BULLETIN OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4874, ISSN (o) 2303-4955
www.imvibl.org /JOURNALS / BULLETIN
Vol. 8(2018), 401–412
DOI: 10.7251/BIMVI1802401K

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

SCATTERING NUMBER AND CARTESIAN PRODUCT OF GRAPHS

Burak Kaval and Alpay Kirlangic

Abstract. In a communication network, the vulnerability is the resistance of the network to disruption of operation after the failure of certain stations or communication links. If a communication network was modelled by a graph, then the scattering number measures vulnerability of the graph. The scattering number of an arbitrary graph $G = (V, E)$ is defined to be $sc(G) = max\{\omega(G-S) - |S| : S \subseteq V(G)$ and $\omega(G - S) \neq 1\}$, where $\omega(G - S)$ denotes the number of connected components of $G - S$. In this paper the scattering number of graphs $K_{1,m} \times K_{1,n}, K_{1,m} \times P_n, K_{1,m} \times C_n$ and $K_2 \times C_n$ is obtained.

1. Introduction

A communication network consists of some centers and links which connect these centers. In a communication network, the vulnerability measures the resistance of the network to disruption of operation after the failure of certain stations or communication links. A communication network is modelled by a graph to measure the vulnerability as centers corresponding to the vertices of a graph and communication links corresponding to the edges of a graph. To measure vulnerability of a graph G, we have some parameters which are connectivity [**6**], toughness [**4**], scattering number [**7**], integrity [**1**]. In this paper, we discuss the scattering number of a graph. The scattering number of a graph was defined by Jung.

Definition 1.1. [**7**] The scattering number sc(G) is $sc(G) = max\{\omega(G - S) - |S| : S \subseteq V(G)$ and $\omega(G - S) \neq 1\}$, where $\omega(G - S)$ denotes the number of connected components of $G - S$.

2010 *Mathematics Subject Classification.* 05C40, 68M10, 68R10.

Key words and phrases. Graph Vulnerability, Scattering Number, Connectivity, Cartesian Product.

A scatter set is an S which achieves this maximum. The scattering number of a graph is closely related to the toughness of a graph. Jung calls the scattering number the additive dual of the toughness. Moreover this parameter can take on both positive and negative values. Note that the scattering number of a complete graph K_n is 2-n [**7**]. On the other hand, Zhang et al.[**9**] prove that the problem of computing the scattering number of a graph is NP-complete. Now we list the following some known results.

For a graph G, let $\alpha(G)$, $\beta(G)$, $\kappa(G)$, $\lambda(G)$ and $\delta(G)$ denote the independence number, covering number, connectivity, edge-connectivity and minimum degree of G, respectively.

THEOREM 1.1. [**8**] *Let G be a noncomplete connected graph of order n. Then*

$$2\alpha(G) - n \leqslant sc(G) \leqslant \alpha(G) - \kappa(G).$$

THEOREM 1.2. [**8**] *Let G be a noncomplete connected graph of order n ($n \geqslant 3$). Then*

$$\begin{aligned} &(a)\ \ 2-\kappa(G) \leqslant sc(G) \leqslant n - 2\kappa(G);\\ &(b)\ \ 2-\lambda(G) \leqslant sc(G) \leqslant n - 2\lambda(G);\\ &(c)\ \ 2-\delta(G) \leqslant sc(G) \leqslant n - 2\delta(G). \end{aligned}$$

THEOREM 1.3. [**8**] *Let G be a noncomplete connected graph of order n ($n \geqslant 4$) and the length of a longest path is p. Then*

$$sc(G) \leqslant n - p.$$

THEOREM 1.4. [**9**] *Let H be a spanning subgraph of a noncomplete connected graph G. Then*

$$sc(H) \geqslant sc(G).$$

DEFINITION 1.2. [**6**] To define the product $G_1 \times G_2$, consider any two points $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $V = V_1 \times V_2$. Then u and v are adjacent in $G_1 \times G_2$ whenever $[u_1 = v_1$ and u_2 adj $v_2]$ or $[u_2 = v_2$ and u_1 adj $v_1]$.

Next we give the following theorem between scattering number and cartesian product.

THEOREM 1.5. [**9**] *Suppose that $n_1, n_2, n_3, ..., n_k$ are k integers not less than 2. Then*

(1) $sc(P_{n_1} \times P_{n_2} \times P_{n_3} \times ... \times P_{n_k}) = 1$, *when all n_i are odd;*
(2) $sc(P_{n_1} \times P_{n_2} \times P_{n_3} \times ... \times P_{n_k}) = 0$, *when some n_i is even.*

To design of interconnection networks in multiprocessor computing systems, graphs as hypercubes, grids are used. These graphs are obtained by using cartesian product. Consequently, these considerations motivated us to investigate the scattering number of some graphs which are obtained by using cartesian product.

