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ON CHARACTERIZATION OF DISTANCE GRAPHS OF A PATH

R. Rajkumar and S. Celine Prabha

ABSTRACT. Let P_n be the path graph on n vertices. In this paper, we consider the distance graphs $G(P_n, D)$, where the distance set $D \subseteq \{1, 2, 3, \ldots, n-1\}$. We characterize the distance set D for which $G(P_n, D)$ is one of path, cycle, wheel, regular, bipartite, acyclic, C_r-free, $K_{1,r}$-free, or having isolated vertices.

1. Introduction

All the graphs we considered in this paper are simple. We use the standard terminologies and notations of graph theory following [**6**]. The degree of a vertex v in a graph Γ is denoted by $d_\Gamma(v)$. The usual shortest path distance between vertices u and v in Γ is denoted by $d(u, v)$ (or $d_\Gamma(u, v)$, if we want to emphasize the graph Γ). K_n and C_n denotes the complete graph and cycle graph, respectively on n vertices. $K_{m,n}$ denotes the complete bipartite graph with partition sizes m and n. A path graph is a simple graph whose vertices can be arranged in a linear sequence in such a way that every two consecutive vertices are adjacent. The path graph on n ($n \geqslant 2$) vertices is denoted by P_n. Through out this paper, we take the vertices of P_n as the linear sequence $v_1, v_2, \ldots, v_n$. A wheel graph on $n \geqslant 4$ vertices is obtained from a cycle graph C_{n-1} by adding a new vertex in such a way that it is adjacent to all the vertices of C_{n-1}.

Let (X, ρ) be a metric space with metric ρ. Then for each set $D \subseteq \{\rho(x, y) | x, y \in X, x \neq y\}$, the *distance graph* of X with respect to the *distance set* D, denoted by $G(X, D)$, is the graph whose vertex set is X and two vertices x, $y \in X$ are adjacent if $\rho(x, y) \in D$.

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The unit distance graphs defined on $\mathbb{R}^n, \mathbb{Q}^n, \mathbb{Z}^n$ with the Euclidean metric are distance graphs, which have been investigated by several authors (see, [**17**] for more details). In [**7**], Eggleton, Erdős and Skilton have studied the distance graphs $G(\mathbb{Z}, D)$, where D is the set of positive integers. The distance graphs on $\mathbb{R}^n$ and $\mathbb{Z}^n$ with the l_p metrics were investigated in many articles (see, for instance [**8**], [**10**], [**13**]).

On the other hand, every graph Γ with the usual shortest path distance d defines a metric space (Γ, d). So for each set $D \subseteq \{d(u,v) | u, v \in V(\Gamma), u \neq v\}$, we can define the distance graph $G(V(\Gamma), D)$. We denote this graph simply by $G(\Gamma, D)$. In literature, there are several papers devoted to the study of distance graphs of graphs. For instance, the n^{th} power graph of a graph Γ is the distance graph $G(\Gamma, \{1, 2, 3, \ldots, n\})$. The graph $\Gamma_n := G(\Gamma, \{n\})$ is called the n^{th} distance graph (or n-distance graph). In [**15**], Simić initiated the study of $n-$distance graph while solving the graph equation $\Gamma_n \cong L(\Gamma)$, where $L(\Gamma)$ is the line graph of Γ. Suzuki [**9**] investigated the $n-$distance graphs of distance regular graphs. Recently, Azimi and Farrokhi [**2**] classified all simple graphs whose 2–distance graphs are either paths or cycles. Note that when the given graph Γ is connected, then $\{d(u,v) | u, v \in V(\Gamma), u \neq v\} = \{n | 1 \leqslant n \leqslant \operatorname{diam}(\Gamma)\}$, where $\operatorname{diam}(\Gamma)$ denotes the diameter of Γ. The distance graph $G(\Gamma, \{\operatorname{diam}(\Gamma)\})$ is called the antipodal graph of Γ, and was introduced by Singleton [**16**]. This graph was further studied by Acharya and Acharya [**1**], Rajendran [**14**], Aravamudhan and Rajendran [**3, 4**], Johns [**12**], and Chartrand et al. [**5**].

For a given graph Γ, the investigation of the structure of the distance graphs $G(\Gamma, D)$ for different choices of the distance set D is a general problem. In this direction, characterizing the distance set D, for which the distance graph $G(\Gamma, D)$ satisfying some graph theoretic properties is a problem of special interest. In this paper, we consider the distance graphs $G(P_n, D)$, where $D \subseteq \{1, 2, \ldots, n-1\}$.

