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FACTOR ALMOST DISTRIBUTIVE LATTICES

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Abstract. In this paper, we introduce the concept of a Factor Almost Distributive Lattice and derive some of its important properties. We also prove that if I is an ideal of B then the Factor Almost Distributive Lattice A/I is a Post Almost Distributive Lattice.

1. Introduction

Emil L. Post [**5**] in 1921 first introduced a general theory of propositions. Later on, E. L. Post [**6**] in 1941 gave a complete list of all closed classes of Boolean functions, nowadays the above list of all closed classes of Boolean functions is called the Post lattice, and moreover, E. L. Post himself also considered the so called Post's lattices with applications to many-valued logical systems, in particular, he proved that each of the above classes has a finite basis and obtained a list of bases for all closed classes. The Post lattice can hence be regarded as a very useful tool in complexity examinations of Boolean circuits and propositional formulas. Recently, the Boolean circuits and Boolean functions attract and deserve a lot of attention in computer science, and the theory behind them is the exhaustively useges in circuit design and various other important fields. In the literature, G. Epstein [**3**] has further studied and developed the lattice theory of Post Algebras. On the other hand, E.L. Post [**6**] considered the two-valued iterative systems of mathematical logic, the postulatees and general theory of Post's lattices and Post algebras are hence formed and developed.

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In 1980, U.M. Swamy and G.C. Rao [**11**] both introduced the concept of an Almost Distributive Lattice (or, simply an ADL) as a common abstraction of most of the existing ring theoretic and lattice theoretic generalizations of a Boolean algebra. In this paper, we introduce the concept of an ideal of Birkhoff center B and derive some of its important properties.

2. Preliminaries

In this section, we recall some necessary notaions and basic results which are needed to make this paper self-contained. However, for more basic information, the reader is referred to the paper [**11**].

Definition 2.1. ([**11**]) An algebra $(A, \vee, \wedge, 0)$ of type $(2, 2, 0)$ is called an Almost Distributive Lattice (ADL) if it satisfies the following axioms: For all $x, y, z \in A$,

(i) $x \vee 0 = x$
(ii) $0 \wedge x = 0$
(iii) $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$
(iv) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
(v) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$
(vi) $(x \vee y) \wedge y = y$.

Theorem 2.1 ([**11**]). *Let m be a maximal element in an ADL $(A, \vee, \wedge, 0)$ and $x \in A$. Then the following statements are equivalent:*

(i) *x is a maximal element of $(A, \leqslant)$.*
(ii) *$x \wedge m = m$.*
(iii) *$x \wedge a = a$, for all $a \in A$.*

Definition 2.2. A non-empty subset I of an ADL $(A, \vee, \wedge, 0)$ is called an ideal of A if and only if it satisfies the following conditions:

(i) $x, y \in I \Longrightarrow x \vee y \in I$.
(ii) $x \in I, a \in A \Longrightarrow x \wedge a \in I$.

Definition 2.3. Let $(A, \vee, \wedge, 0)$ be an ADL with a maximal element m and

$$B(A) = \{x \in A \mid x \wedge y = 0 \text{ and } x \vee y \text{ is maximal for some } y \in A\}.$$

Then $(B(A), \vee, \wedge)$ is a relatively complemented ADL and it is called the Birkhoff center of A. We usually use the symbol B to denote the Birkoff center instead of $B(A)$ when there is no ambiguity.

For any $b \in B$, $b \wedge m$ is a complemented element in the distributive lattice $[0, m]$, whose complement will be denoted by b^m.

For other properties of an ADL, the reader is referred to [**11**].

3. Factor ADL

In our papers [**7, 10**], we introduced the concepts of a P_0-ADL and P_1-ADL respectively and studied their properties. The following definition is taken from [**7**].

DEFINITION 3.1. ([**7**]) Let $(A, \vee, \wedge, 0, m)$ be an ADL with Birkhoff center B. Then, A is said to be a P_0-ADL if there exist elements $0 = e_0, e_1, ..., e_{n-2}, e_{n-1}$ in A such that

(i) $e_{n-1} \wedge m = m$.
(ii) $e_i \wedge e_{i-1} = e_{i-1}$, for $1 \leqslant i \leqslant n-2$.
(iii) For any $x \in A$, there exists $b_i \in B$ such that $x \wedge m = \bigvee_{i=1}^{n-1} (bi \wedge ei \wedge m)$.

