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SOLUTION OF HSU MODEL BY CRANK-NICOLSON METHOD AND SPLITTING TECHNIQUE

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Abstract. An important step in the production of soybean-derived products such as soya milk is the soaking process. Moisture hydration during soaking depends on the time-temperature binomial. The amount of absorbed water increases with soaking time and temperature until it reaches a saturation limit. Both empirical and phenomenological models that represent hydration have been developed to predict the necessary time to obtain the desired moisture content at a certain temperature, representing the dynamic behavior of the soaking process. A distributed parameter phenomenological model which is known as Hsu model is used in this study. As the model involves nonlinear partial differential equations (PDEs) which are inherently 'stiff', it is usually difficult to solve them numerically. To solve Hsu model it is used Crank-Nicolson method and a splitting technique. Besides grain volume variation function is used in the literature, a new exponential type function is used and results are compared.

1. Introduction

Soaking is a prelude to cooking in softening the seeds and gelatinizing the starch before or during cooking. Partial leaching oligosaccharides during soaking reduce gas production in humans and monogastric animals. Pre-soaking brings down the cooking time required to achieve the desired softness making the cooking process convenient and fuel efficient. Maximizing water content and seed size improves profitability in the canning industry; for consumers, it can achieve satiety at the lower calorific value or dry seed weight [**5**].

Water absorption by soybean grain during soaking depends mainly on time-temperature binomial. The amount of absorbed water increases with soaking time and temperature [**3, 4**].

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Models that represent the hydration of grains have been developed to predict the necessary time to obtain the desired moisture content at a certain temperature, representing the dynamic behavior of the soaking process. These models may basically empirical and phenomenological [**3, 4**].

Hsu [**2**] developed well-established distributed parameter model that does not take into account the change in volume expansion due to soaking presented in the literature for the analysis of grain hydration. Coutinho [**3**] obtained experimentally that the increase throughout soybean grains hydration could reach 30 %. After that, Nicolin et al.[**1**] proposed a model with considering the volume variation in hydration of grain by using radius function with respect to time.

In this study, Hsu model with volume variation is investigated by Crank-Nicolson method and a splitting technique. Grain volume variation is represented by two different radius function with respect to time and obtained results are compared.

2. Problem Formulation

Hsu model assumes that seeds are spherical, diffusion takes place only in radial direction, the effect of volume change due to absorption is negligible and the diffusion coefficient is a function of moisture content.

The model in Eq.(2.1) is inspired Fick's diffusion model(1855) that from a mass balance in differential volume element, considering radial diffusivity as exponentially varying with grain moisture content as in Eq.(2.2).

$$\frac{\partial X}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 D \frac{\partial X}{\partial r}\right) \tag{2.1}$$

$$D = D_0 e^{k_1 X} \tag{2.2}$$

where D_0 is a pre-exponential factor that represents the value for water diffusivity in the grain when moisture content (X) equals zero; k_1 is an exponential factor.

The amount of water absorbed by soybean can be obtained from Eq.(2.1) with the following initial and boundary conditions:

$$X = X_0 \quad \text{for any } r \quad \text{and} \quad t = 0 \tag{2.3}$$

$$\frac{\partial X}{\partial r} = 0 \quad \text{for} \quad r = 0 \quad \textit{and} \quad t > 0 \tag{2.4}$$

$$X = \left(1 - e^{-\beta t}\right) X_{eq} + X_0 e^{-\beta t} \quad \text{for } r = R \quad \textit{and} \quad t > 0 \tag{2.5}$$

where X_0 is initial moisture content, X_{eq} is equilibrium moisture content and β is constant saturation rate.

Eq.(2.3) represents the initial condition that assumes moisture content initially uniform throughout the grain. The boundary conditions establish the grain central symmetry in any instant of time in Eq.(2.4) and the moisture content variation at the solid-fluid interface in Eq.(2.5).

In the present study grain volume variation will be represented by the radius versus time functions as in Eq.(2.6) and Eq.(2.7):

$$R_p = R_0 + at^n \tag{2.6}$$

$$R_p = a + bexp(t^c) \tag{2.7}$$

where R_0 is initial radius size and a, b, c, n are constants.