We use Bondy and Murty [**3**] for terminology and notation not defined here and consider only finite, connected and undirected graphs.

2. Scattering Number and Cartesian Product

In this chapter we consider the scattering number of cartesian product of two complete bipartite graphs.

2.1. Scattering Number of $K_{1,m} \times K_{1,n}$.

Firstly we start with a well known Lemma.

LEMMA 2.1. *Let $m, n \in Z^+$ ($m \geqslant 2$, $n \geqslant 2$) and $m \leqslant n$. Then $\alpha(K_{1,m} \times K_{1,n}) = mn + 1$ and $\beta(K_{1,m} \times K_{1,n}) = m + n$.*

THEOREM 2.1. *Let $m, n \in Z^+$ ($m \geqslant 2$, $n \geqslant 2$) and $m \leqslant n$. Then*

$$sc(K_{1,m} \times K_{1,n}) = \alpha(K_{1,m} \times K_{1,n}) - \beta(K_{1,m} \times K_{1,n}) = mn + 1 - (m + n).$$

PROOF. By Theorem 1.1

$$sc(K_{1,m} \times K_{1,n}) \geqslant \alpha(K_{1,m} \times K_{1,n}) - \beta(K_{1,m} \times K_{1,n}) = mn + 1 - (m + n). \tag{2.1}$$

Now we prove that $sc(K_{1,m} \times K_{1,n}) \leqslant \alpha(K_{1,m} \times K_{1,n}) - \beta(K_{1,m} \times K_{1,n})$.

Let A_α be independent set of $K_{1,m} \times K_{1,n}$ and B_β be covering set of $K_{1,m} \times K_{1,n}$. Let vertices of $K_{1,m} \times K_{1,n}$ be $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$ (Figure 1).

If we remove $|S|$=r vertices where $S = \{X \cup Y | X \subseteq A_\alpha$ or/and $Y \subseteq B_\beta\}$, then we have three cases.

Case 1: Let $1 \leqslant |S| = r \leqslant \alpha(K_{1,m} \times K_{1,n})$ and $S \subseteq X$.

- If we remove some/all vertices of A_2, then remaining graph is connected.
- If we remove single vertex of A_1, then remaining graph is connected.
- If we remove both single vertex of A_1 and some/all vertices of A_2 in one copy of $K_{1,m}$ (or $K_{1,n}$), then the remaining graph is disconnected while $m+1 \leqslant r \leqslant \alpha(K_{1,m} \times K_{1,n})$ and so $\omega((K_{1,m} \times K_{1,n}) - S) \leqslant m+n$. Thus

$$sc(K_{1,m} \times K_{1,n}) \leqslant max\{m+n-(m+1)\} = n-1 \leqslant \alpha(K_{1,m} \times K_{1,n}) - \beta(K_{1,m} \times K_{1,n}). \tag{2.2}$$

Case 2: Let $1 \leqslant |S| = r \leqslant \beta(K_{1,m} \times K_{1,n})$ and $S \subseteq Y$. In this case we have two subcases.

Subcase 1: Let $\lfloor \frac{r}{2} \rfloor < m$.

- If r is even, then $\omega((K_{1,m} \times K_{1,n}) - S) \leqslant \frac{r^2}{4} + 1$ and so

$$sc(K_{1,m} \times K_{1,n}) \leqslant \max\{\frac{r^2}{4} + 1 - r\} < \frac{(2m-2)^2}{4} = (m-1)^2. \tag{2.3}$$

On the other hand, $m \leqslant n \Rightarrow m(m-1) \leqslant n(m-1) \Rightarrow m^2 - m \leqslant mn - n \Rightarrow m^2 - 2m + 1 \leqslant mn + 1 - (n + m)$.
By Lemma 2.1,

$$(m-1)^2 \leqslant \alpha(K_{1,m} \times K_{1,n}) - \beta(K_{1,m} \times K_{1,n}). \tag{2.4}$$

