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A NOTE OF NEIGHBOR-TOUGHNESS OF GRAPHS

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Abstract. In this note, we point out some mistakes in Kürkçü and Aksan (2016, [**2**]). We also give the correct definition of neighbor-toughness. Finally, some examples, comments and generalized results related to the computation of the parameter are presented.

1. Introduction

Let $G = (V, E)$ be a graph and $u \in V(G)$. We call $N(u) = \{v \in V(G) | u \neq v, u$ and v are adjacent$\}$ the *open neighborhood* of u, and $N[u] = N(u) \cup \{u\}$ the *closed neighborhood* of u. A vertex u of G is said to be *subverted* if its closed neighborhood $N[u]$ is deleted from G. A set of vertices $S \subseteq V(G)$ is called a *vertex subversion strategy* of G if each of the vertices in S is subverted from G. By G/S we denote the survival subgraph that remains after each vertex of S is subverted from G. A vertex set S is called a *cut strategy* of G if the survival subgraph G/S is disconnected, or is a clique, or is empty.

Kürkçü and Aksan [**2**] claim that they introduce a new vulnerability parameter, neighbor-toughness. The parameter is defined as

$$NT(G) = \min\{\frac{|S|}{\omega(G/S)} : \omega(G/S) \geqslant 1\},$$

where S is any vertex subversion strategy of G and $\omega(G/S)$ is the number of connected components in the graph G/S. By two examples, the authors assert that the neighbor-toughness is a better parameter than the neighbor scattering

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number. This parameter, mentioned above is defined as [**3**]

$$VNS(G) = \max_{S \subseteq V(G)} \{\omega(G/S) - |S|\},$$

where the maximum is taken over all S, the cut-strategy of G, and $\omega(G/S)$ is the number of components of G/S.

We have sufficient reason to show that the above definition and statement in [**2**] are not proper. To the best of our knowledge, the concept of neighbor-toughness appeared firstly in [**4**]. In the next section, we will discuss and revise these items.

2. Main result

In 2013, Wei et al. [**4**] introduced the concept neighbor-toughness (for connected, non-complete graphs) as

$$t_{VN}(G) = \min\{\frac{|S|}{\omega(G/S)}\},$$

where S is any cut strategy of G and $\omega(G/S)$ is the number of components in G/S. A set $S^* \subseteq V(G)$ is called a t_{VN}*-set* of G if

$$t_{VN}(G) = \frac{|S^*|}{\omega(G/S^*)}.$$

For the complete graph, subverting any one vertex will betray the entire graph, its neighbor-toughness is defined to be 0.

The mistake of the definition in [**2**] is that S should be a cut strategy instead of a vertex subversion strategy.

Fig. 1. The cycle graph C_6 and the Petersen graph $P(5,2)$

For example, consider the graph C_6 in Figure 1. By the definition in [**2**], $\{u\}$ is a t_{VN}-set of C_6, since $\frac{|\{u\}|}{\omega(G/\{u\})} = 1 < 2 = \frac{|\{u,v\}|}{\omega(G/\{u,v\})}$. But in [**2**], the authors show that $t_{VN}(C_6) = 2$, a contradiction. In fact, $\{u\}$ is not a t_{VN}-set of C_6, because $C_6/\{u\}$ is P_3, a connected graph. Obviously, $\{u,v\}$ is a t_{VN}-set(cut strategy) of C_6 and $t_{VN}(C_6) = 2$.

On the other hand, consider the Petersen graph $P(5,2)$. Although $\frac{|\{x\}|}{\omega(P(5,2)/\{x\})} = \frac{|\{x,y\}|}{\omega(P(5,2)/\{x,y\})} = 1$, $\{x\}$ is not a t_{VN}-set of $P(5,2)$, since $P(5,2)/\{x\}$ is C_6, a connected graph. By the definition of neighbor-toughness in [**4**], $\{x,y\}$ is a real t_{VN}-set(cut strategy) of $P(5,2)$.

It can be concluded from the above discussion and [**1, 6**] that the definition of neighbor-toughness in [**2**] is wrong, and the definition in [**4**] is correct.

As two new graph parameters, neighbor-toughness and neighbor scattering number can be used to measure the invulnerability of spy networks. Undoubtedly, although formally related, they are independent. Which is a better parameter? It can not be said simply by special examples. In fact, contrary to the author's examples (see [**2**], $VNS(G_1) = VNS(G_2) = 1$, but $NT(G_1) = \frac{2}{3}$, $NT(G_2) = \frac{1}{2}$), there are more examples to show that neighbor scattering number is "better" than neighbor-toughness. Both of the following two graphs are with order 12, and they have equal connectivity and neighbor connectivity 1, as well as equal neighbor-toughness $\frac{1}{2}$, but $VNS(G_1) = 1, VNS(G_2) = 2$.

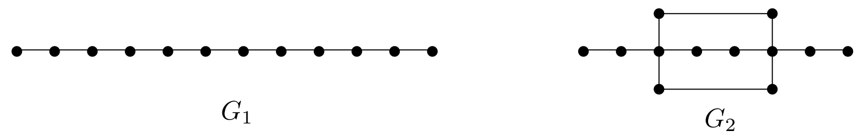

Fig. 2. Two graphs with equal order 12

At last, we generalize a result about the neighbor-toughness of bipartite graphs given in [**2**]. For a bipartite graph $K_{m,n}$, Kürkçü and Aksan prove that

$$t_{VN}(K_{m,n}) = \begin{cases} \frac{1}{m-1}, & \text{if } n < m; \\ \frac{1}{n-1}, & \text{if } n \geqslant m. \end{cases}$$

We show that the above formula is a corollary of the following theorem 2.1 (it is obvious, so we omit the proof).

THEOREM 2.1. *Let $K_{n_1,n_2,\cdots,n_k}$ be a complete k-partite graph, where $n_1 + n_2 + \cdots + n_k \geqslant k + 1$. Then*

$$t_{VN}(K_{n_1,n_2,\cdots,n_k}) = \frac{1}{max\{n_1-1, n_2-1, \cdots, n_k-1\}}.$$

A comet, denoted by $C_{n,k}$, is a graph by coincide an end point of path P_{n-k} with the center point of a star $S_{1,k}$, where $1 \leqslant k \leqslant n-2$ and $n \geqslant 4$. The order of comet $C_{n,k}$ is n.

THEOREM 2.2. *Let $C_{n,k}$ be a comet with order $n(\geqslant 5)$ and $k \leqslant n-2$. Then*

$$t_{VN}(C_{n,k}) = \begin{cases} \frac{1}{k+1}, & \textit{if } k \leqslant n-4; \\ \frac{1}{k}, & \textit{if } k = n-2 \textit{ or } n-3. \end{cases}$$

PROOF. It is easy to know that the vertex in P_{n-k} which is adjacent to the center of star $S_{1,k}$ is a t_{VN}-set of $C_{n,k}$. When $n \geqslant 5$ and $k \leqslant n-4$, $n-k \geqslant 4$, the survival subgraph is a path P_{n-k-3} with k isolated vertex; when $k = n-2$ or $n-3$, the survival subgraph is k isolated vertex, the conclusion holds. □

It is more meaningful to consider the neighbor-toughness computation of general graphs such as trees, Cartesian Product or composition of paths, cycles [**1**]. This is the work we are doing.

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