Now, we use A to denote a P_0-ADL which means that $(A, \vee, \wedge, 0, m)$ is a P_0- ADL with a chain base $\{0 = e_0, e_1, ..., e_{n-2}, e_{n-1} \wedge m = m\}$.

DEFINITION 3.2. ([**9**]) Let A be a P_0-ADL with Birkhoff center B. Then the following statements hold:

(i) A is said to be a P_1-ADL if for every i, $(e_{i+1} \to e_i) = e_i$ exists.
(ii) A is said to be a P_2-ADL if for every i, $(e_{i+1} \Rightarrow e_i) = e_i$ exists.
(iii) A is said to be a Post ADL if for every i, $(e_{i+1} \Rightarrow e_i) = e_i$ exists and $e_{n-2}! = 0$.

REMARK 3.1. ([**7**]) Let A be a P_0-ADL with Birkhoff center B and $x \in A$ such that $x \wedge m = \bigvee_{i=1}^{n-1} (b_i \wedge e_i \wedge m)$ where $b_i \in B$.

(i). If $b_i \wedge b_{i+1} = b_{i+1}$ for all i, then the above representation is called a monotone representation (or, simply mono. rep.) of x.

(ii). If $b_i \wedge b_j = 0$ for $i \neq j$, then the above representation is called a disjoint representation (or, simply dis. rep.) of x.

(iii) Every element of a P_0-ADL has both monotone and disjoint representation.

For other properties of P_0-ADL, P_1-ADL, P_2-ADL and Post ADL, the reader is referred to [**7, 10, 8, 9**] respectively.

As mentioned above, the concepts of a P_0-ADL and its ideals are very interesting concepts in computer Science and in logic .For this reason, we will concentrate on the properties of a Post ADL .

In this paper, we introduce the concept of Factor ADL and then, we will derive some of its its important properties. Now we begin with the following crucial definition.

DEFINITION 3.3. Let A be a P_0-ADL with Birkhoff center B and J be an ideal of B with the following property:

(P): If $b \in B$ **and** $b \wedge e_r \wedge m \leqslant e_{r-1} \wedge m$ **for some** $r > 0$, **then we have** $b \in J$. Now, we call A a Factor ADL.

From hereafter, we use the symbol J to stand for an ideal of B with property **(P)**.

Theorem 3.1. *Let A be a P_0-ADL with ideal J and Birkhoff center B. If an element $x \in A$ has two mono. reps.*

$$x \wedge m = \bigvee_{r=1}^{n-1} b_r \wedge e_r \wedge m = \bigvee_{r=1}^{n-1} c_r \wedge e_r \wedge m.$$

Then $(b_r \wedge {c_r}^m) \vee (c_r \wedge {b_r}^m) \in J$ for $r = 1, 2, ..., n-1$.

Proof. Let J be an ideal of B and $x \wedge m = \bigvee\limits_{r=1}^{n-1} b_r \wedge e_r \wedge m = \bigvee\limits_{r=1}^{n-1} c_r \wedge e_r \wedge m$ be 2 mono reps. Then $x \wedge e_1 \wedge m = \bigvee\limits_{r=1}^{n-1} b_r \wedge e_r \wedge e_1 \wedge m = \bigvee\limits_{r=1}^{n-1} c_r \wedge e_r \wedge e_1 \wedge m$. Thus $x \wedge e_1 \wedge m = \bigvee\limits_{r=1}^{n-1} b_r \wedge e_1 \wedge m = \bigvee\limits_{r=1}^{n-1} c_r \wedge e_1 \wedge m$ and $x \wedge e_1 \wedge m = b_1 \wedge e_1 \wedge \wedge m = c_1 \wedge e_1 \wedge \wedge m$.

Finally, we have

$$\begin{aligned} b_1 \wedge e_1 \wedge {c_1}^m \wedge m = c_1 \wedge e_1 \wedge {c_1}^m \wedge m = 0 &\implies b_1 \wedge {c_1}^m \wedge e_1 \wedge m = 0 \\ &\implies b_1 \wedge {c_1}^m \in J \text{ (by the property } (\mathbf{P})). \end{aligned}$$

Similarly, we derive that $c_1 \wedge {b_1}^m$ and hence $(b_1 \wedge {c_1}^m) \vee (c_1 \wedge {b_1}^m) \in J$.