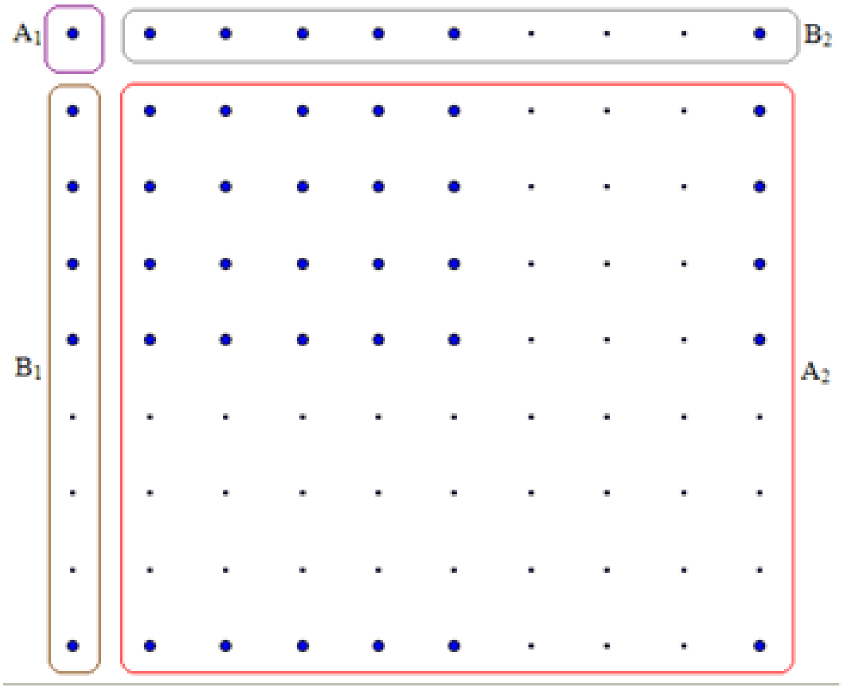

FIGURE 1. Vertices of $K_{1,m} \times K_{1,n}$

By (2.3) and (2.4), we have

$$sc(K_{1,m} \times K_{1,n}) < \alpha(K_{1,m} \times K_{1,n}) - \beta(K_{1,m} \times K_{1,n}). \tag{2.5}$$

- If r is odd, then $\omega((K_{1,m} \times K_{1,n}) - S) \leqslant \frac{r^2+3}{4}$ and so

$$sc(K_{1,m} \times K_{1,n}) \leqslant \max\{\frac{r^2+3}{4} - r\} < \frac{(2m)^2 - 4(2m) + 3}{4} = m^2 - 2m + \frac{3}{4}. \tag{2.6}$$

On the other hand, $m \leqslant n \Rightarrow m(m-1) \leqslant n(m-1) \Rightarrow m^2 - m \leqslant mn - n \Rightarrow m^2 - 2m + \frac{3}{4} \leqslant mn + 1 - (n+m)$.
By Lemma 2.1,

$$m^2 - 2m + \frac{3}{4} \leqslant \alpha(K_{1,m} \times K_{1,n}) - \beta(K_{1,m} \times K_{1,n}). \tag{2.7}$$

By (2.6) and (2.7), we have

$$sc(K_{1,m} \times K_{1,n}) < \alpha(K_{1,m} \times K_{1,n}) - \beta(K_{1,m} \times K_{1,n}). \tag{2.8}$$

By (2.5) and (2.8), we have

$$sc(K_{1,m} \times K_{1,n}) < \alpha(K_{1,m} \times K_{1,n}) - \beta(K_{1,m} \times K_{1,n}). \tag{2.9}$$

Subcase 2: Let $\lfloor \frac{r}{2} \rfloor \geqslant m$, then $\omega((K_{1,m} \times K_{1,n}) - S) \leqslant mr - m^2 + 1$. So

$$sc(K_{1,m} \times K_{1,n}) \leqslant max\{mr - m^2 + 1\} = max\{(m-1)(r-1-m)\}.$$

Since $r \leqslant \beta(K_{1,m} \times K_{1,n}) = m + n$,

$$sc(K_{1,m} \times K_{1,n}) \leqslant (m-1)(m+n-1-m) \leqslant mn + 1 - n - m. \tag{2.10}$$

By Lemma 2.1,

$$(m-1)(m+n-1-m) \leqslant \alpha(K_{1,m} \times K_{1,n}) - \beta(K_{1,m} \times K_{1,n}). \tag{2.11}$$

By (2.10) and (2.11), we have

$$sc(K_{1,m} \times K_{1,n}) < \alpha(K_{1,m} \times K_{1,n}) - \beta(K_{1,m} \times K_{1,n}). \tag{2.12}$$

By (2.9) and (2.12), we have

$$sc(K_{1,m} \times K_{1,n}) < \alpha(K_{1,m} \times K_{1,n}) - \beta(K_{1,m} \times K_{1,n}). \tag{2.13}$$

Case 3: Let $1 \leqslant |S| = r \leqslant \alpha(K_{1,m} \times K_{1,n}) + \beta(K_{1,m} \times K_{1,n})$ and $S \subseteq X \cup Y$. Now we have two subcases.

