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UNIFORM STATISTICAL CONVERGENCE OF DOUBLE SUBSEQUENCES

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Abstract. In the [3] is proven that sequence S_{ij} uniformly statistically converges to L if and only if it there is a subset A of the set $\mathbb{N}\times\mathbb{N}$ uniform density zero and subsequence $S(x)$ defined by, $S_{ij}(x) = S_{ij}$ for $(i,j) \in A^c$, converges to L, in the Pringsheim's sense. In this paper it is proven that analog is valid for subsequence $S(x)$ provided that for each N and $i \leqslant N \vee j \leqslant N$ is a set of all $S_{ij}(x)$ finite set. Is generally valid: If the subsequence $S(x)$ uniformly statistically converges to L, then, there is a subset A of the set $\mathbb{N}\times\mathbb{N}$ uniform density zero and subsequence $S(y)$ defined by, $S_{ij}(y) = S_{ij}(x)$ for $(i,j) \in A^c$, converges to L, in the Pringsheim's sense. If there is a subset A of the set $\mathbb{N}\times\mathbb{N}$ uniform density zero and subsequence $S(y)$ defined by, $S_{ij}(y) = S_{ij}(x)$ for $(i,j) \in A^c$, such that $\lim\limits_{i\to\infty}(\lim\limits_{j\to\infty} S_{ij}(y)) = L$, then, the subsequence $S(x)$ uniformly statistically converges to L.

1. Introduction

The concept of the statistical convergence of a sequences of reals was introduced by H. Fast [12]. Furthemore, Gökhan et al. [15] introduced the notion of pointwise and uniform statistical convergent of double sequences of real-valued function. Çakan and Altay [4] presented multi dimensional analogues of the results presented by Fridy and Orhan [13, 14]. Dündar and Atay [5, 6, 7, 8, 9] investigated the relation between I-convergence of double sequences. Now, we recall that the definitions of concepts of ideal convergence and basic concepts.[1, 2, 10, 11, 16].

The sequence S_{ij} of real numbers converges to L in the Pringsheim's sense, if for any $\varepsilon > 0$ there exists $K > 0$ such that

$$|S_{ij} - L| \leqslant \varepsilon$$

for any $i, j \geqslant K$.

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We write $\lim_{i,j\to\infty} S_{ij} = L$.

Let $K \subset \mathbb{N}\times\mathbb{N}$. Let K_{nm} be the number of $(i,j) \in K$ such that $i \leqslant n, j \leqslant m$. If

$$d_2(K) = \lim_{n,m\to\infty} \frac{K_{nm}}{nm}$$

in the Pringsheim's sense. Then we say that K has double natural density. Let is sequence S_{ij} of real numbers and $\varepsilon > 0$. Let

$$A(\varepsilon) = \{(i,j) \in \mathbb{N}\times\mathbb{N} : |S_{ij} - L| \geqslant \varepsilon\}.$$

The sequence $S = S_{ij}$ statistically converges to $L \in \mathbb{R}$ if

$$(\forall\varepsilon > 0)(d_2(A(\varepsilon)) = 0).$$

In this cese, we write $st - \lim S_{ij} = L$.

Let is set $X \neq \emptyset$. A class I of subsets of X is said to be an ideal in X provided the following statements hold:

(i) $\emptyset \in I$

(ii) $A, B \in I \Rightarrow A \cup B \in I$

(iii) $A \in I$, $B \subset A \Rightarrow B \in I$.

I is nontrivial ideal if $X \notin I$. A nontrivial ideal I is called admissible if $\{x\} \in I$ for any $x \in X$.

In this paper the focus is put on ideal $I_u \subset 2^{\mathbb{N}\times\mathbb{N}}$ defined by: subset A belongs to the I_u if

$$\lim_{p,q\to\infty} \frac{1}{pq} |\{i < p, j < q : (n+i, m+j) \in A\}| = 0$$

uniformly on $n, m \in \mathbb{N}$ in the Pringsheim's sense. That is subset A of the set $\mathbb{N}\times\mathbb{N}$ is uniformly density zero.

The sequence $S = S_{ij}$ uniformly statistically converges to L if for any $\varepsilon > 0$

$$\{(i,j) \in \mathbb{N}\times\mathbb{N} : |S_{ij} - L| \geqslant \varepsilon\} \in I_u.$$

That is sequence $S = S_{ij}$ uniformly statistically converges to L, if any ε, $\varepsilon' > 0$ there exists $K > 0$ such that

$$\frac{1}{pq} |\{i < p, j < q : |S_{n+i,m+j} - L| \geqslant \varepsilon\}| < \varepsilon',\ \forall p, q \geqslant K,\ \forall n, m \in \mathbb{N}.$$

We write $Ust - \lim S_{ij} = L$.