Suppose that $(b_1 \wedge {c_1}^m) \vee (c_1 \wedge {b_1}^m) \in J$ for $1 < r < s \leqslant n-1$, where s is a fixed but arbitrary integer which is greater than 1.

Since $x \wedge m = \{(b_1 \wedge e_1) \vee ... \vee (b_{n-1} \wedge e_{n-1})\} \wedge m =$

$$\{(c_1 \wedge e_1) \vee ... \vee (c_{n-1} \wedge e_{n-1})\} \wedge m$$

$$\implies x \wedge e_s \wedge m = \{(b_1 \wedge e_1) \vee ...(b_{n-1} \wedge e_{n-1})\} \wedge e_s \wedge m =$$

$$\{(c_1 \wedge e_1) \vee ... \vee (c_{n-1} \wedge e_{n-1})\} \wedge e_s \wedge m$$

$$\implies x \wedge e_s \wedge m = \{(b_1 \wedge e_1) \vee ...(b_s \wedge e_s)\} \wedge m =$$

$$\{(c_1 \wedge e_1) \vee ... \vee (c_s \wedge e_s)\} \wedge m$$

$$\begin{aligned} \implies x \wedge {b_s}^m \wedge e_s \wedge m &= ((b_1 \wedge {b_s}^m \wedge e_1) \vee (b_2 \wedge {b_s}^m \wedge e_2) \vee (b_s \wedge {b_s}^m \wedge e_s)) \wedge m \\ &= ((c_1 \wedge {b_s}^m \wedge e_1) \vee (c_2 \wedge {b_s}^m \wedge e_2) \vee ... \vee (c_s \wedge {b_s}^m \wedge e_s)) \wedge m. \end{aligned}$$

Then $c_s \wedge {b_s}^m \wedge e_s \wedge m \leqslant e_{s-1} \wedge m$ and hence $c_s \wedge {b_s}^m \in J$. Similarly, we get that $b_s \wedge {c_s}^m \in J$. By using induction , we deduce that $(c_s \wedge {b_s}^m) \vee (c_s \wedge {b_s}^m) \in J$ for $s = 1, 2, ..., n-1$. □

Definition 3.4. Let $(A, \vee, \wedge, 0, m)$ be an ADL with B and let J be an ideal of B. If

$$x \wedge m = \bigvee_{r=1}^{n-1} f_r \wedge e_r \wedge m \quad \text{and} \quad y \wedge m = \bigvee_{r=1}^{n-1} g_r \wedge e_r \wedge m$$

are mono. reps of $x, y \in A$. We define

$$x \equiv y \iff (b_r \wedge {c_r}^m) \vee (c_r \wedge {b_r}^m) \in J \quad \text{for } r = 1, 2, ..., n-1.$$

Now we are in a good position to state the following theorem.

Theorem 3.2. *The relation $\equiv$ is an equivalence relation.*

PROOF. Let $x \wedge m = \bigvee_{r=1}^{n-1} b_r \wedge e_r \wedge m,\ y \wedge m = \bigvee_{r=1}^{n-1} c_r \wedge e_r \wedge m$ be mono. reps of $x, y \in A$ and $x \equiv y \Longleftrightarrow (b_r \wedge {c_r}^m) \vee (c_r \wedge {b_r}^m) \in J$ for $r = 1, 2, ..., n-1$. Since $0 \in A$ and $0 \vee 0 = 0$, we can easily prove that $(b_r \wedge {b_r}^m) \vee (b_r \wedge {b_r}^m) = 0 \in J$.

$$\begin{aligned}
\text{Let } x, y \in A. \text{ Then } x \equiv y &\Longleftrightarrow (b_r \wedge {c_r}^m) \vee (c_r \wedge {b_r}^m) \in J \\
&\Longleftrightarrow (b_r \wedge {c_r}^m \wedge m) \vee (c_r \wedge {b_r}^m \wedge m) \in J \\
&\Longleftrightarrow ((b_r \wedge {c_r}^m) \vee (c_r \wedge {b_r}^m)) \wedge m \in J \\
&\Longleftrightarrow ((c_r \wedge {b_r}^m) \vee (b_r \wedge {c_r}^m)) \wedge m \in J \\
&\Longleftrightarrow (c_r \wedge {b_r}^m) \vee (b_r \wedge {c_r}^m) \in J \\
&\Longleftrightarrow y \equiv x.
\end{aligned}$$

Hence $\equiv$ is symmetric. By using routine arguments, one can easily prove that $\equiv$ is transitive and hence $\equiv$ is indeed an equivalence relation. □

The proof of the following theorem is direct.