Eq.(2.6) is used in [**4**] for grain volume variation at all temperatures. Also in this study, both functions are used separately to represent the volume variation.

By making the following substitutions [**2**]:

$$X^* = \frac{X - X_0}{X_{eq} - X_0} \qquad r^* = \frac{r}{R}$$

$$t^* = \frac{tD_0^*}{R^2} \qquad D_1 = \frac{D}{D_0^*}$$

$$D_0^* = D_0 e^{k_1 X_0} \qquad k = k_1 \left(X_{eq} - X_0\right)$$

On introducing,

$$S = \int_0^{X^*} D_1 dX^* = \frac{1}{k}\left(e^{kX^*} - 1\right) = \frac{1}{k}(D_1 - 1)$$

By this way, dimensionless form of the model is obtained as Eqs.(2.8)-(2.11):

$$\frac{\partial S}{\partial t^*} = D_1 \left[\frac{\partial^2 S}{\partial r^{*2}} + \frac{2}{r^*}\left(\frac{\partial S}{\partial r^*}\right)\right] \tag{2.8}$$

$$S = 0 \quad \text{for all } r^* \quad \text{and} \quad t^* = 0 \tag{2.9}$$

$$\frac{\partial S}{\partial r^*} = 0 \quad \text{at} \quad t^* = 0 \tag{2.10}$$

$$S = \frac{1}{k}\left(e^{k\left(1 - e^{-Bt^*}\right)}\right) \quad \text{for} \quad r^* = 1 \quad \text{with} \quad B = \frac{\beta R^2}{D_0^*} \tag{2.11}$$

2.1. Crank-Nicolson Method. Discretization of radial and time derivatives of Eq.(2.8) by Crank-Nicolson method yields Eq.(2.12) which is valid for the internal points:

$$\frac{S_{i,j+1} - S_{i,j}}{\Delta t^*} = \frac{1}{2}\left(kS_{i,j} + 1\right)\left[\frac{S_{i+1,j+1} - 2S_{i,j+1} + S_{i-1,j+1}}{\Delta r^{*2}} + \frac{S_{i+1,j} - 2S_{i,j} + S_{i-1,j}}{\Delta r^{*2}}\right] \tag{2.12}$$

$$+\frac{1}{(i-1)\Delta r^*}\left(kS_{i,j} + 1\right)\left[\frac{S_{i+1,j+1} - S_{i-1,j+1}}{2\Delta r^*} + \frac{S_{i+1,j} - S_{i-1,j}}{2\Delta r^*}\right]$$

Initial and boundary conditions are discretized as below:

$$S_i = 0 \quad \text{for all} \quad r^* \quad \text{and} \quad t^* = 0, \quad i = 1, 2, ..., N+1 \tag{2.13}$$

$$S_0 = S_1, \quad r^* = 0 \tag{2.14}$$

$$S_{N+1} = \left[e^{k\left(1-e^{-Bt^*}\right)} - 1 \right] / k, \quad r^* = 1 \quad \text{with} \quad B = \frac{\beta R^2}{D_0^*}. \tag{2.15}$$

Since the radius of soybean changes with respect to time, the average increment of the radius is determined by

$$\Delta r^* = \frac{R_p}{N} \tag{2.16}$$

where N is the number of division of radius.

2.2. A Splitting Technique. Splitting the Eq.(2.8) it is obtained Eq.(2.17) and Eq.(2.18):

$$\frac{1}{2}\frac{\partial S}{\partial t^*} = D_1 \frac{\partial^2}{\partial r^{*2}}, \quad t^j \leqslant t \leqslant t^{j+1/2} \tag{2.17}$$

$$\frac{1}{2}\frac{\partial S}{\partial t^*} = D_1 \frac{2}{r^*}\frac{\partial S}{\partial r^*}, \quad t^{j+1/2} \leqslant t \leqslant t^{j+1} \tag{2.18}$$

