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ON THE h-VECTORS OF CHESSBOARD COMPLEXES

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Abstract. We use a concrete shelling order of chessboard complexes $\Delta_{n,m}$ for $m \geqslant 2n-1$ to describe the type of each facet of $\Delta_{n,m}$ in this order. Further, we find some recursive relations for h-vector, describe the generating facets of shellable $\Delta_{n,m}$ and show that the number of generating facets of $\Delta_{n,m}$ is the value of a special Poisson-Charlier polynomial $p_n(m)$. Some of these results can be extended to chessboard complexes on triangular boards.

1. Introduction

The *chessboard complex* $\Delta_{n,m}$ is an abstract simplicial complex defined on $n\times m$ chessboard. Its vertices are mn squares in this chessboard and $(k-1)$-dimensional faces of $\Delta_{n,m}$ are all configurations of k non-taking rooks on an $n \times m$ chessboard. We label the squares of $n\times m$ table by (i,j), where i represents the rows (numbered top to bottom) while j represents the columns (numbered left to right).

The chessboard complex appears in many situations: as the matching complex of a complete bipartite graph, as a coset complex of certain subgroups of symmetric group $\mathbb{S}_n$, as a complex of injective functions, see in [**6**] for more details.

A simplicial complex Δ is *shellable* if it can be built up inductively in a nice way. To be more precise, its maximal faces (*facets*) can be ordered so that each one of them (except for the first one) intersects the union of its predecessors in a non-empty union of maximal proper faces.

For more information about simplicial complexes, shellability and topological concept we refer the reader to [**1**], [**6**] and [**7**]. Very often the following definition of shelling is useful, see in [**11**].

Definition 1.1. A simplicial complex Δ is shellable if Δ is pure and there exists a linear ordering (*shelling order*) $F_1, F_2, \ldots, F_t$ of facets of Δ such that for

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all $i < j \leqslant t$, there exists some $l < j$ and a vertex v of F_j, such that

$$F_i \cap F_j \subseteq F_l \cap F_j = F_j \smallsetminus \{v\}.$$

For a fixed shelling order $F_1, F_2, \dots, F_t$ of Δ, the *restriction* $\mathcal{R}(F_j)$ of the facet F_j is defined by:

$$\mathcal{R}(F_j) = \{v \text{ is a vertex of } F_j : F_j \smallsetminus \{v\} \subset F_i \text{ for some } 1 \leqslant i < j\}.$$

Geometrically, if we build up Δ from its facets according to the shelling order, then $\mathcal{R}(F_j)$ is the unique minimal new face added at the j-th step. The *type* of the facet F_j in the given shelling order is the cardinality of $\mathcal{R}(F_j)$, that is, $type(F_j) = |\mathcal{R}(F_j)|$.

For a d-dimensional simplicial complex Δ we denote the number of i-dimensional faces of Δ by f_i, and call $f(\Delta) = (f_{-1}, f_0, f_1, \dots, f_d)$ the *f-vector*. The empty set is a face of every simplicial complex, so we have that $f_{-1} = 1$.

For example, the entries of f-vector of $\Delta_{n,m}$ for $m \geqslant n$ are

$$f_{i-1}(\Delta_{n,m}) = \binom{n}{i} \frac{m!}{(m-i)!}, \quad \text{for } i = 1, 2, \dots, n.$$

An important invariant of a simplicial complex Δ is the *h-vector* $h(\Delta) = (h_0, h_1, \dots, h_{d+1})$ defined by the formula

$$h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{d+1-i}{d+1-k} f_{i-1}.$$

If a simplicial complex Δ is shellable, then

$$h_k(\Delta) = |\{F \text{ is a facet of } \Delta : type(F) = k\}|$$

is a nice combinatorial interpretation of $h(\Delta)$.

The establishing of shellability of a simplicial complex gives us many information about algebraic, combinatorial and topological properties of this complex, see [**2**] or [**3**].

THEOREM 1.1. *If a d-dimensional simplicial complex Δ is shellable, then Δ is either contractible or homotopy equivalent to a wedge of h_{d+1} spheres of the dimension d.*

A set of maximal simplices from a simplicial complex Δ are *generating simplices* if the removal of their interiors makes Δ contractible.

