BULLETIN OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4874, ISSN (o) 2303-4955
www.imvibl.org /JOURNALS / BULLETIN
Vol. 8(2018), 385-389
DOI: 10.7251/BIMVI1802385R

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

LEFT ZEROID AND RIGHT ZEROID ELEMENTS OF SEMIRINGS

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Abstract. In this paper, we introduce the notion of a left zeroid and a right zeroid elements of semirings. We prove that a left zeroid μ of a simple semiring M is regular if and only if M is a regular semiring and studied some of their properties.

1. Introduction and Preliminaries

The notion of a semiring is an algebraic structure with two associative binary operations where one of them distributes over the other, was first introduced by Vandiver [6] in 1934 but semirings had appeared in studies on the theory of ideals of rings.A universal algebra $S = (S, +, \cdot)$ is called a semiring if and only if $(S, +), (S, \cdot)$ are semigroups which are connected by distributive laws, i.e., $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$, for all $a, b, c \in S$. In structure, semirings lie between semigroups and rings. The results which hold in rings but not in semigroups hold in semirings, since semiring is a generalization of ring. The study of rings shows that multiplicative structure of ring is an independent of additive structure whereas in semiring multiplicative structure of semiring is not an independent of additive structure of semiring. The additive and the multiplicative structure of a semiring play an important role in determining the structure of a semiring. The theory of rings and theory of semigroups have considerable impact on the development of theory of semirings. Semirings play an important role in studying matrices and determinants. Semirings are useful in the areas of theoretical computer science as well as in the solution of graph theory, optimization theory, in particular for studying automata,coding theory and formal languages. Semiring theory has many applications in other branches.

2010 *Mathematics Subject Classification.* Primary 16Y60, 03G25.;
Key words and phrases. left zeroid, right zeroid, idempotent, semiring, division semiring.

Clifford and Miller [3] studied zeroid elements in semigroups. Dawson [4] studied semigroups having left or right zeroid elements. The zeroid of a semiring was introduced by Bourne and Zassenhaus.In this paper,we extend the concept of left or right zeroid elements of semigroup to semiring. We prove that, a left zeroid μ of a simple semiring M is regular if and only if M is a regular.

An element u of a semigroup M is called a zeroid element of M if, for each element a of M, there exist x and y in M such that $ax = ya = u$.

A semiring $(M, +, \cdot)$ is said to be division semiring if $(M \setminus 0, \cdot)$ is a group.

2. A left zeroid and a right zeroid elements in semirings.

In this section we introduce the notion of a left zeroid and a right zeroid elements in semirings and we study their properties.

Definition 2.1. An element x of a semiring M is called a left zeroid (right zeroid) if for each $y \in M$, there exists $a \in M$ such that $ay = x$ $(ya = x)$.

Theorem 2.1. *Let M be a semiring with a left zeroid element x of M and an idempotent e of M. Then $xe = x$.*

Proof. Let x be a left zeroid element and e be an idempotent of M . Then there exists $a \in M$ such that $ae = x$. Therefore $xe = aee = ae = x$. □

Corollary 2.1. *Let M be a semiring with a right zeroid element x and an idempotent e. Then $ex = x$.*

Theorem 2.2. *Let M be a semiring and e be a left zeroid element of M. Then xe is a left zeroid of M for all $x \in M$.*

Proof. Let $y \in M$. Then there exists $t \in M$ such that $ty = e$, since e is a left zeroid of M. Thus $xty = xe$. Hence xe is a left zeroid of M. □

Corollary 2.2. *Let M be a semiring, e be a left zeroid of M. Then every element of Me is a left zeroid of M.*

Definition 2.2. Let M be a semiring and $a \in M$. If there exists $b \in M$ such that $b + a = b$ $(a + b = b)$ then a is said to be additively left(right) zeroid of M.

Theorem 2.3. *Let M be a semiring with identity $a + ab = a$ for all $a, b \in M$. If x is a left zeroid of M then x is an additively left zeroid of M.*

Proof. Suppose $x \in M$ is a left zeroid, $c \in M$. Then there exists $b \in M$ such that $bc = x$. Thus $b + bc = b + x$ and $b = b + x$. Therefore x is an additively left zeroid. Hence the Theorem. □

Theorem 2.4. *Let M be a semiring with identity $a + ab = a$ for all $a, b \in M$ and $(M, +)$ be left cancellative. If x is an additively left zeroid of M, then x is a left zeroid of M.*

Proof. Suppose x is an additively left zeroid of M. Then there exists $b \in M$ such that $b = b + x$. Thus $b + bc = b + x$ for all $c \in M$ and $bc = x$. Hence the Theorem. □

THEOREM 2.5. *Let M be a semiring. If semiring M has both a left zeroid and a right zeroid.Then every left or right zeroid of M is a zeroid of M.*

PROOF. Suppose μ and μ' are a left zeroid a right zeroid of M respectively. Then there exist $y, z \in M$ such that $yx = \mu$ and $xz = \mu'$. Thus $xzy = \mu' y$ and $xzyx = \mu' yx$. Finally $xzyx = \mu'\mu$. Hence $\mu'\mu$ is a zeroid of M. Similarly we can prove $\mu\mu'$ is a zeroid.