Discretization of radial and time derivatives by finite difference yields Eq.(2.18) and Eq.(2.19):

$$\begin{aligned} S_{i,j+1/2} = {} & 2\left(kS_{i,j-1/2} + 1\right) r_{j+1/2} \left(S_{i+1,j-1/2} + S_{i-1,j-1/2}\right) \\ & + (1 - 4(kS_{i,j-1/2} + 1)) r_{j+1/2} S_{i,j-1/2} \end{aligned} \tag{2.19}$$

$$S_{i,j+1} = S_{i,j+1/2} + \frac{4}{(i-1)}\left(kS_{i,j+1/2} + 1\right) r_{j+1} (S_{i+1,j+1/2} - S_{i-1,j+1/2}) \tag{2.20}$$

where $r_{j+1/2}$ and r_{j+1} are depend on r^* and t^*. Where Eq.(2.19) is solved for a time interval of $\Delta t^*/2$ using the initial condition of Eq.(2.1). The solution of Eq.(2.19) is used as the initial condition of Eq.(2.20) and Eq.(2.20) is solved for a time interval of $\Delta t^*/2$.

Initial and boundary conditions are discretized as in Eq.(2.21)-(2.23):

$$S_i = 0 \quad \text{for all} \quad r^* \quad \text{and} \quad t^* = 0 \tag{2.21}$$

$$S_{2,j+1} = S_{1,j+1} \quad \text{for} \quad r^* = 0 \tag{2.22}$$

$$S_{n+1,j} = \frac{1}{k}\left[e^{k\left(1-e^{-Bt^*}\right)} - 1 \right] \quad \text{for} \quad r^* = 1 \quad \text{with} \quad B = \frac{\beta R^2}{D_0^*}. \tag{2.23}$$

3. Problem Solution

Hsu model is solved by Crank-Nicolson method and a splitting technique by using Eq.(2.6) and Eq.(2.7) and the adjusted parameters at temperature 10^o C are given in Table 1. The number of divisions of the radius (N) is taken as 20.

Table 1. Adjusted parameters (Nicolin et al.(2012))

$T(^oC)$	D_0x10^{10}	k_1	$\beta x10^3$	
10	4.810 ± 0.085	0.018 ± 0.0001	5.078	
20	7.859 ± 0.087	0.021 ± 0.0002	4.026	
30	11.250 ± 0.017	0.027 ± 0.0002	4.602	
40	17.349 ± 0.077	0.033 ± 0.0006	5.535	
50	4.810 ± 0.085	0.039 ± 0.0006	8.732	

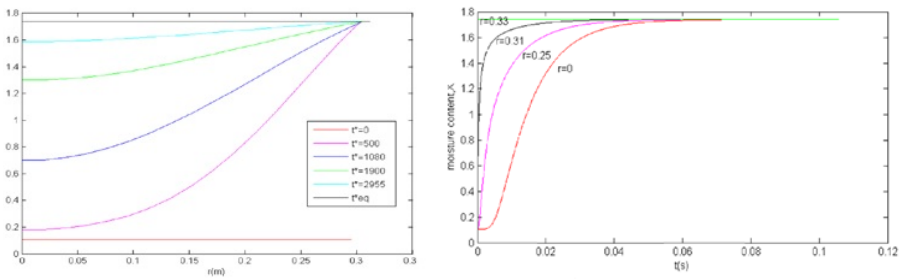

Figure 1. Moisture content versus radius (on the left) and moisture content versus time (on the right) profiles at $T = 10^oC$, $R_p = R_0 + at^n$

Figure 1 illustrates the moisture content profiles as a function of radius for various times (on the left) and moisture content profiles as a function of time for the various radial position (on the right) by using Crank-Nicolson method.

In Figure 2, moisture versus radius (on the left) and moisture versus time (on the right) profiles are given by using the splitting technique. In both of the calculations in Figures 1 and 2, grain volume variation which is represented by the radius versus time function given by Eq.(2.6).

Figure 3 and Figure 4 show the moisture content profiles as a function of radius for various times (left on) and moisture content profiles as a function of time for the various radial position (right on) by using Crank-Nicolson method and the splitting technique, respectively. In these calculations, Eq.(2.7) is used to represent volume variation.

