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A NOTE ON AVERAGE DOMINATION AND AVERAGE INDEPENDENT DOMINATION NUMBERS IN GRAPHS

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Abstract. Henning[**6**] introduced the concept of average domination and average independent domination. The domination number $\gamma_v(G)$ of G relative to v is the minimum cardinality of a dominating set containing v. The average domination number of G is $\gamma_{av}(G) = \frac{1}{|V(G)|}\sum_{v\in V(G)} \gamma_v(G)$. The independent domination number $i_v(G)$ of G relative to v is the minimum cardinality of a maximal independent set in G that contains v. The average independent domination number of G is $i_{av}(G) = \frac{1}{|V(G)|}\sum_{v\in V(G)} i_v(G)$. In this note, we look at these parameters in a different point of view and hence simplify the results.

1. Introduction

Domination and its variations in graphs are well studied and the literature on this subject has been surveyed and detailed in the books [**4**], [**5**]. For notation and graph theory terminology we in general follow [**4**]. Specifically, let $G = (V,E)$ be a graph with a vertex set V of order n and edge set E of size m, and let v be a vertex in V. The open neighborhood of v is $N(v) = \{u \in V | uv \in E\}$ and the closed neighborhood of v is $N[v] = \{v\} \cup N(v)$. For a set $S \subseteq V$, its *open neighborhood* $N(S) = \cup_{v\in S} N(v)$ and its *closed neighborhood* $N[S] = N(S) \cup S$. A *leaf* is a vertex of degree one and its neighbor is called a *support vertex.* A vertex v is said to be a *full degree vertex* if $deg(v) = |V(G)| - 1$.

A subset $S \subseteq V$ of vertices is a *dominating set* of G if every vertex in $V - S$ is adjacent to at least one vertex of S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set in G. A subset $I \subseteq V$ of vertices is an *independent*

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set of G if no two vertices are adjacent in I. The *independence number* $\beta(G)$ of G is the maximum cardinality of an independent set in G, while the *independent domination number i(G)* of G is the minimum cardinality of maximal independent set of G.

DEFINITION 1.1. Let $\mu(G)$ be a numerical invariant of a graph G defined in such a way that it is the minimum or maximum number of vertices of a set $S \subseteq V$ with a given property P. A set with property P and with $\mu(G)$ vertices in G is called a μ - set of G. A vertex v of a graph G is defined to be a

(1) ([**3**]) μ - good vertex if it belongs to some μ - set of G;
(2) ([**3**]) μ - bad vertex if it belongs to no μ - set of G;
(3) ([**7**]) μ - fixed vertex if it belongs to every μ - set of G;
(4) ([**7**]) μ - free vertex if it belongs to some μ - set but not to all μ - sets of G.

DEFINITION 1.2. ([**8**]) A graph G is μ-excellent if every vertex of G belongs to some μ-set.

DEFINITION 1.3. The corona $H^+ = H \circ K_1$ is the graph constructed from a copy of H, where for each vertex $v \in V(H)$, a new vertex v' and a pendant edge vv' are added.

DEFINITION 1.4. The binomial tree of order $n \geqslant 0$ with root R is the tree B_n defined as follows.

(1) If $n = 0, B_n = B_0 = R$, i.e., the binomial tree of order zero consists of a single vertex R.
(2) If $n > 0, B_n = R, B_0, B_1, \cdots, B_{n-1}$, i.e., the binomial tree of order $n > 0$ comprises of the root R, and n binomial sub trees, $B_0, B_1, \cdots, B_{n-1}$.

NOTE 1.1. *From above definition,* $B_n = B_{n-1}^+$.

DEFINITION 1.5. Let $p_1, p_2, \cdots, p_n$ be non-negative integers and G be a graph with $|V(G)| = n$. The thorn graph of a graph, with parameters $p_1, p_2, \cdots, p_n$, is obtained by attaching p_i new vertices of degree 1 to the vertex u_i of the graph G, $i = 1, 2, \cdots, n$. The thorn graph of the graph G will be denoted by G^* or by $G^*(p_1, p_2, \cdots, p_n)$.

DEFINITION 1.6. A complete k-ary tree with depth n is, all leaves with the same depth and all internal vertices have exactly k children. A complete k-ary tree has $\dfrac{k^{n+1}-1}{k-1}$ vertices and $\dfrac{k^{n+1}-1}{k-1} - 1$ edges.