We denote with X a set of all double sequences of 0's and 1's, i.e.

$$X = \{x = x_{ij} : x_{ij} \in \{0,1\}, i, j \in \mathbb{N}\}.$$

Let sequence $S = S_{ij}$ and $x \in X$. Then with $S(x)$ we denote a sequence defined following way

$$S_{ij}(x) = S_{ij}, \text{ for } x_{ij} = 1.$$

which we refer to as subsequence of sequence S.

The subsequence $S(x)$ of sequence S uniformly statistically converges to L, if for any ε, $\varepsilon' > 0$ there exists $K > 0$ such that for every p, $q \geqslant K$ and for all $n, m \in \mathbb{N}$ provided that $x_{nm} = 1$ we have

$$\frac{|\{i < p, j < q : |S_{n+i,m+j} - L| \geqslant \varepsilon, x_{n+i,m+j} = 1\}|}{|\{i < p, j < q : x_{n+i,m+j} = 1\}|} \leqslant \varepsilon'.$$

We write $Ust - \lim S_{ij}(x) = L$.

2. New results

THEOREM 2.1. *Let notions and notations as in above. Then, we have*

$$\lim_{i,j\to\infty} S_{ij} = L \Longrightarrow Ust - \lim S_{ij} = L \Longrightarrow st - \lim S_{ij} = L.$$

PROOF. If $\lim_{i,j\to\infty} S_{ij} = L$, then for any $\varepsilon > 0$ there exists $K \in \mathbb{N}$ such that for all $i, j \geqslant K$, we have $|S_{ij} - L| \leqslant \varepsilon$. Then

$$\begin{aligned}
&\frac{1}{pq}\left|\{i < p, j < q : |S_{n+i,m+j} - L| \geqslant \varepsilon\}\right| = \\
&\frac{1}{pq}\left|\{i < p, j < q : |S_{n+i,m+j} - L| \geqslant \varepsilon, n+i < K \vee m+j < K\}\right| + \\
&\frac{1}{pq}\left|\{i < p, j < q : |S_{n+i,m+j} - L| \geqslant \varepsilon, n+i, m+j \geqslant K\}\right| = \\
&\frac{1}{pq}\left|\{i < p, j < q : |S_{n+i,m+j} - L| \geqslant \varepsilon, n+i < K \vee m+j < K\}\right|.
\end{aligned}$$

If $n, m < K$, then, $\forall \varepsilon, \varepsilon' > 0, \exists K_1$, such that for $\forall p, q \geqslant K_1$, we have

$$\begin{aligned}
&\frac{1}{pq}\left|\{i < p, j < q : |S_{n+i,m+j} - L| \geqslant \varepsilon\}\right| \leqslant \\
&\frac{1}{pq}\left[q(K-n) + p(K-m)\right] \leqslant \varepsilon', \forall p, q \geqslant K_1.
\end{aligned}$$

If $n < K$, then, $\forall \varepsilon, \varepsilon' > 0, \exists K_1$, such that for $\forall p, q \geqslant K_2$, we have

$$\frac{1}{pq}\left|\{i < p, j < q : |S_{n+i,m+j} - L| \geqslant \varepsilon\}\right| \leqslant \frac{1}{pq} q(K-n) \leqslant \varepsilon', \forall p, q \geqslant K_2.$$

If $m < K$, then, $\forall \varepsilon, \varepsilon' > 0, \exists K_1$, such that for $\forall p, q \geqslant K_2$, we have

$$\frac{1}{pq}\left|\{i < p, j < q : |S_{n+i,m+j} - L| \geqslant \varepsilon\}\right| \leqslant \frac{1}{pq} p(K-m) \leqslant \varepsilon', \forall p, q \geqslant K_3.$$

Hence, $\forall\, \varepsilon, \varepsilon' > 0, \exists K_4 = \max\{K, K_1, K_2, K_3\}$ such that for $\forall p, q \geqslant K_4, \forall n, m \in \mathbb{N}$, we have

$$\frac{1}{pq}\left|\{i < p, j < q : |S_{n+i,m+j} - L| \geqslant \varepsilon\}\right| \leqslant \varepsilon',$$

respectively, $Ust - \lim S_{ij} = L$.