THEOREM 3.3. *Let $x \equiv y$ and $z \equiv w$. Then $x \vee z \equiv y \vee w$ and $x \wedge z \equiv y \wedge w$.*

PROOF. Let
$x \wedge m = \bigvee_{r=1}^{n-1} a_r \wedge e_r \wedge m,\ y \wedge m = \bigvee_{r=1}^{n-1} b_r \wedge e_r \wedge m,\ z \wedge m = \bigvee_{r=1}^{n-1} c_r \wedge e_r \wedge m$ and $w \wedge m = \bigvee_{i=1}^{n-1} d_r \wedge e_r \wedge m$. Then

$$(x \vee z) \wedge m = \bigvee_{i=1}^{n-1} (a_r \vee c_r) \wedge e_r \wedge m,\ (y \vee w) \wedge m = \bigvee_{r=1}^{n-1} (b_r \vee d_r) \wedge m,$$

and

$$x \wedge z \wedge m = \bigvee_{r=1}^{n-1} a_r \wedge c_r \wedge e_r \wedge m,$$
$$y \wedge w \wedge m = \bigvee_{r=1}^{n-1} b_r \wedge d_r \wedge e_r \wedge m.$$

Since $x \equiv y,\ z \equiv w$, we have $((a_r \wedge {b_r}^m) \vee (b_r \wedge {a_r}^m)) \wedge m \in J$ and $((c_r \wedge {d_r}^m) \vee (d_r \wedge {c_r}^m)) \wedge m \in J$. Now we show that $x \vee z \equiv y \vee w$ that is $((a_r \wedge {b_r}^m \wedge {d_r}^m) \vee (b_r \wedge {a_r}^m \wedge {c_r}^m) \vee (c_r \wedge {b_r}^m \wedge {d_r}^m) \vee (d_r \wedge {a_r}^m \wedge {c_r}^m)) \wedge m \in J$. Again, since ${b_r}^m \wedge {d_r}^m \in B$, and J is an ideal of B, we get that ${b_r}^m \wedge {d_r}^m \wedge ((a_r \wedge {b_r}^m) \vee (b_r \wedge {a_r}^m)) \wedge m \in J$ and hence $a_r \wedge {b_r}^m \wedge {d_r}^m \wedge m \in J$. Similarly, we get that $b_r \wedge {a_r}^m \wedge {c_r}^m \wedge m \in J, c_r \wedge {b_r}^m \wedge {d_r}^m \wedge m \in J, d_r \wedge {a_r}^m \wedge {c_r}^m \wedge m \in J$ and hence $((a_r \wedge {b_r}^m \wedge {d_r}^m) \vee (b_r \wedge {a_r}^m \wedge {c_r}^m) \vee (c_r \wedge {b_r}^m \wedge {d_r}^m) \vee (d_r \wedge {a_r}^m \wedge {c_r}^m)) \wedge m \in J$. Therefore $x \vee z \equiv y \vee w$. Analogously, we can prove $x \wedge z \equiv y \wedge w$. □

REMARK 3.2. We take $[x] = \{$The abstract class of $\equiv$, containing $x \in A\}$. The set of all classes $[x]$, where x runs over A will be denoted by A/J and the set A/J becomes an ADL with $[e_0] = 0$ and $[e_{n-1} \wedge m] = m$ where $[x] \vee [y] = [x \vee y]$ and $[x] \wedge [y] = [x \wedge y]$. Also, B/J is an ADL and B/J is the center of of A/J.

THEOREM 3.4. *Let A be a P_0-ADL with Birkhoff center B and J be an ideal of B. Then the Factor ADL A/J is a Post ADL.*

PROOF. Clearly A/I is a P_0-ADL and by property of ideal J of B, we can immediately prove that A/J is a Post ADL. □

COROLLARY 3.1. *Let $(A; e_0, e_1, ..., e_{n-1})$ be a Post ADL with a maximal element m, Birkhoff center B. Let J be an ideal of B. Then the Factor ADL A/I is a Post ADL.*

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