2. Average Domination Number

For a vertex v of G, the domination number $\gamma_v(G)$ of G relative to v is defined as the minimum cardinality of a dominating set containing v. The average domination number of G is

$$\gamma_{av}(G) = \frac{1}{|V(G)|} \sum_{v \in V(G)} \gamma_v(G).$$

We call a dominating set of cardinality $\gamma_v(G)$ containing v a γ_v - set. Let $S \subseteq V$ be a γ - set of G. Then for every vertex $v \in V(G)$, either $v \in D$ or $v \notin D$. If $v \in D$, then $\gamma_v = \gamma$, if $v \notin D$, then $\gamma_v = \gamma + 1$. Therefore, $\gamma \leqslant \gamma_v \leqslant \gamma + 1$ for all $v \in V(G)$ and thus $\gamma \leqslant \gamma_{av} \leqslant \gamma + 1$.

Definition 2.1. Let G be a graph. Let V_g be the set of all γ-good vertices in G and V_b be the set of all γ-bad vertices. Then, $|V_b| = |V| - |V_g|$. Let $g_\gamma = |V_g|$ and $b_\gamma = |V_b|$. Therefore, $g_\gamma = n - b_\gamma$.

Proposition 2.1. *For any graph G,* $\gamma_{av}(G) = \gamma(G) + \dfrac{b_\gamma(G)}{n}$.

Proof. By definition of $\gamma_{av}(G)$,

$$\gamma_{av}(G) = \frac{1}{|V|}\left(\textstyle\sum_{v \in V(G)} \gamma_v\right) = \frac{1}{n}\left(\sum_{v \in V_g} \gamma_v + \sum_{v \in V_b} \gamma_v\right) = \frac{1}{n}\Big(\gamma|V_g| + (\gamma+1)|V_b|\Big) = \frac{1}{n}(n\gamma + b_\gamma) = \gamma(G) + \frac{b_\gamma}{n}.$$

□

Corollary 2.1. $\gamma \leqslant \gamma_{av} < \gamma + 1$.

Corollary 2.2. *G is γ-excellent iff* $\gamma_{av} = \gamma$.

Corollary 2.3. $\gamma_{av}(C_n) = \gamma(C_n) = \left\lceil \dfrac{n}{3} \right\rceil$, $\gamma_{av}(K_n) = \gamma(K_n) = 1$, $\gamma_{av}(K_{m,n}) = \gamma(K_{m,n}) = 2$, *for* $m, n \geqslant 2$.

Corollary 2.4. *If* $G = H^+$, *then* $\gamma_{av}(G) = \gamma(G) = |V(H)|$, *for some* H.

Corollary 2.5 ([**1**]). $\gamma_{av}(B_n) = \gamma(B_n) = 2^{n-1}$.

Proof. Since $B_n = B^+_{n-1}$, $\gamma_{av}(B_n) = \gamma(B_n) = 2^{n-1}$. □

Corollary 2.6 ([**6**]). $\gamma_{av} \leqslant \gamma + 1 - \dfrac{\gamma}{n}$, *with equality iff G has a unique γ - set.*

Proposition 2.2. $$b_\gamma(P_n) = \begin{cases} \dfrac{2n}{3} & \text{if } n \equiv 0 \ (mod\ 3) \\ 0 & \text{if } n \equiv 1 \ (mod\ 3) \\ \dfrac{n-2}{3} & \text{if } n \equiv 2 \ (mod\ 3). \end{cases}$$

Corollary 2.7 ([**6**]). $$\gamma_{av}(P_n) = \begin{cases} \dfrac{n+2}{3} - \dfrac{2}{3n} & \text{if } n \equiv 2 \ (mod\ 3) \\ \dfrac{n+2}{3} & \text{otherwise.} \end{cases}$$

Proposition 2.3. *Let G be a connected graph of order n and G^* be the thorn graph of G. Then*

$$b_\gamma(G^*) = \begin{cases} 0 & \text{if } p_i = 1 \ \text{ for all } \ i \\ \sum_i p_i & \text{if } p_i > 1 \ \text{ for all } \ i. \end{cases}$$

Proof. For $p_i = 1$ for all i, G^* is γ-excellent and hence $b_\gamma(G^*) = 0$. For $p_i > 1$ for all i, $b_\gamma(G^*) = \sum_i p_i$. □

Corollary 2.8 ([**1**]).

$$\gamma_{av}(G^*) = \begin{cases} n & \text{if } p_i = 1 \text{ for all } i \\ n + 1 - \dfrac{n}{n + \sum_i p_i} & \text{if } p_i > 1 \text{ for all } i. \end{cases}$$

Proposition 2.4. *Let G be a complete k-ary tree with depth n. Then*

$$\gamma(G) = \begin{cases} \dfrac{k^2(k^n - 1)}{k^3 - 1} + 1 & n \equiv 0 (mod\ 3) \\ \\ \dfrac{k^{n+2} - 1}{k^3 - 1} & n \equiv 1 (mod\ 3) \\ \\ \dfrac{k(k^{n+1} - 1)}{k^3 - 1} & n \equiv 2 (mod\ 3). \end{cases}$$