Subcase 1: If $\lfloor \frac{r}{2} \rfloor < m$ then $\omega((K_{1,m} \times K_{1,n}) - S) < \begin{cases} \frac{r^2}{4} + 1, & \text{if r is even} \\ \frac{r^2+3}{4}, & \text{if r is odd} \end{cases}.$

Subcase 2: If $\lfloor \frac{r}{2} \rfloor \geqslant m$ then $\omega((K_{1,m} \times K_{1,n}) - S) < mr - m^2 + 1$.

The rest of the proof is very similar to that of Case 2. Thus in either of these two subcases, we have

$$sc(K_{1,m} \times K_{1,n}) \leqslant \alpha(K_{1,m} \times K_{1,n}) - \beta(K_{1,m} \times K_{1,n}). \tag{2.14}$$

By (2.2), (2.13) and (2.14), the proof is completed. □

2.2. Scattering Number of $K_{1,m} \times P_n$.

Now we give the following Theorem for $sc(K_{1,m} \times P_n)$.

THEOREM 2.2. *Let $m, n \in Z^+$ ($m \geqslant 2$, $n \geqslant 2$). Then*

$$sc(K_{1,m} \times P_n) = \begin{cases} m-1, & \textit{if n is even} \\ m-2, & \textit{if n is odd} \end{cases}.$$

PROOF. Let the vertices of $K_{1,m}$ and P_n be $v_i (1 \leqslant i \leqslant m+1)$ and $u_j (1 \leqslant j \leqslant n)$, respectively. Hence the vertices of $K_{1,m} \times P_n$ is denoted by (v_i, u_j). We shall abbreviate (v_i, u_j) as $w_{i,j}$ for $1 \leqslant i \leqslant m+1$ and $1 \leqslant j \leqslant n$. It is obvious that $2 \leqslant |S| = r \leqslant \beta(K_{1,m} \times P_n) = \frac{n(m+1)}{2} + 1 - m$.

For the proof we have four cases according to m and n.

Case 1: Let $m > n$ and n be odd. Let $|S| = r$ be the number of removing vertices of graph $K_{1,m} \times P_n$.

- If $2 \leqslant r \leqslant n-1$, then let $S_1 = \{w_{i,2} | 2 \leqslant i \leqslant m+1\}$ and $S_2 = \{w_{i,n-1} | 2 \leqslant i \leqslant m+1\}$. If S consist of vertices $w_{1,1}$ and at least one element of S_1 or consist of vertices $w_{1,n}$ and at least one element of S_2, then

$$\omega((K_{1,m} \times P_n) - S) \leqslant r \text{ and so } sc(K_{1,m} \times P_n) \leqslant max\{r - r\} = 0. \tag{2.15}$$

Otherwise $\omega((K_{1,m} \times P_n) - S) < \alpha(K_{1,m} \times P_n)$ and so

$$sc(K_{1,m} \times P_n) < max\{\alpha(K_{1,m} \times P_n) - \beta(K_{1,m} \times P_n)\} = m - 1. \tag{2.16}$$

- If $n \leqslant r \leqslant \beta(K_{1,m} \times P_n) - 1$, then let $S_3 = \{v_{j,k} | (j,k) \in I \times J, I = \{2, , m+1\}$ and $J = \{2, 4, 6, , n-2\}\}$. If S consist of all vertices of $w_{1,i} (1 \leqslant i \leqslant n)$ or consist of all vertices of $w_{1,i} (1 \leqslant i \leqslant n)$ and at least one element of S_3, then $\omega((K_{1,m} \times P_n) - S) \leqslant r + m - n$. So

$$\omega((K_{1,m} \times P_n) - S) \leqslant max\{r + m - n - r\} = m - n. \tag{2.17}$$

Otherwise $\omega((K_{1,m} \times P_n) - S) < \alpha(K_{1,m} \times P_n)$ and so

$$sc(K_{1,m} \times P_n) < max\{\alpha(K_{1,m} \times P_n) - \beta(K_{1,m} \times P_n)\} = m - 1. \tag{2.18}$$

- Let $r = \beta(K_{1,m} \times P_n)$. If S is the minimum covering set of $K_{1,m} \times P_n$, then $\omega((K_{1,m} \times P_n) - S) = \alpha(K_{1,m} \times P_n)$. Therefore

$$sc(K_{1,m} \times P_n) = max\{\alpha(K_{1,m} \times P_n) - \beta(K_{1,m} \times P_n)\} = m - 1. \tag{2.19}$$

Otherwise $\omega((K_{1,m} \times P_n) - S) < \alpha(K_{1,m} \times P_n)$ and so

$$sc(K_{1,m} \times P_n) < max\{\alpha(K_{1,m} \times P_n) - \beta(K_{1,m} \times P_n)\} = m - 1. \tag{2.20}$$

By (2.15), (2.16), (2.17), (2.18), (2.19) and (2.20), we have

$$sc(K_{1,m} \times P_n) = m - 1. \tag{2.21}$$

The proofs of Case 2, Case 3 and Case 4 are done similar to the proof in Case 1. The values $|S|$, $\omega(K_{1,m} \times P_n)$ and $sc(K_{1,m} \times P_n)$ required for Case 2 are given in Table 1. Similarly, the values in Table 2 and Table 3 are given for the proof of Case 3 and Case 4, respectively.