Let x be a left zeroid of M . Then there exists $a \in M$ such that $a\mu\mu' = x$. Therefore x is a right zeroid. Since $(a\mu)\mu'$ is a zeroid. Hence the Theorem. □

THEOREM 2.6. *If e is an idempotent of a semiring M then e is the left identity of eM and e is the right identity of Me.*

PROOF. Let $ex \in eM$. Then $eex = ex$. Hence e is the left identity of eM. Similarly we can prove e is the right identity of Me. □

THEOREM 2.7. *If e is an idempotent left zeroid of a semiring M then eM is a division semiring.*

PROOF. Obviously eM is a subsemiring of M and e is the left identity of eM. Suppose $eb \in eM$ there exists $c \in M$ such that $c(eb) = e$. Thus $(ec)(eb)= ee$. Therefore $(ec)(eb) = e$. Finally, e is the left zeroid of eM. Hence ec is the left inverse of eb. Thus eM is a division semiring. □

THEOREM 2.8. *Let U be a non empty set of all left zeroids of semiring M. Then U is a left ideal of M.*

PROOF. Suppose $x_1, x_2 \in U$, $a \in M$ and $x \in M$. Then there exist $y, z \in M$ such that $yx = x_1$ and $zx = x_2$. Thus $(y + z)x = x_1 + x_2$. Therefore $x_1 + x_2$ is a left zeroid of M. By Theorem 2.2, ax_1 is a left zeroid of M. Hence U is a left ideal of M. □

COROLLARY 2.3. *Let M be a semiring. If M has a left zeroid and right zeroid. Then U is an ideal of M.*

COROLLARY 2.4. *Let M be a simple semiring. If M has a left zeroid and a right zeroid then every element of M is a zeroid.*

THEOREM 2.9. *Let M be a semiring and e be an idempotent left zeroid of M. Then a mapping $f : M \to eM$, defined by $f(x) = ex$ is an onto homomorphism.*

PROOF. Let $x_1, x_2 \in M$. Then

$$f(x_1 + x_2) = e(x_1 + x_2) = ex_1 + ex_2 = f(x_1) + f(x_2)$$

$$f(x_1x_2) = e(x_1x_2) = (ex_1)x_2 = [(ex_1)e]x_2 = (ex_1)(ex_2) = f(x_1)f(x_2).$$

Hence f is a homomophism from *M to eM*. Obviously f is onto. Hence the Theorem. □

THEOREM 2.10. *If e is an idempotent left zeroid of a semiring M then Me is a regular semiring.*

Proof. Obviously e is a right identity of Me. Suppose $ze \in Me$. There exists $g \in M$ such that $gze = e$ and $e = ee = e(gze) = (eg)(ze)$. Therefore e is a left zeroid of Me. Suppose $x \in Me$ then there exists $y \in Me$ such that $yx = e$. Then $xyx = xe = x$. Thus Me is a regular semiring. □

Theorem 2.11. *Let M be asemiring. If e is the only idempotent of M, which is a left zeroid of M then e is a zeroid of M.*

Proof. Let e be the only idempotent of a semiring M, which is a left zeroid of M. Then by Theorem 2.10, Me is regular. Suppose $b \in Me$. Then there exists $x \in Me$ such that $b = bxb$. Therefore bx is an idempotent of M. Hence $bx = e$. Each element of Me has right inverse and e is a right identity of Me. Therefore Me is a division semiring.

Let $c \in M$, then $ce \in Me$. There exists $de \in Me$, such that $(ce)(de) = e$. Then $c(ede) = e$. Therefore e is a right zeroid of M. Thus e is a zeroid of M. □

We define a relation $\leqslant$ on the non-empty set of idempotents of a semiring M as follows: $e \leqslant f \Leftrightarrow ef = e$.

Theorem 2.12. *Let M be a semiring. If e is a unique least idempotent and the left (right) zeroid of M then e is a zeroid of M.*

Proof. Suppose e is the least idempotent and the left zeroid of M. Let M contains an idempotent f, which is a left zeroid of M. By Theorem 2.1, $fe = f$. Then $f \leqslant e$. Since e is the unique least idempotent, we have $f = e$. Therefore by Theorem 2.11, e is a zeroid of M.

Suppose that e is a right zeroid of M. Let M contains an idempotent f which is a right zeroid of M. By Corollary 2.1, we have $fe = f$. Therefore $f \leqslant e$. Hence $e = f$. Thus e is the only idempotent of M which is a right zeroid of M. By Theorem 2.11, e is a zeroid of M. Hence the Theorem. □

Theorem 2.13. *A semiring M with a left zeroid μ contains a left zeroid idempotent if and only if μ is a regular of M.*

Proof. Suppose left zeroid μ is regular element of M. Then there exists $x \in M$ such that $\mu = \mu x \mu$. Then $x\mu = x\mu x\mu$. Hence $x\mu$ is a left zeroid idempotent. Conversely suppose that e is a left zeroid idempotent of M. We can prove e is a left zeroid of $M\mu$. By Theorem 2.10 $M\mu e$ is regular. Therefore $M\mu e = M(\mu e) = M\mu$. Hence $M\mu$ is regular. Thus μ is regular. □

Theorem 2.14. *Let M be a simple semiring. Then a left zeroid μ of a simple semiring M is regular if and only if M is a regular semiring.*

Proof. Suppose M is a simple semiring with a regular left zeroid μ of M. Since μ is regular, there exists $x \in M$ such that $\mu = \mu x \mu$. Then $x\mu$ is an idempotent of M. Suppose $b \in M$. Then there exists $c \in M$ such that $cb = \mu$ and there exists $d \in M$ such that $dc = \mu$. Then $\mu b = dcb = d\mu$. Therefore $M\mu b = M(d\mu) = (Md)\mu \subseteq M\mu$. Thus $M\mu$ is a right ideal of M. Obviously $M\mu$ is a left ideal of M. Hence $M\mu = M$, since M is simple. Every element of M is a left zeroid of M. Thus

$x\mu$ is a left zeroid idempotent of M. by Theorem 2.10 $Mx\mu$ is regular. We have $Mx\mu = M\mu x\mu = M\mu = M$.

Converse is obvious. □

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Received by editors 27.05.2017; Revised version 20.01.2018; Available online 29.01.2018.

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