Proposition 2.5. *Let G be the complete k-ary tree with depth n. Then*

$$b_\gamma(G) = \begin{cases} |V(G)| - \gamma(G) - k & \text{if } n \equiv 0 (mod\ 3) \\ |V(G)| - \gamma(G) & \text{otherwise.} \end{cases}$$

Proof. Let G be the complete *k-ary* tree with depth n. Then all the vertices on the levels $n - 1 - 3i \leqslant 3$, $0 \leqslant i \leqslant \dfrac{n-4}{3}$, are γ-fixed vertices.

If $n \equiv 1(mod\ 3)$, then the root vertex is γ-fixed. If $n \equiv 2(mod\ 3)$, then the vertices at level one are γ-fixed. Therefore, γ-set is unique and hence $b_\gamma(G) = |V(G)| - \gamma(G)$.

If $n \equiv 0(mod\ 3)$, then the root vertex and the vertices at level one are γ-free vertices and thus $b_\gamma(G) = |V(G)| - \gamma(G) - k$. □

Corollary 2.9 ([**1**]).

$$\gamma_{av}(G) = \begin{cases} \gamma(G) + 1 - \dfrac{\gamma(G) + k}{|V(G)|} & \text{if } n \equiv 0(mod\ 3) \\ \gamma(G) + 1 - \dfrac{\gamma}{|V(G)|} & \text{otherwise.} \end{cases}$$

Proposition 2.6. *Let G be a graph with full-degree vertices. Then, $b_\gamma = n - r$, where r denotes the number of full degree vertices.*

Corollary 2.10. *Let G be a graph with r full-degree vertices. Then, $\gamma_{av}(G) = 2 - \dfrac{r}{n}$.*

Corollary 2.11 ([**1**]). *Let r_1, r_2 be the number of full-degree vertices of two graphs G_1, G_2 respectively. Then, $\gamma_{av}(G_1 + G_2) = 2 - \dfrac{r_1 + r_2}{n_1 + n_2}$, where $n_1 = |V(G_1)|, n_2 = |V(G_2)|$.*

COROLLARY 2.12 ([**1**]). *Let G_1 and G_2 be two graphs without full-degree vertex. Then,* $\gamma_{av}(G_1 + G_2) = 2$.

3. Average Domination Number of Some Middle Graphs

DEFINITION 3.1. The middle graph M(G) is the graph obtained from G by inserting a new vertex into every edge of G and by joining edges those pairs of these new vertices which lie on adjacent edges of G. The new inserted vertices are called middle vertices. If G is a (n,m)-graph, then $|V(M(G))| = n + m$.

PROPOSITION 3.1. $\gamma(M(P_n)) = \left\lceil \frac{n}{2} \right\rceil$ *and* γ*-set is unique when* n *is even.*

PROPOSITION 3.2. $\gamma(M(C_n)) = \left\lceil \frac{n}{2} \right\rceil$.

PROPOSITION 3.3. $\gamma(M(K_n)) = \left\lceil \frac{n}{2} \right\rceil$.

PROPOSITION 3.4. $\gamma(M(W_n)) = \left\lceil \frac{n}{2} \right\rceil$.

PROPOSITION 3.5. $\gamma(M(K_{1,n})) = n$.

PROPOSITION 3.6. *Let* $M(C_n)$ *be the middle graph of* C_n. *Then*

$$b_\gamma(M(C_n)) = \begin{cases} n & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

PROOF. When n is even, the n middle vertices of $M(C_n)$ are γ-free vertices and the remaining n vertices are γ-bad. When n is odd, we need $\frac{n-1}{2}$ middle vertices and 1 non middle vertex to dominate $V(M(C_n))$ and hence the vertices in each γ-set are γ-free. Therefore, $b_\gamma(M(C_n)) = 0$, when n is odd. □

COROLLARY 3.1 ([**2**]). $\gamma_{av}(M(C_n)) = \frac{n+1}{2}$.

PROPOSITION 3.7. *Let* $M(P_n)$ *be the middle graph of* P_n. *Then*

$$b_\gamma(M(P_n)) = \begin{cases} \frac{3n-2}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

PROOF. Let $P_n : v_1, v_2, \cdots, v_n$ be a path. When n is even, γ-set is unique and hence $b_\gamma(M(P_n)) = |V(M(P_n))| - \gamma(M(P_n)) = \frac{3n-2}{2}$. When n is odd, the vertices $v_2, v_4, \cdots, v_{n-1}$ are γ - bad vertices of $M(P_n)$. Therefore, $b_\gamma(M(P_n)) = \frac{n-1}{2}$. □

COROLLARY 3.2 ([**2**]). $\gamma_{av}(M(P_n)) = \frac{n^2+n-1}{2n-1}$.