From Table 1, we have

$$sc(K_{1,m} \times P_n) = m - 2. \tag{2.22}$$

From Table 2, we have

$$sc(K_{1,m} \times P_n) = m - 1. \tag{2.23}$$

From Table 3, we have

$$sc(K_{1,m} \times P_n) = m - 2. \tag{2.24}$$

By (2.21), (2.22), (2.23) and (2.24), the proof is completed.

□

TABLE 1. Case 2

<table>
<tr><th colspan="5">Case 2: Let m>n and n be even.</th></tr>
<tr><th>|S|</th><th>S</th><th colspan="2">ω((K1,m x Pn)-S)</th><th>sc(K1,m x Pn)</th></tr>
<tr><td rowspan="3">$2 \leq r \leq n-1$</td><td>• $w_{1,1}$ and at least one of $w_{i,2}$ (i=2,…,m+1)</td><td>= r</td><td rowspan="3">≤ r</td><td rowspan="3">≤ 0</td></tr>
<tr><td>• $w_{1,n}$ and at least one of $w_{i,n-1}$ (i=2,…,m+1)</td><td>= r</td></tr>
<tr><td>• Otherwise</td><td>< r</td></tr>
<tr><td rowspan="3">$n \leq r < \frac{n.(m+1)}{2}+1-m$</td><td>• All vertices of $w_{1,i}$ (i=1,…,n)</td><td>= r+m-n</td><td rowspan="3">≤ r+m-n</td><td rowspan="3">≤ m-n</td></tr>
<tr><td>• All vertices of $w_{1,i}$ (i=1,…,n) and at least one of $S_1=\{w_{j,k} \mid (j,k)\in I \times J$, I={2,…,m+1} and J={2,4,…,n-2} vertices</td><td>= r+m-n</td></tr>
<tr><td>• Otherwise</td><td>< r+m-n</td></tr>
<tr><td rowspan="2">$\frac{n.(m+1)}{2}+1-m \leq r \leq \beta(K_{1,m} \times P_n)-1$</td><td>• All vertices of $w_{1,i}$(i=1,…,n) and $S_1=\{w_{j,k} \mid (j,k)\in I \times J$, I={2,…,m+1} and J={2,4,…,n-2}}</td><td>$=\frac{n.(m+1)}{2}-1$</td><td rowspan="2">$\leq \frac{n.(m+1)}{2}-1$</td><td rowspan="2">≤ m-2</td></tr>
<tr><td>• Otherwise</td><td>$<\frac{n.(m+1)}{2}-1$</td></tr>
<tr><td rowspan="2">$r = \beta(K_{1,m} \times P_n)$</td><td>• Minimum covering set of $K_{1,m} \times P_n$</td><td>$= \alpha(K_{1,m} \times P_n)$</td><td rowspan="2">$\leq \alpha(K_{1,m} \times P_n)$</td><td rowspan="2">≤ 0</td></tr>
<tr><td>• Otherwise</td><td>$< \alpha(K_{1,m} \times P_n)$</td></tr>
</table>

TABLE 2. Case 3

<table>
<tr><th colspan="5">Case 3: Let m≤n and n be odd.</th></tr>
<tr><th>|S|</th><th>S</th><th colspan="2">ω((K1,m x Pn)-S)</th><th>sc(K1,m x Pn)</th></tr>
<tr><td rowspan="3">$2 \leq r \leq \beta(K_{1,m} \times P_n)-1$</td><td>• $w_{1,1}$ and at least one of $w_{i,2}$ (i=2,…,m+1) vertices</td><td>= r</td><td rowspan="3">≤ r</td><td rowspan="3">≤ 0</td></tr>
<tr><td>• $w_{1,n}$ and one of $w_{i,n-1}$ (i=2,…,m+1) vertices</td><td>= r</td></tr>
<tr><td>• Otherwise</td><td>< r</td></tr>
<tr><td rowspan="2">$r = \beta(K_{1,m} \times P_n)$</td><td>• Minimum covering set of $K_{1,m} \times P_n$</td><td>$= \alpha(K_{1,m} \times P_n)$</td><td rowspan="2">$\leq \alpha(K_{1,m} \times P_n)$</td><td rowspan="2">≤ m-1</td></tr>
<tr><td>• Otherwise</td><td>$< \alpha(K_{1,m} \times P_n)$</td></tr>
</table>

2.3. Scattering Number of $K_{1,m} \times C_n$.

In this section, to obtain the $sc(K_{1,m} \times C_n)$ we need the following Theorems.