Let $Ust - \lim S_{ij} = L$, then, $\forall\, \varepsilon, \varepsilon' > 0, \exists K > 0$ such that for $\forall p, q \geqslant K, \forall n, m \in \mathbb{N}$, we have

$$\frac{1}{pq}\left|\{i < p, j < q : |S_{n+i,m+j} - L| \geqslant \varepsilon\}\right| \leqslant \varepsilon'.$$

Specially, for $n=m=1$ and for any $p,\ q\geqslant K$, we have

$$\frac{1}{pq}\left|\{i\leqslant p, j\leqslant q : |S_{i,j}-L|\geqslant\varepsilon\}\right|\leqslant\varepsilon',$$

ie. $st-\lim S_{ij}=L$. □

EXAMPLE 2.1. Let $S=S_{nm}$ defined as

$$S_{nm}=\begin{cases}1, & 1+1+2+\cdots+k<n,m\leqslant 1+1+2+\cdots+k+k+1,\\ & k=1,2,\cdots\\ 0, & \text{otherwise}\end{cases}.$$

Let

$$1+1+2+\cdots+k\leqslant p<1+1+2+\cdots+k+k+1,$$
$$1+1+2+\cdots+k<q.$$

Then, for any $\varepsilon,\varepsilon'>0$, there exists $k_0\in\mathbb{N}$, such that for all

$$p,q>1+\frac{k_0\left(k_0+1\right)}{2}$$

is true

$$\frac{1}{pq}\left|\{i\leqslant p, j\leqslant q : |S_{i,j}-0|\geqslant\varepsilon\}\right|\leqslant$$
$$\frac{1}{pq}\left[2^2+3^2+\cdots+(k-1)^2+(p-1-1-\cdots-k)(k+1)\right]=$$
$$\frac{1}{pq}\left[\frac{(k-1)k(2k-1)}{6}-1+\left(p-1-\frac{k(k+1)}{2}\right)(k+1)\right]\leqslant$$
$$\frac{\dfrac{(k-1)k(2k-1)}{6}-1+\left(\dfrac{(k+1)(k+2)}{2}-\dfrac{k(k+1)}{2}\right)(k+1)}{\left(1+\dfrac{k(k+1)}{2}\right)\left(1+\dfrac{(k+1)(k+2)}{2}\right)}\leqslant\varepsilon'.$$

Hence, $st-\lim S_{ij}=0$.

For all $k\in\mathbb{N}$ and for any $\varepsilon>0$, $n=1+1+2+\cdots+k$, we have

$$\frac{1}{(k+1)^2}\left|\{i,j<k+1 : |S_{n+i,n+j}-0|\geqslant\varepsilon\}\right|=1.$$

Respectively, sequence $S=S_{nm}$ does not uniformly statistically converge.

In the [3] is proved theorem: If $S=S_{ij}$ is a double sequence, then

$$Ust-\lim S_{ij}=L$$

if and only if there exists $A\subset\mathbb{N}\times\mathbb{N}$ uniformly density zero, such that $\lim\limits_{i,j\to\infty}S_{ij}=L$ in the Pringsheim's sense, for

$$x_{ij}=\begin{cases}1, & (i,j)\notin A\\ 0, & (i,j)\in A\end{cases}$$

Following theorem is a generalization of subsequences.

Let $x \in X$. Let is an ideal $I_u(x) \subset 2^{\mathbb{N}\times\mathbb{N}}$ defined by: the subset A of set $\{(i,j) : x_{ij} = 1\}$ belongs to the $I_u(x)$ if for all $\varepsilon > 0$ there exists $K > 0$ such that for all $p, q \geqslant K$ and for all $n, m \in \mathbb{N}$ provided that $x_{nm} = 1$, we have

$$\frac{|\{i < p, j < q : (n+i, m+j) \in A\}|}{|\{i < p, j < q : x_{n+i,m+j} = 1\}|} \leqslant \varepsilon.$$

THEOREM 2.2. (a) *Let $x \in X$ and $Ust - \lim S_{ij}(x) = L$. Then there is a set $A \in I_u(x)$, such that subsequence $S(y)$ of the sequence S converges to L in the Pringsheim's sense, for*

$$y_{ij} = \begin{cases} 1, & (i,j) \notin A, x_{ij} = 1 \\ 0, & (i,j) \in A, x_{ij} = 0 \end{cases}.$$

(b) *If there is a set $A \in I_u(x)$ such that for subsequence $S(y)$ of the sequence S valid $\lim\limits_{i\to\infty}(\lim\limits_{j\to\infty} S_{ij}(y)) = L$, for*

$$y_{ij} = \begin{cases} 1, & (i,j) \notin A, x_{ij} = 1 \\ 0, & (i,j) \in A, x_{ij} = 0 \end{cases}$$

then $Ust - \lim S_{ij}(x) = L$.