PROPOSITION 3.8. *Let $M(W_n)$ be the middle graph of W_n. Then*

$$b_\gamma(M(W_n)) = \begin{cases} n & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

PROOF. When n is even, the $2(n$ - $1)$ middle vertices of $M(W_n)$ are γ-free vertices and the remaining n vertices are γ-bad. When n is odd, $\dfrac{n-1}{2}$ middle vertices of $M(W_n)$ dominates all but one non middle vertex. Therefore all the vertices are γ-free. Hence, $b_\gamma(M(W_n)) = 0$. □

COROLLARY 3.3 ([**2**]). $\gamma_{av}(M(W_n)) = \begin{cases} \dfrac{3n^2}{6n-4} & \text{if } n \text{ is even} \\ \dfrac{n+1}{2} & \text{if } n \text{ is odd} \end{cases}$

PROPOSITION 3.9. *Let $M(K_n)$ be the middle graph of K_n. Then*

$$b_\gamma(M(K_n)) = \begin{cases} n & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

PROOF. When n is even, the middle vertices on the perfect matching of K_n are the γ - sets of K_n. Therefore, the n vertices in K_n are γ-bad vertices of $M(K_n)$. When n is odd, all the vertices are γ-free. Therefore, $b_\gamma(M(K_n)) = 0$, when n is odd. □

COROLLARY 3.4 ([**2**]). $\gamma_{av}(M(K_n)) = \begin{cases} \dfrac{n^2+n+4}{2n+2} & \text{if } n \text{ is even} \\ \dfrac{n+1}{2} & \text{if } n \text{ is odd.} \end{cases}$

PROPOSITION 3.10. *Let $M(K_{1,n})$ be the middle graph of $K_{1,n}$. Then*

$$b_\gamma(M(K_{1,n})) = 1.$$

PROOF. Since the non pendant vertex of $K_{1,n}$ is the only γ-bad vertex of $M(K_{1,n})$, $b_\gamma(M(K_{1,n})) = 1$. □

COROLLARY 3.5 ([**2**]). *Let $M(K_{1,n})$ be the middle graph of $K_{1,n}$. Then*

$$\gamma_{av}(M(K_{1,n})) = \frac{2n^2+n+1}{2n+1}.$$

4. Average independent domination in graphs

For a vertex v of G, the independent domination number $i_v(G)$ of G relative to v is the minimum cardinality of a maximal independent set in G that contains v. The average independent domination number of G is

$$i_{av}(G) = \frac{1}{|V(G)|} \sum_{v \in V(G)} i_v(G).$$

DEFINITION 4.1. Let G be a graph. Let V_{g_i} and V_{b_i} be the set of all i-good vertices and the set of all i-bad vertices in G respectively. Let $b_i = |V_{b_i}|$.

THEOREM 4.1. $i_{av} \leqslant i + \dfrac{(\beta - i)b_i}{n}$.

PROOF. By definition,

$$i_{av} = \frac{1}{|V(G)|} \sum_{v \in V} i_v = \frac{1}{|V(G)|} \left(\sum_{v \in V_{g_i}} i_v + \sum_{v \in V_{b_i}} i_v \right) \leqslant \frac{1}{n}\Big(i|V_{g_i}| + \beta |V_{b_i}| \Big) =$$
$$\frac{1}{n}\Big(i(n - |V_{b_i}|) + \beta |V_{b_i}| \Big) = i + \frac{(\beta - i)b_i}{n}.$$ □

COROLLARY 4.1. $i_{av} \leqslant \beta - \dfrac{i(\beta - i)}{n}$.

PROOF. Since, $b_i \leqslant n - i$, the result follows. □

THEOREM 4.2 ([**6**]). $i_{av} = \beta - \dfrac{i(\beta - i)}{n}$ *iff G is well-covered or G has a unique i-set and for each vertex not in $i(G)$ - set, every maximal independent set containing it has cardinality $\beta(G)$.*

