right\}$ then S is a bi-ideal of $\Gamma-$semiring M.

(ii) If $M = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in Q \right\}$. Then M is a semiring with respect to usual addition of matrices and usual matrix multiplication and

$$A = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid 0 \neq a, 0 \neq b \in Q \right\}.$$

Then A is a left bi-quasi ideal and A is not a bi-ideal of semiring M.

3. Fuzzy left and right bi-quasi ideals

In this section we introduce the notion of fuzzy right (left) bi-quasi ideal as a generalization of fuzzy bi-ideal of semiring M.and study the properties of fuzzy right bi-quasi ideals

Definition 3.1. A fuzzy subset μ of semiring M is called a fuzzy left (right) bi-quasi ideal if

(i) $\mu(x + y) \geqslant \min\{\mu(x), \mu(y)\}$ for all $x, y \in M$.

(ii) $\chi_M \circ \mu \cap \mu \circ \chi_M \circ \mu \subseteq \mu\big(\mu \circ \chi_M \cap \mu \circ \chi_M \circ \mu \subseteq \mu\big)$.

A fuzzy subset μ of semiring M is called a fuzzy bi-quasi ideal if it is both a fuzzy left and a fuzzy right bi-quasi ideal of M.

Example 3.1. Let Q be the set of all rational numbers,

$$M = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in Q \right\}.$$

Then M is a semiring with respect to usual addition of matrices and ternary operation is defined as the usual matrix multiplication and

$$A = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, 0 \neq b \in Q \right\}.$$

Then A is a right bi-quasi ideal but not a bi-ideal of semiring M. Define $\mu : M \to [0, 1]$ such that $\mu(x) == \begin{cases} 1 & \text{if } x \in A, \\ 0, & \text{otherwise.} \end{cases}$ Then μ is a fuzzy right bi-quasi ideal of M.

Theorem 3.1. *Every fuzzy right ideal of semiring M is a fuzzy right bi-quasi ideal of M.*

Proof. Let μ be a fuzzy right ideal of semiring M and $x \in M$.

$$\mu \circ \chi_M(x) = \sup_{x=ab} \min\{\mu(a), \chi_M(b)\} = \sup_{x=ab} \mu(a) \leqslant \sup_{x=ab} \mu(ab) = \mu(x).$$

Therefore $\mu \circ \chi_M(x) \leqslant \mu(x)$. Now

$$\mu \circ \chi_M \circ \mu(x) = \sup_{x=uvs} \min\{\mu \circ \chi_M(uv), \mu(s)\} \leqslant \sup_{x=uvs} \min\{\mu(uv), \mu(s)\} = \mu(x).$$

Now

$$\begin{aligned}\mu \circ \chi_M \cap \mu \circ \chi_M \circ \mu(x) &= \min\{\mu \circ \chi_M(x), \mu \circ \chi_M \circ \mu(x)\}\\ &\leqslant \min\{\mu \circ \chi_M(x), \mu(x)\}\\ &\leqslant \mu(x).\end{aligned}$$

Hence μ is a fuzzy right bi-quasi ideal of semiring M. □

THEOREM 3.2. *Every fuzzy left ideal of semiring M is a fuzzy left bi-quasi ideal of M.*

PROOF. Let μ be a fuzzy left ideal of semiring M and $x \in M$.

$$\begin{aligned}\chi_M \circ \mu(x) &= \sup_{x=a\alpha b} \min\{\chi_M(a), \mu(b)\} = \sup_{x=ab} \min\{1, \mu(b)\} = \sup_{x=ab} \mu(b)\\ &\leqslant \sup_{x=ab} \mu(ab) = \sup_{x=ab} \mu(x) = \mu(x)\end{aligned}$$

So, we have $\chi_M \circ \mu(x) \leqslant \mu(x)$. Now

$$\begin{aligned}\mu \circ \chi_M \circ \mu(x) &= \sup_{x=uvs} \min\{\mu \circ \chi_M(uv), \mu(s)\}\\ &\leqslant \sup_{x=uvs} \min\{\mu(uv), \mu(s)\}\\ &= \mu(x)\end{aligned}$$

and

$$\begin{aligned}\chi_M \circ \mu \cap \mu \circ \chi_M \circ \mu(x) &= \min\{\chi_M \circ \mu(x), \mu \circ \chi_M \circ \mu(x)\}\\ &\leqslant \min\{\mu(x), \mu(x)\}\\ &= \mu(x).\end{aligned}$$

Therefore $\chi_M \circ \mu \cap \mu \circ \chi_M \circ \mu \subseteq \mu$. Hence μ is a fuzzy left bi-quasi ideal of M. □