THEOREM 2.3. [**2**] *Let $G = T \times C_n$ be the cartesian product of an n-cycle C_n and a tree T with the maximum degree $\Delta(T) \geqslant 2$. Then G possesses a Hamiltonian cycle if and only if $\Delta(T) \geqslant n$.*

TABLE 3. Case 4

<table>
<tr><th colspan="5">Case 4: Let $m \leq n$ and n be even.</th></tr>
<tr><th>\|S\|</th><th>S</th><th colspan="2">$\omega((K_{1,m} \times P_n)-S)$</th><th>$sc(K_{1,m} \times P_n)$</th></tr>
<tr><td rowspan="3">$2 \leq r \leq \frac{n.(m+1)}{2} + 1 - m$</td><td>• $w_{1,1}$ and one of $w_{i,2}$ (i=2,…,m+1) vertices</td><td>$= r$</td><td rowspan="3">$\leq r$</td><td rowspan="3">≤ 0</td></tr>
<tr><td>• $w_{1,n}$ an done of $w_{i,n-1}$ (i=2,…,m+1) vertices</td><td>$= r$</td></tr>
<tr><td>• Otherwise</td><td>$< r$</td></tr>
<tr><td rowspan="2">$\frac{n.(m+1)}{2} + 1 - m \leq r \leq \beta(K_{1,m} \times P_n)$-1</td><td>• All vertices of $w_{1,i}$(i=1,…,n) and $S_1=\{w_{j,k} \mid (j,k) \in I \times J$, I={2,…,m+1} and J={2,4,…,n-2}}</td><td>$= \frac{n.(m+1)}{2} - 1$</td><td rowspan="2">$\leq \frac{n.(m+1)}{2} - 1$</td><td rowspan="2">$\leq$ m-2</td></tr>
<tr><td>• Otherwise</td><td>$< \frac{n.(m+1)}{2} - 1$</td></tr>
<tr><td rowspan="2">$r = \beta(K_{1,m} \times P_n)$</td><td>• Minimum covering set of $K_{1,m} \times P_n$</td><td>$= \alpha(K_{1,m} \times P_n)$</td><td rowspan="2">$\leq \alpha(K_{1,m} \times P_n)$</td><td rowspan="2">$\leq 0$</td></tr>
<tr><td>• Otherwise</td><td>$< \alpha(K_{1,m} \times P_n)$</td></tr>
</table>

THEOREM 2.4. [4] *Let $G = (V, E)$ be a graph. If G is hamiltonian then $\tau(G) \geqslant 1$.*

THEOREM 2.5. [5] *Let $G = (V, E)$ be a graph. If $\tau(G) \geqslant 1$ then $sc(G) \leqslant 0$.*

THEOREM 2.6. *Let $m, n \in Z^+$ ($m \geqslant 2$, $n \geqslant 2$). Then*

$$sc(K_{1,m} \times C_n) = \begin{cases} 0, & \text{if } m \leqslant n \text{ and } n \text{ is even} \\ -1, & \text{if } m < n \text{ and } n \text{ is odd} \\ m - n, & \text{if } m \geqslant n \end{cases}.$$

PROOF. To prove the Theorem we have three cases.

Case 1: Let $m \leqslant n$ and n be even.

By Theorem 1.2, since $\alpha(K_{1,m} \times C_n) = \beta(K_{1,m} \times C_n) = \frac{n(m+1)}{2}$, then

$$sc(K_{1,m} \times C_n) \geqslant 0. \tag{2.25}$$

Now we prove that $sc(K_{1,m} \times C_n) \leqslant 0$. If we choose $T \cong K_{1,m}$ in Theorem 2.4, then $\Delta(T) = \Delta(K_{1,m}) = m$, $m \geqslant 2$ and $m \leqslant n$. Therefore $K_{1,m} \times C_n$ is hamiltonian. By Theorem 2.5 we have $\tau(K_{1,m} \times C_n) \geqslant 1$ and by Theorem 2.6

$$sc(K_{1,m} \times C_n) \leqslant 0. \tag{2.26}$$

By (2.25) and (2.26),

$$sc(K_{1,m} \times C_n) = 0.$$

Case 2: Let $m < n$ and n be odd. $K_{1,m} \times C_n$ consists of $(m+1)$ copies of C_n and n copies of $K_{1,m}$ (Figure 2).