For a given shelling order of a complex Δ we have that

$$\{F \in \Delta : F \text{ is a facet and } \mathcal{R}(F) = F\}$$

is a set of generating facets of Δ. A facet F is in this set if and only if

$$\forall v \in F \text{ there exists a facet } F' \text{ before } F \text{ such that } F \cap F' = F \smallsetminus \{v\}.$$

G. Ziegler in [**10**] proved that chessboard complexes $\Delta_{n,m}$ are vertex decomposable for $m \geqslant 2n - 1$. As vertex-decomposability is a stronger property than shellability (see in [**1**]), he established that these complexes are shellable.

2. Recursive relations for h-vector

G. Ziegler noted in [**10**] that the natural lexicographical order of facets of $\Delta_{n,m}$ is not a shelling order. A concrete linear order of the facets that is a shelling of $\Delta_{n,m}$ (for $m \geqslant 2n-1$) can be found in [**5**]. Here we restate this shelling order inductively and describe the type of each facet of $\Delta_{n,m}$ in this order.

REMARK 2.1. First, we note that $\Delta_{1,m}$ is a shellable 0-dimensional complex for all $m \in \mathbb{N}$. Assuming that the complexes $\Delta_{k,r}$ are shellable (whenever $k < n$ and $r \geqslant 2k-1$), we describe a shelling order for $\Delta_{n,m}$ (for $m \geqslant 2m-1$). The facets of $\Delta_{n,m}$ are ordered by the following criterias:

(1) The position of the rook in the first row.

Note that each facet of $\Delta_{n,m}$ contains exactly one rook in the first row. Our shelling order starts with the facets of $\Delta_{n,m}$ having a rook at the position $(1,1)$, then follow up the facets with a rook at the position $(1,2)$, and so on up to the facets that contain a rook at $(1,m)$.

All facets of $\Delta_{n,m}$ that contain a rook at $(1,1)$ span a subcomplex that is isomorphic to $\Delta_{n-1,m-1}$, which is shellable by inductive assumption. We use the assumed shelling order of $\Delta_{n-1,m-1}$ to define the linear order of the facets of $\Delta_{n,m}$ containing a rook at $(1,1)$.

To order the facets of $\Delta_{n,m}$ that have rook at $(1,i)$ for $i > 1$ we consider:

(2) The number of occupied columns immediately before the i-th column.

The shelling order of the facets containing the rook at $(1,i)$ starts with facets that do not contain a rook in the column $(i-1)$. These facets span a subcomplex of $\Delta_{n,m}$ that is isomorphic to $\Delta_{n-1,m-2}$. By the inductive assumption this subcomplex is shellable. We order the above described facets of $\Delta_{n,m}$ in the same way as their corresponding facets are ordered in the assumed shelling of $\Delta_{n-1,m-2}$.

The order of the facets of $\Delta_{n,m}$ that contain a rook at the position $(1,i)$ continues with the facets that have a rook in the column $i-1$ but not in the column $i-2$. Note that the subcomplex of $\Delta_{n,m}$ spanned by the facets that contain the rooks at $(1,i)$ and $(j,i-1)$ (for a fixed $j > 1$), but do not contain a rook in the column $i-2$ is isomorphic to $\Delta_{n-2,m-3}$ (we just delete two rows and three columns). Again, we use the assumption of shellability of $\Delta_{n-2,m-3}$ to define the order of corresponding facets of $\Delta_{n,m}$.

Our shelling order of the facets containing a rook at $(1,i)$ continues further in the same manner. The facets that have a rook at $(1,i)$, contain the rooks in the columns $i-1,\dots,i-k+1$ (at fixed positions), but not in the column $i-k$ (here we assume that $k < i$) span the subcomplex of $\Delta_{n,m}$ isomorphic to $\Delta_{n-k,m-k-1}$. For a fixed configuration of the rooks in the columns $i-1,\dots,i-k+1$ (there are $(n-1)\cdots(n-k+1)$ of such configurations) the shelling order for $\Delta_{n-k,m-k-1}$ defines the order of corresponding facets of $\Delta_{n,m}$.

Note that for a fixed i, $1 < i < n$, the part of the shelling of $\Delta_{n,m}$ that lists the facets containing $(1,i)$ ends with the facets containing the rooks in columns $1,2,\dots,i-1$. There are $\frac{(n-1)!}{(n-i)!}$ ways to distribute the rooks in the first $i-1$ columns. For a fixed distribution of the rooks in the first i column, all of these facets span a subcomplex isomorphic to shellable complex $\Delta_{n-i,m-i}$.