PROOF. Let $i_{av} = \beta - \dfrac{i(\beta - i)}{n}$. Suppose i - set is not unique. Then there exists a vertex $w \in V_{b_i}$ such that $i_w = i$ and so $n\ i_{av} \leqslant i(n - b_i) + \beta(b_i - 1) + i$. Therefore, $i_{av} < \beta - \dfrac{i(\beta - i)}{n}$. If some vertex $w' \in V_{b_i}$ such that v belongs to a maximal independent set of cardinality less than β, then $n\ i_{av} \leqslant i(n - b_i) + \beta(b_i - 1) + (\beta - 1)$. Therefore, $i_{av} < \beta - \dfrac{i(\beta - i)}{n}$. Hence, if $i_{av} = \beta - \dfrac{i(\beta - i)}{n}$, then G must have a unique $i(G)$ - set and each vertex not in $i(G)$ - set is such that every maximal independent set containing it has cardinality $\beta(G)$. Conversely if G is well-covered or G has a unique i-set and for each vertex not in $i(G)$ - set, every maximal independent set containing it has cardinality $\beta(G)$, then it is easy to verify $i_{av} = \beta - \dfrac{i(\beta - i)}{n}$. □

COROLLARY 4.2. $i_{av}(T) \leqslant n - 2 + \dfrac{2}{n}$.

5. (γ_{av}, i_{av})-graphs

THEOREM 5.1. *Let G be a graph. Then, $\gamma_{av}(G) = i_{av}(G)$ iff $\gamma = i$, $b_\gamma = b_i$ and $i_v = i + 1$ for all i - bad vertices.*

PROOF. Let $\gamma_{av} = i_{av}$. Since, $\gamma_v \leqslant i_v$ for all v and $\sum_v \gamma_v = \sum_v i_v$, $\gamma_v = i_v$ for all v. Since $\gamma = \gamma_{v_1}$ for some v_1, $\gamma = \gamma_{v_1} = i_{v_1} \geqslant i$. Since $\gamma \leqslant i$, $\gamma = i$. We have $\gamma_{av} = \gamma + \dfrac{b_\gamma}{n}$ and $i_{av} = i + \dfrac{\sum_{v \in V_{b_i}} (i_v - i)}{n}$. Since $\gamma_{av}(G) = i_{av}(G)$ and $\gamma = i$, $b_\gamma = \sum_{v \in V_{b_i}} (i_v - i)$. Since $\gamma = i$, cardinality of every i-good vertex is a γ-good vertex. Therefore, $|V_{g_i}| \leqslant |V_g|$ and hence $|V_{b_i}| \geqslant |V_b| = b_\gamma$. Since $i_v - i \geqslant 1$ for all $v \in V_{b_i}$, $b_\gamma = \sum_{v \in V_{b_i}} (i_v - i) \geqslant \sum_{v \in V_{b_i}} 1 = |V_{b_i}|$. Thus, $b_\gamma = |V_{b_i}|$. Now, $V_{b_i} = \phi$ iff $V_b = \phi$. Let $V_{b_i} \neq \phi$. Suppose $i_{v_1} - i \geqslant 2$, for some $v_1 \in V_{b_i}$ then,

$b_\gamma = \sum_{v \in V_{b_i}} (i_v - i) \geqslant 2 + \sum_{v \in V_{b_i} - \{v_1\}} (i_v - 1) \geqslant 2 + |V_{b_i}| - 1 = b_\gamma + 1$, which is a contradiction. Therefore, $i_v - i \leqslant 1$ for all $v \in V_{b_i}$. Hence $i_v = i + 1$ for all $v \in V_{b_i}$. It is easy to verify the converse. □

References

1. E. Aslan, A. Kirlagic. The average lower domination number of graphs. *Bull. Int. Math. Virtual Inst.*, **3**(2)(2013), 155 – 160.
2. D. Dogan. Average lower domination number for some middle graphs. *International J. Math. Combin.*, **4**(2012), 58 – 67.
3. G. H. Fricke, T. W. Haynes, S. M. Hedetniemi, S. T. Hedetniemi and R. C. Laskar. Excellent trees. *Bull. Inst. Comb. Appl.*, **34**(2002), 27 – 38.
4. T. W. Haynes, S. T. Hedetniemi, P. J. Slater, *F*undamentals of Domination in Graphs. Marcel Dekker, New York 1998.
5. T. W. Haynes, S. T. Hedetniemi, P. J. Slater. *D*omination in graphs : Advanced Topics. Marcel Dekker, New York 1998.
6. M. A. Henning. Trees with equal average domination and independent domination numbers. *Ars. Combin.*, **71**(2004), 305 – 318.
7. E. Sampathkumar and P. S. Neeralagi. Domination and neighborhood critical fixed, free and totally free points. *Sankhya*, **54** (1992), 403 – 407.
8. M. Yamuna. *Excellent - Just Excellent - Very Excellent graphs.* Ph.D Thesis, Alagappa University, Karaikudi 2003.

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