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ON EXPONENTIAL BOUNDS OF HYPERBOLIC COSINE

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Abstract. In this note, natural exponential bounds for *coshx* are established. The inequalities thus obtained are interesting and sharp.

1. Introduction

The well-known Lazarević inequality [1, 2] states that

$$coshx < \left(\frac{sinhx}{x}\right)^p ; x > 0 \text{ if and only if } p \geqslant 3. \tag{1.1}$$

Chen, Zhao and Qi [3] obtained the inequality -

$$coshx \leqslant \left(\frac{\pi^2 + 4x^2}{\pi^2 - 4x^2}\right);\ x \in [0, \pi/2). \tag{1.2}$$

which is Redheffer - type [4].

The inequality (1.2) later was generalised and sharpened by Zhu and Sun [5] as follows -

$$\left(\frac{r^2 + x^2}{r^2 - x^2}\right)^\alpha \leqslant coshx \leqslant \left(\frac{r^2 + x^2}{r^2 - x^2}\right)^\beta \text{ for } 0 \leqslant x < r \tag{1.3}$$

if and only if $\alpha \leqslant 0$ and $\beta \geqslant \frac{r^2}{4}$.

Below are the bounds of *coshx* given in [6] -

$$\left(\frac{1}{cosx}\right)^{2/3} < coshx < \frac{1}{cosx};\ x \in (0, \pi/4). \tag{1.4}$$

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Yupei Lv, Wang et al. [7] give the refinement of (1.4) as follows -

$$\left(\frac{1}{cosx}\right)^a < coshx < \frac{1}{cosx};\ x \in (0, \pi/4) \text{ and } a \approx 0.811133. \tag{1.5}$$

For $x \in (0, 1)$ the following inequality [6, 8] -

$$\frac{3}{3-x^2} \leqslant coshx \leqslant \frac{2}{2-x^2} \tag{1.6}$$

holds.

In this paper, we shall obtain more sharp bounds than given in the above inequalities (1.1) - (1.6) for $coshx$ by using natural exponential function.

2. Main Results

We obtain our main results by using the following l'Hôpital's Rule of Monotonicity [9, Thm. 1.25] -

LEMMA 2.1. *Let $f, g : [a, b] \to \mathbb{R}$ be two continuous functions which are differentiable on (a, b) and $g' \neq 0$ in (a, b). If f'/g' is increasing (or decreasing) on (a, b), then the functions $\frac{f(x)-f(a)}{g(x)-g(a)}$ and $\frac{f(x)-f(b)}{g(x)-g(b)}$ are also increasing (or decreasing) on (a, b). If f'/g' is strictly monotone, then the monotonicity in the conclusion is also strict.*

Now we give our Main results.

THEOREM 2.1. *If $x \in (0, 1)$ then*

$$e^{ax^2} < coshx < e^{x^2/2} \tag{2.1}$$

with the best possible constants $a \approx 0.433781$ and $1/2$.

PROOF. Let $e^{ax^2} < coshx < e^{bx^2}$, which implies that, $a < \frac{log(coshx)}{x^2} < b$.

$$\text{Then } f(x) = \frac{log(coshx)}{x^2} = \frac{f_1(x)}{f_2(x)},$$

where $f_1(x) = log(coshx)$ and $f_2(x) = x^2$ with $f_1(0) = f_2(0) = 0$. By differentiation we get

$$\frac{f_1'(x)}{f_2'(x)} = \frac{tanhx}{2x} = \frac{f_3(x)}{f_4(x)}$$

where $f_3(x) = tanhx$ and $f_4(x) = 2x$, with $f_3(0) = f_4(0) = 0$. Again differentiation gives us -

$$\frac{f_3'(x)}{f_4'(x)} = \frac{sech^2x}{2},$$

which is clearly strictly decreasing in (0, 1). By Lemma 2.1, $f(x)$ is strictly decreasing in (0, 1). Consequently, $a = f(1) = log(cosh1) \approx 0.4333781$ and $b = f(0+) = 1/2$ by l'Hôpital's Rule. □

Remark 2.1. For $-r < x < r$,

$$e^{Ax^2} \leqslant coshx \leqslant e^{x^2/2}, \text{where } A = \frac{log(coshr)}{r^2}. \tag{2.2}$$

Proof. For any $r > 0$, clearly $sech^2x$ is strictly increasing in $(-r, 0)$ and strictly decreasing in $(0, r)$. Applying Lemma 2.1, we get, $A \approx log(coshr)/r^2$. □

For the application of Thm. 2.1, we give another proof of the following theorem [6, Thm.1.2] :

Theorem 2.2. *If* $x \in (0, 1)$ *then*

$$\frac{1}{coshx} < \frac{x^2}{sinh^2x} < \left(\frac{1}{coshx}\right)^{1/2}. \tag{2.3}$$

Proof. As $e^{-ax^2} < e^{-x^2/3}$, for $a \approx 0.433781$
and by theorem 3 in [10] -

$$e^{-x^2/3} < \tfrac{x^2}{sinh^2x} < e^{-bx^2}$$

where $x \in (0, 1)$ and $b \approx 0.322878$. Using these inequalities with (2.1) , it is clear that -

$$\tfrac{1}{coshx} < \tfrac{x^2}{sinh^2x} < e^{-bx^2} < e^{-x^2/4} < \left(\tfrac{1}{coshx}\right)^{1/2}.$$

This completes the proof. □

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