THEOREM 3.3. *Every fuzzy left ideal of semiring M is a fuzzy right bi-quasi ideal of M.*

PROOF. Let μ be a fuzzy left ideal of semiring M and $x \in M$. By Theorem 3.2, we have $\chi_M \circ \mu \subseteq \mu$. Then

$$\begin{aligned}\mu \circ \chi_M \circ \mu(x) &= \sup_{x=uvs} \min\{\mu(s), \chi_M \circ \mu(vs)\}\\ &\leqslant \sup_{x=uvs} \min\{\mu(s), \mu(vs)\}\\ &= \mu(x).\end{aligned}$$

Now

$$\begin{aligned}\mu \circ \chi_M \cap \mu \circ \chi_M \circ \mu(x) &= \min\{\mu \circ \chi_M(x), \mu \circ \chi_M \circ \mu(x)\}\\ &\leqslant \min\{\mu \circ \chi_M(x), \mu(x)\}\\ &\leqslant \mu(x).\end{aligned}$$

Hence μ is a fuzzy right bi-quasi ideal of M. □

Corollary 3.1. *Every fuzzy right ideal of semiring M is a fuzzy left bi-quasi ideal of M.*

Corollary 3.2. *Every fuzzy right (left) ideal of semiring M is a fuzzy bi-quasi ideal of M.*

Theorem 3.4. *Let M be a semiring and μ be a non-empty fuzzy subset of M. A fuzzy subset μ is a fuzzy left bi-quasi ideal of semiring if and only if the level subset μ_t of μ is a left bi-quasi ideal of semiring M for every $t \in [0,1]$, where $\mu_t \neq \phi$.*

Proof. Let M be a semiring and μ be a non-empty fuzzy subset of M. Suppose μ is a fuzzy left bi-quasi ideal of semiring, $\mu_t \neq \phi, t \in [0,1]$ and $a, b \in \mu_t$, Then $\mu(a) \geqslant t, \mu(b) \geqslant t$. Thus $\mu(a+b) \geqslant \min\{\mu(a), \mu(b)\} \geqslant t$ and $a+b \in \mu_t$.

Let $x \in M\mu_t \cap \mu_t M \mu_t$. Then $x = ba = cde$, where $b, d \in M, a, c, e \in \mu_t$ Thus

$$\chi_M \circ \mu(x) \geqslant t \text{ and } \mu \circ \chi_M \circ \mu(x) \geqslant t \text{ and } \mu(x) \geqslant (\chi_M \circ \mu \cap \mu \circ \chi_M \circ \mu)(x) \geqslant t.$$

Therefore $x \in \mu_t$. Hence μ_t is a left bi-quasi ideal of M.

Conversely suppose that μ_t is a left bi-quasi ideal of semiring M, for all $t \in Im(\mu)$. Let $x, y \in M, \mu(x) = t_1, \mu(y) = t_2$ and $t_1 \geqslant t_2$. Then $x, y \in \mu_{t_2}$. Thus $x + y \in \mu_{t_2}$ and $x, y \in \mu_{t_2}$ and $\mu(x+y) \geqslant t_2 = \min\{t_1, t_2\} = \min\{\mu(x), \mu(y)\}$. Therefore $\mu(x+y) \geqslant t_2 = \min\{\mu(x), \mu(y)\}$. We have $M\mu_l \cap \mu_l M \mu_l \subseteq \mu_t$, for all $l \in Im(\mu)$.

Suppose $t = \min\{Im(\mu)\}$. Then $M\mu_t \cap \mu_t M \mu_t \subseteq \mu_t$. Therefore

$$\chi_M \circ \mu \cap \mu \circ \chi_M \circ \mu \subseteq \mu.$$

Hence μ is a fuzzy left bi-quasi ideal of semiring M. □

Corollary 3.3. *Let M be a semiring and μ be a non-empty fuzzy subset of M. A fuzzy subset μ is a fuzzy right bi-quasi ideal of semiring if and only if the level subset μ_t of μ is a right bi-quasi ideal of $\Gamma-$semiring M for every $t \in [0,1]$, where $\mu_t \neq \phi$.*

Theorem 3.5. *Let I be a non-empty subset of a semiring M and χ_I be the characteristic function of I. Then I is a right bi-quasi ideal of semiring M if and only if χ_I is a fuzzy right bi-quasi ideal of semiring M.*

Proof. Let I be a non-empty subset of a semiring M and χ_I be the characteristic function of I. Suppose I is a right bi-quasi ideal of semiring M. Obviously χ_I is a fuzzy subsemiring of M. We have $IM \cap IMI \subseteq I$. Then

$$\chi_I \circ \chi_M \cap \chi_I \circ \chi_M \circ \chi_I = \chi_{IM} \cap \chi_{IMI} = \chi_{IM \cap IMI} \subseteq \chi_I.$$

Therefore χ_I is a fuzzy right bi-quasi ideal of semiring M.