As figures show the speed of convergence in the splitting technique is faster than Crank-Nicolson method.

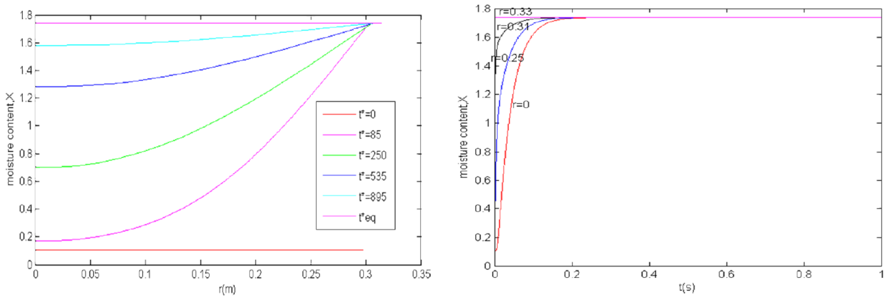

FIGURE 2. Moisture content versus radius (on the left) and moisture content versus time (on the right) profiles at $T = 10^oC$, $R_p = R_0 + at^n$

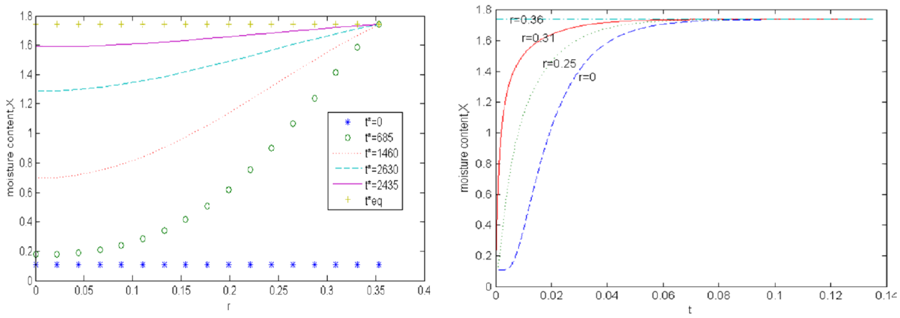

FIGURE 3. Moisture content versus radius (on the left) and moisture content versus time (on the right) profiles at $T = 10^oC$, $R_p = a + bexp(t^c)$

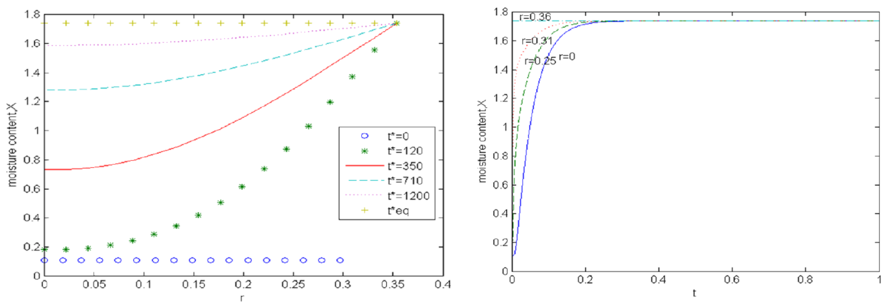

FIGURE 4. Moisture content versus radius (on the left) and moisture content versus time (on the right) profiles at $T = 10^oC$, $R_p = a + bexp(t^c)$

4. Conclusion

The number of division of radius is chosen as 60 in Nicolin et al.[**1**] and they observed that there is no significant difference among the profiles obtained for $50 \leqslant N \leqslant 70$. Although we use 20 spatial points, the same trend is obtained with Nicolin et al.[**1**]. This shows that our approach reduces the number of mesh points drastically and gives the good approximation.

For both methods, the maximum error is $1.444x10^{-8}$ is obtained. The speed of convergence in the splitting technique is faster than Crank- Nicolson method and when it is used Eq.(2.7) computational time is less.

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