PROOF. a) Let $Ust - \lim S_{ij}(x) = L$. Then for all $k \in \mathbb{N}$ there exists $r_k > 0$, such that for all p, $q \geqslant r_k$ and for all $n, m \in \mathbb{N}$ provided $x_{nm} = 1$, we have

$$\frac{\left|\left\{i < p, j < q : |S_{n+i,m+j} - L| \geqslant \frac{1}{k},\ x_{n+i,m+j} = 1\right\}\right|}{|\{i < p, j < q : x_{n+i,m+j} = 1\}|} \leqslant \frac{1}{k^2}.$$

Let $A = \bigcup\limits_{k=2}^{\infty} \bigcup\limits_{n,m=1}^{\infty} \{(n+i, m+j) :$

$$i, j \geqslant r_k, i < r_{k+1} \vee j < r_{k+1}, |S_{n+i,m+j} - L| \geqslant \frac{1}{k}, x_{n+i,m+j} = 1\}.$$

For all $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that for $\forall k > k_0$ we have

$$\sum_{k=k_0}^{\infty} \frac{1}{k^2} \leqslant \frac{\varepsilon}{2}, \frac{1}{(k_0-1)^2} \leqslant \frac{\varepsilon}{2}.$$

Then, for all $p, q \geqslant r_{k_0}$ and for all $n, m \in \mathbb{N}$ provided that $x_{nm} = 1$, we have

$$\frac{|\{i < p, j < q : (n+i, m+j) \in A\}|}{|\{i < p, j < q : x_{n+i,m+j} = 1\}|} =$$

$$\frac{|\{i < p, j < q : i, j \geqslant r_{k_0}, (n+i, m+j) \in A\}|}{|\{i < p, j < q : x_{n+i,m+j} = 1\}|} +$$

$$\frac{|\{i < p, j < q : i < r_{k_0} \vee j < r_{k_0}, (n+i, m+j) \in A\}|}{|\{i < p, j < q : x_{n+i,m+j} = 1\}|} \leqslant$$

$$\frac{|\{i<p,j<q:i,j\geqslant r_{k_0},(n+i,m+j)\in A\}|}{|\{i<p,j<q:x_{n+i,m+j}=1\}|}+$$
$$\frac{|\{i<p,j<q:i,j\geqslant r_{k_0+1},(n+i,m+j)\in A\}|}{|\{i<p,j<q:x_{n+i,m+j}=1\}|}+\ldots+$$

$$\frac{|\{i<p,j<q:i<r_{k_0}\vee j<r_{k_0},(n+i,m+j)\in A\}|}{|\{i<p,j<q:x_{n+i,m+j}=1\}|}\leqslant$$

$$\textstyle\sum_{k=k_0}^{\infty}\frac{1}{k^2}+\frac{1}{(k_0-1)^2}\leqslant\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon,$$

Let

$$y_{ij}=\begin{cases}1, & (i,j)\notin A, x_{ij}=1\\ 0, & (i,j)\in A, x_{ij}=0\end{cases}.$$

Then, for all $\varepsilon>0$ there exists $k_0\in\mathbb{N}$ such that for $y_{n+i,m+j}=1, n+i, m+j\geqslant r_{k_0}$ we have

$$|S_{n+i,m+j}(y)-L|=|S_{n+i,m+j}-L|\leqslant\frac{1}{k_0}\leqslant\varepsilon$$

which implies that for all $n,m\geqslant r_{k_0}$ we have

$$|S_{nm}(y)-L|\leqslant\varepsilon.$$

Respectively, $\lim\limits_{i,j\to\infty}S_{ij}(y)=L$ in the Pringsheim's sense.

b) For all $\varepsilon>0$ there exists $n_0,m_0\in\mathbb{N}$ such that for $n\geqslant n_0\vee m\geqslant m_0$ we have

$$|S_{nm}(y)-L|\leqslant\varepsilon.$$

Then

$$\frac{|\{i<p,j<q:|S_{n+i,m+j}-L|\geqslant\varepsilon,x_{n+i,m+j}=1\}|}{|\{i<p,j<q:x_{n+i,m+j}=1\}|}=$$

$$\frac{|\{i<p,j<q:n+i<n_0,m+j<m_0,|S_{n+i,m+j}-L|\geqslant\varepsilon,x_{n+i,m+j}=1\}|}{|\{i<p,j<q:x_{n+i,m+j}=1\}|}+$$

$$\frac{|\{i<p,j<q:n+i\geqslant n_0\vee m+j\geqslant m_0,|S_{n+i,m+j}-L|\geqslant\varepsilon,x_{n+i,m+j}=1\}|}{|\{i<p,j<q:x_{n+i,m+j}=1\}|}\leqslant.$$

$$\frac{n_0m_0}{|\{i<p,j<q:x_{n+i,m+j}=1\}|}+\frac{|\{i<p,j<q:(n+i,m+j)\in A\}|}{|\{i<p,j<q:x_{n+i,m+j}=1\}|}.$$

The first summand is smaller than $\dfrac{\varepsilon}{2}$ for all $p,q \geqslant N$ and for all $n,m \in \mathbb{N}$ such that $x_{n+i,m+j} = 1$.