Conversely suppose that χ_I is a fuzzy right bi-quasi ideal of M. Then I is a subsemiring of M. We have $\chi_I \circ \chi_M \cap \chi_I \circ \chi_M \circ \chi_I \subseteq \chi_I$. Then $\chi_{IM} \cap \chi_{IMI} \subseteq \chi_I$ and $\chi_{IM \cap IMI} \subseteq \chi_I$ Therefore $IM \cap IMI \subseteq I$. Hence I is a right bi-quasi ideal of semiring M. □

Theorem 3.6. *If μ and λ are fuzzy left bi-quasi ideals of semiring M, then $\mu \cap \lambda$ is a fuzzy left bi-quasi ideal of semiring M.*

Proof. Let μ and λ be fuzzy bi- quasi ideals of semiring M. Then

$$\begin{aligned}\mu\cap\lambda(x+y) &= \min\{\mu(x+y),\lambda(x+y)\}\\ &\geqslant \min\{\min\{\mu(x),\mu(y)\},\min\{\lambda(x),\lambda(y)\}\}\\ &= \min\{\min\{\mu(x),\lambda(x)\},\min\{\mu(y),\lambda(y)\}\}\\ &= \min\{\mu\cap\lambda(x),\mu\cap\lambda(y)\}\end{aligned}$$

and

$$\begin{aligned}\chi_M\circ\mu\cap\lambda(x) &= \sup_{x=ab}\min\{\chi_M(a),\mu\cap\lambda(b)\}\\ &= \sup_{x=ab}\min\{\chi_M(a),\min\{\mu(b),\lambda(b)\}\}\\ &= \sup_{x=ab}\min\{\min\{\chi_M(a),\mu(b)\},\min\{\chi_M(a),\lambda(b)\}\}\\ &= \min\{\sup_{x=ab}\min\{\chi_M(a),\mu(b)\},\sup_{x=ab}\min\{\chi_M(a),\lambda(b)\}\}\\ &= \min\{\chi_M\circ\mu(x).\chi_M\circ\lambda(x)\}\\ &= \chi_M\circ\mu\cap\chi_M\circ\lambda(x)\end{aligned}$$

Therefore $\chi_M\circ\mu\cap\lambda=\chi_M\circ\mu\cap\chi_M\circ\lambda$. Further on from $\mu\cap\lambda\circ\chi_M\circ\mu\cap\lambda(x)$, we have

$$\begin{aligned}&\mu\cap\lambda\circ\chi_M\circ\mu\cap\lambda(x)\\ &\quad= \sup_{x=abc}\min\{\mu\cap\lambda(a),\chi_M\circ\mu\cap\lambda(bc)\}\\ &\quad= \sup_{x=abc}\min\{\mu\cap\lambda(a),\chi_M\circ\mu\cap\chi_M\circ\lambda(bc)\}\\ &\quad= \sup_{x=abc}\min\{\min\{\mu(a),\lambda(a)\},\min\{\chi_M\circ\mu(bc),\chi_M\circ\lambda(bc)\}\}\\ &\quad= \sup_{x=abc}\min\{\min\{\mu(a),\chi_M\circ\mu(bc)\},\min\{\lambda(a),\chi_M\circ\lambda(bc)\}\}\\ &\quad= \min\{\sup_{x=abc}\min\{\mu(a),\chi_M\circ\mu(bc)\},\sup_{x=abc}\min\{\lambda(a),\chi_M\circ\lambda(bc)\}\}\\ &\quad= \min\{\mu\circ\chi_M\circ\mu(x),\lambda\circ\chi_M\circ\lambda(x)\}\\ &\quad= \mu\circ\chi_M\circ\mu\cap\lambda\circ\chi_M\circ\lambda(x).\end{aligned}$$

Therefore $\mu\cap\lambda\circ\chi_M\circ\mu\cap\lambda=\mu\circ\chi_M\circ\mu\cap\lambda\circ\chi_M\circ\lambda$. Hence

$$\chi_M\circ\mu\cap\lambda\cap\mu\cap\lambda\circ\chi_M\circ\mu\cap\lambda=(\chi_M\circ\mu)\cap(\mu\circ\chi_M\circ\mu)\cap(\chi_M\circ\lambda)\cap(\lambda\circ\chi_M\circ\lambda)\subseteq\mu\cap\lambda.$$

Thus $\mu\cap\lambda$ is a left fuzzy bi-quasi ideal of M.