The order of the facets that contain a rook at $(1,n)$ finishes with $(n-1)!$ facets containing the rooks in each of the first $n-1$ columns. A similar situation is with the facets that contain the rook at $(1,i)$ for $m \geqslant i \geqslant n$.

The rigorous proof that the above defined linear order is a shelling can be found in [**5**] (see Theorem 4.4).

Remark 2.2. Now, we use the above defined shelling to determine the type of a given facet and to discuss when a facet of $\Delta_{n,m}$ is a generating facets.

(i) Assume that a facet F containing a rook at the position $(1,i)$ also contains the rooks at each of the first $i-1$ columns. This is possible for $i \leqslant n$.

In that case we have $F = S \cup T$, where $S = \{(a_1,1),\dots,(a_{i-1},i-1),(1,i)\}$ and T is a facet of $\Delta_{n-i,m-i}$. Note that F cannot be a generating facet, because $F \setminus \{(1,i)\}$ is not contained in any of facets that precede F in the shelling order defined in Remark 2.1. Further, for any j such that $1 \leqslant j < i$ the vertex (a_j,j) belongs to the restriction of F. There is an empty column after the i-th column. If we assume that the p-th column is empty for $p > i$, then $F' = F \setminus \{(a_j,j)\} \cup \{(a_j,p)\} \supset F \setminus \{(a_j,j)\}$, and F' precedes F in the described shelling order. Therefore, we obtain that

$$type(F) = |S| - 1 + type(T).$$

(ii) If a facet F contains the rooks at the squares $(1,i)$, $(a_{i-1},i-1)$,…, $(a_{i-k+1},i-k+1)$ and F does not have a rook in the column $i-k$ for $k < i$, then $F = S \cup T$. Here $S = \{(1,i),(a_{i-1},i-1),\dots,(a_{i-k+1},i-k+1)\}$ and T is a facet of $\Delta_{n-k,m-k-1}$.

In this situation, any of the rooks from a square contained in S can be moved to the empty column $i-k$, and therefore we obtain

$$type(F) = |S| + type(T).$$

Note that F is a generating facet of $\Delta_{n,m}$ if and only if T is a generating facet of $\Delta_{n-k,m-k-1}$. Further, if $i > n$, then any facet $F = \{(1,i)\} \cup T$ (here T is a facet of $\Delta_{n-1,m-1}$) is a generating facet of $\Delta_{n,m}$ if and only if T is a generating facet for $\Delta_{n-1,m-1}$.

Now, we describe some recursive relations for the entries of h-vector $\Delta_{n,m}$.

Theorem 2.1. *For fixed $n,m \in \mathbb{N}$, $m \geqslant 2n-1$, and for all $k = 1,2,\dots,n-1$ we have that*

$$h_k(\Delta_{n,m}) = \sum_{i=1}^{k} \frac{(n-1)!}{(n-i)!}\Big[h_{k+1-i}(\Delta_{n-i,m-i}) + (k+1-i)h_{k-i}(\Delta_{n-i,m-i-1})\Big] + \\ + \frac{(n-1)!}{(n-k-1)!} + (m-k-1)h_{k-1}(\Delta_{n-1,m-1}).$$

PROOF. Assume that k is fixed. For $i = 1, 2, \dots, k+1$ there are

$$\frac{(n-1)!}{(n-i)!} h_{k+1-i}(\Delta_{n-i,m-i})$$

facets of $\Delta_{n,m}$ of the type k described in (i) of Remark 2.2. Therefore, we obtain that for $i = k+1$ there are $\frac{(n-1)!}{(n-k-1)!}$ of these facets.

Also, for $i = 1, 2, \dots, k$ there are

$$(k+1-i)\frac{(n-1)!}{(n-i)!} h_{k-i}(\Delta_{n-i,m-i-1})$$

facets of the type k described in (ii) of Remark 2.2 (the rook at the first row is at the position $(i, 1)$ for $1 < i < n$, and there is an empty column before).