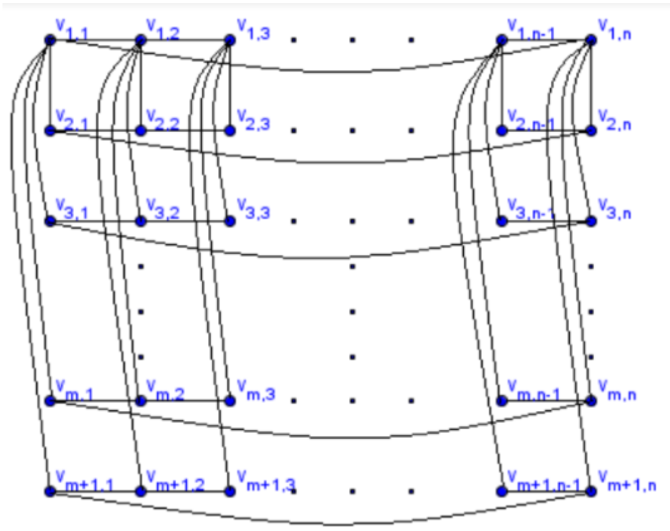

FIGURE 2. $K_{1,m} \times C_n$

Let $A = \{v_{1,1}, v_{1,2}, v_{1,3}, ..., v_{1,n}\}$ and $B = V(K_{1,m} \times C_n) \setminus A$. Let $S = S_0 \cup S_1$, where $S \subset V(K_{1,m} \times C_n)$, $S_0 \subseteq A$ and $S_1 \subseteq B$. Since S is a cut set, we have two cases.

- $|S_0| \geqslant 1$ and $|S_1| \geqslant 2$

  or
- $S_0 = A$ and $|S_1| \geqslant 0$.

Subcase 1: Let $|S_0| \geqslant 1$ and $|S_1| \geqslant 2$. Then $\delta(K_{1,m} \times C_n) = \kappa(K_{1,m} \times C_n) = 3$ and so $deg(v) = 3$ for every vertex v in B. Therefore if we remove three vertices that incident with vertex v (one in A and two in B), then the remaining graph have two components C_0 and C_1, such that C_0 is a isolated vertex and C_1 is a connected graph. Thus $|S| = 3$ and $\omega((K_{1,m} \times C_n) - S) = 2$. For each vertex v of C_1, $deg(v) \geqslant 2$.

Consider a vertex v_1 of C_1 where $deg(v_1) = 2$. If we remove two vertices that are incident with v_1 then the remaining graph have two isolated vertices and a connected graph C_2. Therefore, $|S| = 5$ and $\omega((K_{1,m} \times C_n) - S) = 3$.

Now we consider the graph C_2. For each vertex v in C_2, $deg(v) \geqslant 2$. Let v_2 be a vertex of C_2 where $deg(v_2) = 2$. If we remove two vertices that are incident with v_2

then the remaining graph have three isolated vertices and a connected graph C_3. Therefore, $|S| = 7$ and $\omega((K_{1,m} \times C_n) - S) = 4$.

Similarly, if we continue removing vertices from the every components $C_n (n \geqslant 4)$, we obtain $\omega((K_{1,m} \times C_n) - S) \leqslant r - 1$, where $|S| = r$. Hence

$$sc(K_{1,m} \times C_n) \leqslant max\{r - 1 - r\} = -1. \tag{2.27}$$

Subcase 2: Let $S_0 = A$ and $|S_1| \geqslant 0$. Therefore, $|S| \geqslant n + k$ and $\omega((K_{1,m} \times C_n) - S) \leqslant m + k$ $(k \in Z^+)$,

$$sc(K_{1,m} \times C_n) \leqslant max\{m + k - (n + k)\} = m - n. \tag{2.28}$$

On the other hand,

$$\text{if } m < n \text{ then } m - n \leqslant -1. \tag{2.29}$$

By (2.27), (2.28) and (2.29), we have

$$sc(K_{1,m} \times C_n) \leqslant -1. \tag{2.30}$$

Also, if we choose $G \cong K_{1,m} \times C_n$ in Theorem 1.3(a), we have

$$sc(K_{1,m} \times C_n) \geqslant 2 - \kappa(K_{1,m} \times C_n) = 2 - 3 = -1. \tag{2.31}$$

By (2.30) and (2.31), we have

$$sc(K_{1,m} \times C_n) = -1. \tag{2.32}$$

Case 3: Let $m \geqslant n$. The proof follows directly from Case 2. Thus we have

$$sc(K_{1,m} \times C_n) \leqslant m - n. \tag{2.33}$$

Now we can choose $S = A$, where $S \subset V(K_{1,m} \times C_n)$, $|S| = n$ and $\omega((K_{1,m} \times C_n) - S) = m$. Hence

$$sc(K_{1,m} \times C_n) = max\{\omega((K_{1,m} \times C_n) - S) - |S|\} = m - n. \tag{2.34}$$