Hence the theorem. □

Corollary 3.4. *If μ and λ are fuzzy right bi-quasi ideals of semiring M then $\mu\cap\lambda$ is a fuzzy right bi-quasi ideal of semiring M.*

Theorem 3.7. *Let μ and λ be fuzzy right ideal and fuzzy left ideal of semiring M respectively. Then $\mu\cap\lambda$ is a left fuzzy bi-quasi ideal of semiring M.*

PROOF. Let μ and λ be fuzzy right ideal and fuzzy left ideal of semiring M respectively. Then

$$\begin{aligned}\mu\cap\lambda(x+y) &= \min\{\mu(x+y),\lambda(x+y)\}\\ &\geqslant \min\{\min\{\mu(x),\mu(y)\},\min\{\lambda(x),\lambda(y)\}\}\\ &= \min\{\min\{\mu(x),\lambda(x)\},\min\{\mu(y),\lambda(y)\}\}\\ &= \min\{\mu\cap\lambda(x),\mu\cap\lambda(y)\}\end{aligned}$$

By Theorem 3.6 have $\chi_M\circ(\mu\cap\lambda)=\chi_M\circ\mu\cap\chi_M\circ\lambda$ and $\mu\cap\lambda\circ\chi_M\circ\mu\cap\lambda=\mu\circ\chi_M\circ\mu\cap\lambda\circ\chi_M\circ\lambda$. Thus

$$\begin{aligned}&\chi_M\circ(\mu\cap\lambda)\cap\mu\cap\lambda\circ\chi_M\circ\mu\cap\lambda\\ &\qquad=(\chi_M\circ\mu)\cap(\mu\circ\chi_M\circ\mu)\cap(\chi_M\circ\lambda)\cap(\lambda\circ\chi_M\circ\lambda)\subseteq\mu\cap\lambda.\end{aligned}$$

Hence $\mu\cap\lambda$ is a left fuzzy bi-quasi ideal of semiring M. □

COROLLARY 3.5. *Let μ and λ be fuzzy right ideal and fuzzy left ideal of semiring M respectively. Then $\mu\cap\lambda$ is a right fuzzy bi-quasi ideal of semiring M.*

Proof of the following theorems are similar to theorems in [14]. So we omit the proofs.

THEOREM 3.8. *If μ is a fuzzy quasi-ideal of a regular semiring M then μ is a fuzzy ideal of M.*

THEOREM 3.9. *M is a regular semiring if and only $AB=A\cap B$, for any right ideal A and left ideal B of semiring M.*

THEOREM 3.10. *A semiring M is a regular if and only if $\lambda\circ\mu=\lambda\cap\mu$, for any fuzzy right ideal λ and fuzzy left ideal μ of M.*

THEOREM 3.11. *Let M be a regular semiring. Then μ is a fuzzy left bi-quasi ideal of M if and only if μ is a fuzzy quasi ideal of M.*

PROOF. Let μ be a fuzzy left bi-quasi ideal of semiring M and $x\in M$. Then

$$\chi_M\circ\mu\cap\mu\circ\chi_M\circ\mu\subseteq\mu.$$

Suppose $\chi_M\circ\mu(x)>\mu(x)$ and $\mu\circ\chi_M(x)>\mu(x)$. Since M is a regular, there exists $y\in M$ such that $x=xyx$. Thus

$$\mu\circ\chi_M(x)=\sup_{x=xyx}\min\{\mu(x),\chi_M(yx)\}=\sup_{x=xyx}\min\{\mu(x),1\}=\sup_{x=xyx}\mu(x)=\mu(x).$$

Therefore $\mu\circ\chi_M\cap\chi_M\circ\mu\subseteq\mu$. and

$$\mu\circ\chi_M\circ\mu(x)=\sup_{x=xyx}\min\{\mu\circ\chi_M(x),\mu(yx)\})>\sup_{x=xyx}\min\{\mu(x),\mu(yx)\}=\mu(x)$$

which is a contradiction. Hence μ is a fuzzy quasi ideal of M. It is clear that the converse is true. □

COROLLARY 3.6. *Let M be a regular semiring. Then μ is a fuzzy right bi-quasi ideal of M if and only if μ is a fuzzy quasi ideal of M.*

THEOREM 3.12. ([15]) *Let M be a semiring. If M is a regular semiring if and only if $B = MB \cap BMB$ for every left bi-quasi ideal B of M.*