When a facet contains a rook at the position $(1, i)$ for $i > k+1$, then $(1, i)$ belongs to the restriction of this facet. The number of these facets of type k is

$$(m-k-1)h_{k-1}(\Delta_{n-1,m-1}).$$

By adding all possibilities we get the formula for $h_k(\Delta_{n,m})$.

□

It follows from Theorem 1.1 that for $x \geqslant 2n-1$ the complex $\Delta_{n,x}$ is homotopy equivalent to a wedge of $(n-1)$-dimensional spheres. We let $p_n(x) = h_n(\Delta_{n,x})$ to denote the number of these spheres, i. e., $p_n(x)$ counts the number of generating facets of $\Delta_{n,x}$. Note that $p_n(x)$ (for a fixed $n \in \mathbb{N}$ and variable $x \in \mathbb{N}, x \geqslant 2n-1$) is a function $p_n : \mathbb{N}_{\geqslant 2n-1} \to \mathbb{N}$. It is well-known fact that $p_n(x)$ is the reduced Euler characteristic:

$$p_n(\Delta_{n,x}) = \widetilde{\chi}(\Delta_{n,x}) = \sum_{i=0}^{n}(-1)^{n-i} f_{i-1}(\Delta_{n,x}) = \sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i} x^{(i)},$$

where $x^{(i)} = x(x-1)\cdots(x-i+1)$ and $x^{(0)} = 1$. Therefore, the functions $p_n(x)$ coincides with special Poisson-Charlier polynomials, see Chapter 10 in [**8**].

The first few polynomials $p_n(x)$ are:

$$p_1(x) = x-1, p_2(x) = x^2-3x+1, p_3(x) = x^3-6x^2+8x-1,$$
$$p_4(x) = x^4-10x^3+29x^2-24x+1, p_5(x) = x^5-15x^4+75x^3-145x^2+89x-1,$$
$$p_6(x) = x^6-21x^5+160x^4-545x^3+814x^2-415x+1,$$
$$p_7(x) = x^7-28x^6+301x^5-1575x^4+4179x^3-5243x^2+2372x-1.$$

There are some well-known recursive relations for special Poisson-Charlier polynomials, see [**8**] and [**9**]:

$$p_n(x) = xp_{n-1}(x-1) - p_{n-1}(x),\ p_n(x) = p_n(x-1) + np_{n-1}(x-1). \tag{2.1}$$

If we interpret $p_n(x)$ as the reduced Euler characteristic $\widetilde{\chi}(\Delta_{n,x})$, both of these relations follow from recursive relations for f-vector of $\Delta_{n,x}$:

$$f_i(\Delta_{n,x}) = xf_{i-1}(\Delta_{n-1,x-1}) + f_i(\Delta_{n-1,x}) = f_i(\Delta_{n,x-1}) + nf_{i-1}(\Delta_{n-1,x-1}).$$

The second relation in (2.1) also follows easily from Remark 2.2. There are $p_n(x-1)$ generating facets of $\Delta_{n,x}$ in which the last column is empty, and $np_{n-1}(x-1)$ generating facets that contain a rook in the last column.

Some other recursive relations for $p_n(x) = h_n(\Delta_{n,x})$ can be obtained in a similar way as in Theorem 2.1. We list these relations without proof.

Proposition 2.1. *For all $n, x \in \mathbb{N}$ such that $x \geqslant 2n-1$ the numbers $p_n(x)$ satisfy the following recursive relations*

$$p_n(x) = (x-n)p_{n-1}(x-1) + \sum_{k=1}^{n-1} \frac{(n-1)!}{(n-k-1)!} p_{n-k}(x-k-1),$$

$$p_n(x) = (x-n)(n-1)! + \sum_{k=1}^{n-1} \frac{(n-1)!}{(n-k)!}(x-k)p_{n-k}(x-k-1),$$

$$p_{n-1}(x-1) = (n-1)! + \sum_{k=1}^{n-1} \frac{(n-1)!}{(n-k)!} p_{n-k}(x-k).$$

3. Chessboard complex on a triangular board

For given $a_1, \ldots, a_n \in \mathbb{N}$, the triangular board $\Psi_{a_n,\ldots,a_1}$ is defined in [**4**] as a left justified board with a_i rows of length i. E. Clark and M. Zeckner in [**4**] consider chessboard complex $\Sigma(\Psi_{a_n,\ldots,a_1})$ whose faces correspond with non-taking rooks configurations on this triangular board.