Therefore by (2.33) and (2.34), we have

$$sc(K_{1,m} \times C_n) = m - n.$$

The proof is completed. □

2.4. Scattering Number of $K_2 \times C_n$.

THEOREM 2.7.

$$sc(K_2 \times C_n) = \begin{cases} 0, & \text{if } n \text{ is even} \\ -1, & \text{if } n \text{ is odd} \end{cases}.$$

PROOF. Since $K_2 \times P_n$ is a spanning subgraph of $K_2 \times C_n$, we have $sc(K_2 \times P_n) \geqslant sc(K_2 \times C_n)$ by Theorem 1.5. On the other hand, since $sc(K_2 \times P_n) \leqslant 0$ by Theorem 1.4, then

$$sc(K_2 \times C_n) \leqslant 0. \tag{2.35}$$

Now we have two cases according to the parity of n.

Case 1: Let n be even. Then $\alpha(K_2 \times C_n) = \beta(K_2 \times C_n) = n$. By Theorem 1.2,

$$sc(K_2 \times C_n) \geqslant 2\alpha(K_2 \times C_n) - 2n = 2n - 2n = 0. \tag{2.36}$$

By (2.35) and (2.36), we have $sc(K_2 \times C_n) = 0$.

Case 2: Let n be odd. If we remove r vertices from $K_2 \times C_n$ then $\omega((K_2 \times C_n) - S) \leqslant r - 1$. Thus

$$sc(K_2 \times C_n) \leqslant max\{r - 1 - r\} = -1. \tag{2.37}$$

On the other hand, by Theorem 1.3, we have

$$sc(K_2 \times C_n) \geqslant 2 - \delta(K_2 \times C_n) = 2 - 3 = -1. \tag{2.38}$$

By (2.37) and (2.38), we have

$$sc(K_2 \times C_n) = -1.$$

The proof is completed.

□

3. CONCLUSION

When the obtained results are examined, it can be seen that the scattering number is equal to $\alpha - \beta$ in the following graphs.

- The graph $K_{1,m} \times K_{1,n}$ while n is odd.
- The graph $K_{1,m} \times P_n$ while n is odd.
- The graph $K_{1,m} \times C_n$ while $m < n$ and n is even.
- The graph $K_2 \times C_n$ while n is even.

In addition, the scattering number of $K_2 \times C_n$ is -1 and $\alpha - \beta = -2$. Furthermore, the difference of scattering number and $\alpha - \beta$ depends on m in the following graphs.

- The graph $K_{1,m} \times P_n$ while n is even.
- The graph $K_{1,m} \times C_n$ while $m < n$ and n is odd.
- The graph $K_{1,m} \times C_n$ while $m \geqslant n$.

In other words, we observe that the scattering number approachs (equals in some cases) the lower bound in Theorem 1.2 while the value of m decrease and distancing otherwise.

References

1. C. A. Barefoot, R. C. Entringer and H. Swart. Vulnerability in graphs - A comparative survey. *J. Combin. Math. Combin. Comput.*, **1**(1987), 13–22.
2. V. Batagelj and T. Pisanski. Hamiltonian cycles in the cartesian product of a tree and a cycle. *Discrete Math.*, **38**(2-3)(1982), 311–312.
3. J. A. Bondy and U. S. R. Murty. *Graph Theory with Applications*. American Elsevier Publishing Co., Inc., New York, NY 1976.
4. V. Chvatal. Tough graphs and Hamiltonian circuits. *Discrete Math.*, **5**(1973), 215–228.

5. P. Dankelmann, W. Goddard, C. A. McPillan and H. C. Swart. A note on extremal values of the scattering number. *Taiwanese J. Math.*, **17**(5)(2013), 1651-1658.
6. F. Harary. *Graph theory.* Reading, Mass.: Addison-Wesley Pub. Co. (1969).
7. H. A. Jung. On a class of posets and the corresponding comparability graphs. *J. Combin. Theory, Ser. B*, **24**(2)(1978), 125-133.
8. S. Zhang, and Z. Wang. Scattering number in graphs. *Networks*, **37**(2)(2001), 102–106.
9. S. Zhang, X. Li. and X. Han. Computing the scattering number of graphs. *Int. J. Comput. Math.*, 79(2)(2002), 179–187.

Received by editors 09.10.2017; Revised version 17.02.2018; Available online 12.03.2018.

Department of Mathematics, Ege University, Izmir, Turkey
E-mail address: burakaval@gmail.com

Department of Mathematics, Ege University, Izmir, Turkey
E-mail address: alpay.kirlangi@ege.edu.tr