The second summand is smaller than $\dfrac{\varepsilon}{2}$ for all $p,q \geqslant M$ and for all $n,m \in \mathbb{N}$ such that $x_{nm} = 1$. Therefore, for all $p,q \geqslant \max\{N,M\}$ and for all $n,m \in \mathbb{N}$ provided that $x_{nm} = 1$ we have that

$$\frac{|\{i < p, j < q : |S_{n+i,m+j} - L| \geqslant \varepsilon, x_{n+i,m+j} = 1\}|}{|\{i < p, j < q : x_{n+i,m+j} = 1\}|} \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

it is $Ust - \lim S_{ij}(x) = L$. □

We denote

$$X' = \{x \in X : \{(i,j) : i \leqslant N \vee j \leqslant N, x_{ij} = 1\} \; is \; finite \; set \; for \; \forall N \in \mathbb{N}\}.$$

Corollary 2.1. *Let sequence $S = S_{ij}$ and $x \in X'$. Then, $Ust-\lim S_{ij}(x) = L$ if and only if there is a set $A \in I_u(x)$, such that subsequence $S(y)$ of the sequence S converges to L in the Pringsheim's sense, for*

$$y_{ij} = \begin{cases} 1, & (i,j) \notin A, x_{ij} = 1 \\ 0, & (i,j) \in A, x_{ij} = 0 \end{cases}.$$

References

[1] B. Altay and F. Başar. Some new spaces of double sequences. *J. Math. Anal. Appl.*, **309**(1)(2005), 70-90.

[2] M. Balcerzak, K. Dems and A. Komisarski. Statistical convergence and ideal convergence for sequences of functions. *J. Math. Anal. Appl.*, **328**(1)(2007), 715-729.

[3] F. Čunjalo and F. Destović. Subsequence characterization of uniform statistical convergence of double sequences. *Res. Comm. Math. Math. Sci.*, (accept at November 27, 2017).

[4] C. Çakan and B. Altay. Statistically boundedness and statistical core of double sequences. *J. Math. Anal. Appl.*, **317**(2)(2006), 690-697.

[5] E. Dündar. On rough I_2-convergence of Double Sequences. *Numer. Funct. Anal. and Optimiz*, **37**(4)(2016), 480-491.

[6] E. Dündar and B. Atay. I_2-convergence of Double Sequences of Function. *El. J. Math. Anal. Appl.*, **3**(1)(2015), 111-121.

[7] E. Dündar and B. Atay. I_2-convergence and I_2-Cauchy of Double Sequences. *Acta Math. Sci. Ser. B Engl. Ed.*, **34B**(2)(2014), 343-353.

[8] E. Dündar and B. Atay. I_2-uniform convergence of double sequences of function. *Filomat*, **30**(5)(2016), 1273–1281.

[9] E. Dündar and B. Atay. On some properties of I_2-convergence and I_2-Cauchy of double sequences. *Gen. Math. Notes*, **7**(1)(2011), 1-12.

[10] E. Dündar and C. Çakan. Rough convergence of double sequences. *Gulf J. Math.*, **2**(1)(2014), 45–51.

[11] E. Dündar and Y. Sever. Multipliers for bounded statistical convergence of double sequences. *Int. Math. Forum*, **7**(52)(2012), 2581–2587.

[12] H. Fast. Sur la convergenc statistique. *Collog. Math.*, **2**(1951), 241-244.

[13] J. A. Fridy. On statistical convergence. *Analysis*, **5**(4)(1985), 301-313.

[14] J. A. Fridy and C. Orhan. Statistical limit superior and inferior. *Proc. Amer. Math. Soc.*, **125**(12)(1997), 3625-3631.

[15] A. Gökhan, M. Güngör and M. Et. Statistical convergence of double sequences of real-valued function. *Int. Math. Forum*, **2**(8)(2007), 365-374.

[16] E. Tas and T. Yurdakadmin. Characterization of uniform statistical convergence for double sequences. *Miskolc Math. Notes*, **13**(2)(2012), 543-553.

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