Theorem 3.1 (Theorem 3.1, [**4**]). *If $a_i \geqslant i$ for all $i = 1, \ldots, n$ then $\Sigma(\Psi_{a_n,\ldots,a_1})$ is vertex decomposable.*

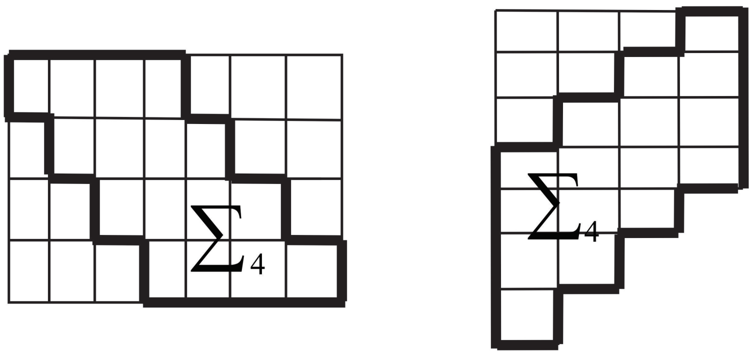

Figure 1. An admissible 4-shape in a triangular board $\Psi_{4,1,1,1}$

The admissible k-shape (Definition 3.1. in [**10**]) is a subset of $k \times (2k-1)$ chessboard:

$$\Sigma_k = \{(i,j) \in [k] \times [2k-1] : 0 \leqslant j - i \leqslant k-1\},$$

see the left side of Figure 1.

G. Ziegler proved (see Theorem 3.3. in [**10**]) that if a set of squares $A \subset \mathbb{Z}^2$ contains isomorphic copy of an admissible k-shape, then the $(k-1)$-skeleton of the chessboard complex on A is vertex decomposable. Note that the board $\Psi_{a_n,\dots,a_1}$ contains a transposed admissible n-shape (rotated for 90°, see Figure 1) if $a_n \geqslant n$ and $a_i \geqslant 1$ for $i = 1, \dots, n-1$. Therefore, we obtain that the complex $\Sigma(\Psi_{a_n,\dots,a_1})$ is vertex decomposable if $a_i \geqslant 1$ for all $i = 1, 2, \dots, n-1$, and $a_n \geqslant n$.

As every vertex decomposable complex is shellable, we obtain the following theorem.

Theorem 3.2. *Let $a_1 \geqslant 1$ for all $i = 1, 2, \dots, n-1$, and let $a_n \geqslant n$. Then the simplicial complex $\Sigma(\Psi_{a_n,\dots,a_1})$ is shellable.*

Remark 3.1. It is possible to consider the chessboard complex on the table $\Psi_{a_n,\dots,a_1}$ for $a_i \in \mathbb{N}_0$. Some examples of triangular chessboard complexes when $a_i = 0$ for some i were analyzed in [**4**]. For given $a_1, \dots, a_n \in \mathbb{N}_0$, the table $\Psi_{a_n,\dots,a_1}$ contains an admissible k-shape if and only if

$$a_i + a_{i+1} + \cdots + a_n \geqslant 2n - i, \text{ for all } i = 1, 2, \dots, n. \tag{3.1}$$

Therefore, if $a_1, a_2, \dots, a_n \in \mathbb{N}_0$ satisfy the above conditions the corresponding complex $\Sigma(\Psi_{a_n,\dots,a_1})$ is vertex decomposable.

Theorem 3.2 can be proved by a slight variation of the shelling order defined in Remark 2.1. Note that $c_i = a_i + a_{i+1} + \cdots + a_n$ in relation (3.1) is just the number of squares in the i-th column of $\Psi_{a_n,\dots,a_1}$. The table $\Psi_{a_n,\dots,a_1}$ is uniquely determined by the sequence $c_1, c_2, \dots, c_n$. Therefore, we can use $\Sigma_{c_1,c_2,\dots,c_n}$ instead $\Sigma(\Psi_{a_n,\dots,a_1})$ to denote the chessboard complex on $\Psi_{a_n,\dots,a_1}$.

Again, we define our shelling order of $\Sigma_{c_1,c_2,\dots,c_n}$ recursively. First, we consider

(1) The position of the rook in the last column.