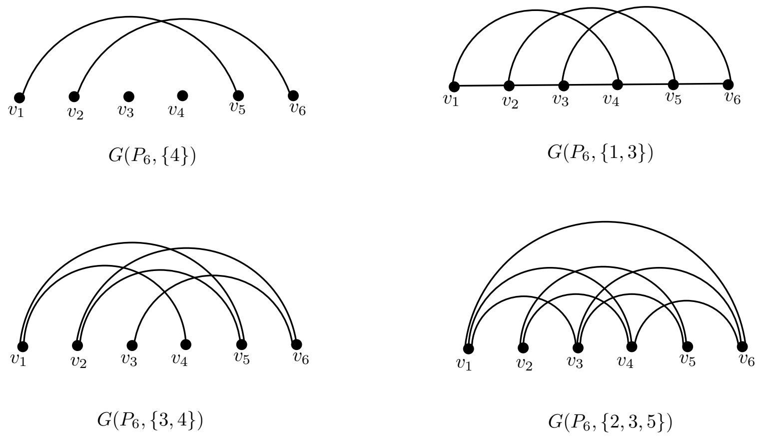

FIGURE 1. Some distance graphs of P_6

In Figure 1, we describe the structure of $G(P_6,\{4\})$, $G(P_6,\{1,3\})$, $G(P_6,\{3,4\})$ and $G(P_6,\{2,3,5\})$. Even though the path graph has a very simple structure, the distance graphs $G(P_n,D)$ gets a complicated structure for the different choices of the distance set D. In [**11**], Murali and Harinath investigated the laceability properties of the distance graphs $G(P_n,D)$. In the next section, we mainly characterize the distance set D for which $G(P_n,D)$ is one of path, cycle, regular, bipartite, wheel, acyclic, C_r-free, $K_{1,r}$-free, or having isolated vertices.

2. Main results

THEOREM 2.1. *(1)* $G(P_n,D)$ *is a path if and only if* $D=\{1\}$ *or* $\{r,n-r+1 \mid g.c.d\ (r,n-r+1)=1\}$.
(2) $G(P_n,D)$ *is a cycle if and only if* $D=\{r,n-r \mid g.c.d\ (r,n-r)=1\}$.

PROOF. The proof is divided into several cases.

Case (1): Let $|D|=1$.

Then $D=\{k\}$, where $1\leqslant k\leqslant n-1$. If $k=1$, then $G(P_n,D)=P_n$. Now we assume that $k\neq 1$.

(i): Let n be even and $k\leqslant \frac{n}{2}$ (resp. n be odd and $k\leqslant \lceil\frac{n}{2}\rceil-1$). If k divides n, then $G(P_n,D)$ is the disjoint union of k paths: $v_1,v_{k+1},v_{2k+1},\dots,v_{n-k+1}$; $v_2,v_{k+2},v_{2k+2},\dots,v_{n-k+2}$; $\dots$; $v_k,v_{2k},\dots,v_n$. If k is not a divisor of n, then $G(P_n,D)$ is the disjoint union of k paths: $v_1,v_{k+1},v_{2k+1},\dots,v_{mk+1}$; $v_2,v_{k+2},v_{2k+2},\dots,v_{mk+2}$; $\dots$; $v_k,v_{2k},\dots,v_{mk}$, where m is the quotient when n is divided by k.

(ii): Let n be even and $k\geqslant \frac{n}{2}+1$ (resp. n be odd and $k\geqslant\lceil\frac{n}{2}\rceil$). Then $G(P_n,D)$ has exactly k components, in which $n-k$ components are the path P_2 and the remaining $2k-n$ components are isolated vertices. These $n-k$ paths are v_1,v_{k+1}; v_2,v_{k+2}; $\dots$; v_{n-k},v_n and the remaining isolated vertices are $v_{n-k+1},v_{n-k+2},\dots,v_k$.

Case (2): Let $|D|=2$.

Then $D=\{r,l\}$, where $1\leqslant r<l\leqslant n-1$. If $r=1$, then $v_1,v_2,\dots,v_{l+1},v_1$ is a cycle in $G(P_n,D)$. If $l<n-1$, then this cycle is a proper subgraph of $G(P_n,D)$. If $l=n-1$, then $G(P_n,D)$ becomes this cycle. Note that if $l=n-1$, then v_1 and v_n are adjacent in $G(P_n,D)$, so $G(P_n,D)$ is a cycle only when $r=1$.

So hereafter, we assume that $r\neq 1$ and $l<n-1$. Now we take n to be an even integer. The arguments given below also holds when n is an odd integer, if we replace $\frac{n}{2}$ by $\lceil\frac{n}{2}\rceil$. We need to consider the following subcases:

Subcase (2a): Let $r+l<n$.

Clearly $r<\frac{n}{2}$. There are two possibilities.

(i): Let $r<\frac{n}{2}$ and $l<\frac{n}{2}$. Then v_r,v_{n-r},v_{l-1} are adjacent to $v_{\frac{n}{2}}$ in $G(P_n,D)$ and so $d_{G(P_n,D)}(v_{\frac{n}{2}})\geqslant 3$. Therefore, $G(P_n,D)$ is neither a path nor a cycle.