THEOREM 3.13. *Let M be a semiring. Then M is a regular if and only if $\mu = \mu \circ \chi_M \cap \mu \circ \chi_M \circ \mu$, for any fuzzy right bi-quasi ideal μ of semiring M.*

PROOF. Let μ be a fuzzy right bi-quasi ideal of regular semiring M. Then $\mu \circ \chi_M \ \cap \mu \circ \chi_M \circ \mu \subseteq \mu$. Thus

$$\begin{aligned}
\mu \circ \chi_M(x) &= \sup_{x=xyx} \min\{\mu(x), \chi_M(yx)\} = \mu(x) \\
\mu \circ \chi_M \circ \mu(x) &= \sup_{x=xyx} \min\{\mu(x), \chi_M \circ \mu(yx)\} \\
&= \sup_{x=xyx} \min\{\mu(x), \sup_{yx=rs} \min\{\chi_M(s), \mu(r)\}\} \\
&= \sup_{x=xyx} \min\{\mu(x), \sup_{yx=ras} \min\{1, \mu(s)\}\} \\
&\geqslant \sup_{x=xx} \min\{\mu(x), \mu(x)\} = \mu(x).
\end{aligned}$$

Therefore $\chi_M \circ \mu \cap \mu \circ \chi_M \circ \mu = \mu$.

Conversely suppose that $\mu = \mu \circ \chi_M \cap \mu \circ \chi_M \circ \mu$, for any fuzzy left bi-quasi ideal μ of semiring M. Let B be a right bi-quasi ideal ofsemiring M. Then by Theorem 3.8, χ_B be a fuzzy right bi-quasi ideal of semiring M. Therefore

$$\chi_B = \chi_M \circ \chi_B \cap \chi_B \circ \chi_M \circ \chi_B = \chi_{MB} \cap \chi_{BMB}$$

and $B = MB \cap BMB$. By Theorem 3.12, M is a regular semiring. □

THEOREM 3.14. *Let M be a semiring. Then M is a regular if and only if $\mu \cap \gamma \subseteq \gamma \circ \mu \cap \mu \circ \gamma \circ \mu$, for every fuzzy left bi-quasi ideal $\mu$and every fuzzy ideal γ of semiring M.*

PROOF. Let M be a semiring. Suppose M is regular and $x \in M$. Then there exists $y \in M$ such that $xy = xyxy$.

$$\begin{aligned}
\mu \circ \gamma \circ \mu(x) &= \sup_{x=yz} \min\{\mu \circ \gamma(y), \mu(z)\} \\
&\geqslant \sup_{x=xyx} \min\{\mu \circ \gamma(x\alpha y), \mu(x)\} \\
&= \min\{\sup_{x=xyx} \min\{\mu(a), \gamma(b)\}, \mu(x)\} \\
&\geqslant \min\{\sup_{xy=xyxy} \min\{\mu(a), \gamma(yxy)\}, \mu(x)\} \\
&\geqslant \min\{\min\{\mu(x), \gamma(x)\}\} \\
&= \min\{\mu(x), \gamma(x)\} \\
\gamma \circ \mu(x) &= \sup_{x=ab} \min\{\gamma(a), \mu(b)\} \\
&= \sup_{x=xyx} \min\{\gamma(y), \mu(x)\} \\
&\geqslant \min\{\gamma(x), \mu(x)\} = \mu \cap \gamma(x).
\end{aligned}$$

Therefore $\mu \cap \gamma \subseteq \mu \circ \gamma \circ \mu$. Hence $\mu \cap \gamma \subseteq \gamma \circ \mu \cap \mu \circ \gamma \circ \mu$.

Conversely suppose that the condition holds. Let μ be a fuzzy left bi-quasi ideal of semiring M. Then $\mu \cap \chi_M \subseteq \chi_M \circ \mu \cap \mu \circ \chi_M \circ \mu$ and $\mu \subseteq \chi_M \circ \mu \cap \mu \circ \chi_M \circ \mu$. By Theorem 3.12, M is a regular semiring. □

COROLLARY 3.7. *Let M be a semiring. Then M is a regular if and only if $\mu \cap \gamma \subseteq \mu \circ \gamma \cap \mu \circ \gamma \circ \mu$, for every fuzzy right bi-quasi ideal μ and every fuzzy ideal γ ofsemiring M.*

4. Conclusion

In this paper, we introduced the notion of fuzzy right (left) bi-quasi ideal of semiring and characterized the regular semiring in terms of fuzzy right(left) bi-quasi ideals of semiring and studied some of their algebraical properties.

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