Note that any facet of $\Sigma = \Sigma_{c_1,c_2,\dots,c_n}$ has to contain a rook at one of $c_n = a_n$ position in the last column. The shelling order of Σ begins with the facets that contain a rook at $(1, n)$, then follow the facets with a rook at $(2, n)$, and our linear order ends with the facets that contain a rook at (a_n, n). Note that all of facets of Σ that contain a rook at $(1, n)$ span a complex isomorphic to $\Sigma_{c_1-1,c_2-1,\dots,c_{n-1}-1}$. This complex is shellable, and we prescribe its shelling order (and add $(1, n)$ in each of its facets) to obtain the beginning part of our shelling.

To order the facets of Σ that have a rook at (i, n) (the fixed position in the n-th column) we consider

(2) The number of occupied rows that precede the i-th row.

The order of these facets starts with the facets of Σ that do not contain a rook in the row $i-1$. The subcomplex of Σ spanned by these facets is isomorphic to $(n-2)$-dimensional complex $\Sigma_1 = \Sigma(\Psi_{c_1-2,c_2-2\dots,c_{n-1}-2})$. Obviously, we have that $c_i - 2 \geqslant 2(n-1) - i$ and by the assumption Σ_1 is shellable. We will use this shelling order of Σ_1 to define the linear order of the corresponding facets of Σ.

For ordering the facets of Σ that contain the rooks at (i, n),$(i-1, s_1)$,…, $(i-k+1, s_{k-1})$ and the row $i-k$ is empty (here we assume $k < i$), we consider

$S = \{s_1, s_2, \dots, s_{k-1}\} \subseteq [n-1]$. Now, we let Σ_S to denote the subcomplex spanned with these facets. After we delete $k+1$ consecutive rows $i, i-1, \dots, i-k$ (the last deleted row is empty) and k columns labelled by $s_1, s_2, \dots, s_{k-1}, n$, we obtain that $\Sigma_S \cong \Sigma_{b_1-k-1, b_2-k-1, \dots, b_{n-k}-k-1}$, where

$$\{b_1, b_2, \dots, b_{n-k}\} = \{c_1, c_2, \dots, c_{n-1}\} \setminus \{c_{s_1}, \dots, c_{s_{k-1}}\}. \tag{3.2}$$

As we have that

$$b_i - k - 1 \geqslant c_{i+k-1} - k - 1 \geqslant 2n - (k+i-1) - k - 1 = 2(n-k) - i$$

by inductive assumption Σ_S is shellable. The facets of Σ with the rooks at fixed positions are ordered in our shelling order as their corresponding facets of Σ_S.

The facets of Σ that contain the rooks at (k, n),$(k-1, s_1)$,..., $(1, s_{i-1})$ (the first k rows are occupied) span a subcomplex $\Sigma_{\overline{S}}$. Note that $\Sigma_{\overline{S}} \cong \Sigma_{b_1-k, b_2-k, \dots, b_{n-k}-k}$, where b_i are defined as in (3.2).

A similar reasoning as for standard chessboard complexes gives us the recursive formula for h-vector of $\Sigma_{c_1, c_2, \dots, c_n}$ if $c_i \geqslant 2n - i$ for all $i = 1, 2, \dots, n$. For all $n > k > 0$ the entries of h-vector of $\Sigma = \Sigma_{c_1, c_2, \dots, c_n}$ satisfy

$$h_k(\Sigma) = \sum_{i=1}^{k} (i-1)! \sum_{S \subset [n-1], |S| = i-1} \big((k+1-i) h_{k-i}(\Sigma_S) + h_{k+1-i}(\Sigma_{\overline{S}})\big) +$$

$$+ \frac{(n-1)!}{(n-k-1)!} + (c_n - k - 1) h_{k-1}(\Sigma_{c_1-1, c_2-1, \dots, c_{n-1}-1}).$$

The Betti number of $\Sigma = \Sigma_{c_1, c_2, \dots, c_n}$ can be computed as

$$h_n(\Sigma) = \sum_{i=1}^{n} (n-i)(i-1)! \sum_{S \subset [n-1]} h_{n-i}(\Sigma_S) + (c_n - n) h_{n-1}(\Sigma_{c_1-1, \dots, c_{n-1}-1}).$$

The complexes Σ_S and $\Sigma_{\overline{S}}$ that appear in these formulas are above defined smaller triangular chessboard complexes.

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