(ii): Let $r<\frac{n}{2}$ and $l>\frac{n}{2}$. Then v_n,v_{n-2r},v_{n-r-l} are adjacent to v_{n-r} in $G(P_n,D)$ and so $d_{G(P_n,D)}(v_{n-r})\geqslant 3$. Therefore, $G(P_n,D)$ is neither a path nor a cycle.

Subcase (2b): Let $r+l>n$.

Subsubcase (2b)I: Let $r+l=n+1$.

Let m be the quotient when n is divided by r.

(i): Let $g.c.d(r,l) = k > 1$. Let $s = |\{a|1 \leqslant a \leqslant n, a \equiv i \mod k\}|$. Note that $s = \frac{n+1}{k}$. For each fixed $i, 1 \leqslant i \leqslant k-1$, let $x_1^i = i$ and define

$$x_t^i = \begin{cases} x_{t-1} + r & \text{if} \quad x_{t-1} + r \leqslant n \\ x_{t-1} - l & \text{if} \quad x_{t-1} + r > n \end{cases}$$

where $t = 2, 3, \ldots, s$. Then $v_{x_1^i}, v_{x_2^i}, v_{x_3^i}, \ldots, v_{x_s^i}, v_{x_1^i}$ form a cycle in $G(P_n, D)$. Let $y_1 = r$ and define

$$y_t = \begin{cases} y_{t-1} + r & \text{if} \quad y_{t-1} + r \leqslant n \\ y_{t-1} - l & \text{if} \quad y_{t-1} + r > n \end{cases}$$

where $t = 2, 3, \ldots, s$. Then $v_{y_1}, v_{y_2}, \ldots, v_{y_s}$ form a path on s vertices in $G(P_n, D)$. These $k-1$ cycles and the path are the only components of $G(P_n, D)$, since v_n is in the $k-1^{th}$ cycle and $a + (k-1)(a+1) = ak + k - 1 = n$, where a is the quotient when n is divided by k.

(ii): Let $g.c.d(r,l) = 1$.

Let $x_0 = 0$ and define $x_k = x_{k-1} + r$ if $x_{k-1} + r \leqslant n$, and $x_k = x_{k-1} - l$ if $x_{k-1} + r > n$ for $k = 1, \ldots, n$. Then $x_k = a_k r - b_k l$ for some $a_k, b_k \geqslant 0$. Suppose $x_k = x_{k'}$ for some $1 \leqslant k < k' \leqslant n$. Then $(a_{k'} - a_k)r = (b_{k'} - b_k)l$, $a_{k'} > a_k$, and $b_{k'} > b_k$. Since $\gcd(r,l) = 1$, r divides $b_{k'} - b_k$ and l divides $a_{k'} - a_k$. Hence $k' - k = a_{k'} - a_k + b_{k'} - b_k \geqslant r + l = n + 1$, which is a contradiction. This shows that $v_{x_1}, v_{x_2}, \ldots, v_{x_n}$ is a path with n distinct vertices, from which it follows that $G(P_n, D)$ is a path, as required.

Subsubcase (2b)II: Let $r + l > n + 1$.

Then there are three possibilities.

(i): Let $r = \frac{n}{2}$ and $l > \frac{n}{2} + 1$. Then $v_{\frac{n}{2}}, v_{\frac{n}{2}-1}, v_{\frac{n}{2}+1}$ are pendent vertices in $G(P_n, D)$, and so $G(P_n, D)$ is neither a path nor a cycle.

(ii): Let $r > \frac{n}{2}$ and $l > \frac{n}{2} + 1$. Then $v_{\frac{n}{2}}$ is an isolated vertex in $G(P_n, D)$, since $d_{P_n}(v_{\frac{n}{2}}, v_i) < d_{P_n}(v_{\frac{n}{2}}, v_n) = \frac{n}{2} < r < l$ for all $i = \frac{n}{2} + 1, \ldots, n-1$, and $d_{P_n}(v_{\frac{n}{2}}, v_i) < d_{P_n}(v_{\frac{n}{2}}, v_1) = \frac{n}{2} - 1 < r < l$ for all $i = 2, 3, \ldots, \frac{n}{2} - 1$. So $G(P_n, D)$ is neither a path nor a cycle.

(iii): Let $r < \frac{n}{2}$ and $l > \frac{n}{2} + 1$. Then v_r, v_l, v_{l-1} are isolated vertices in $G(P_n, D)$, and so $G(P_n, D)$ is neither a path nor a cycle.

Subcase (2c): Let $r + l = n$.

Let m be the quotient when n is divided by r.

(i): Let $g.c.d(r,l) = k > 1$. Let $s = |\{a|1 \leqslant a \leqslant n, a \equiv i \mod k\}|$. Note that $s = \frac{n}{k}$. For each fixed $i, 1 \leqslant i \leqslant k-1$, let $x_1^i = i$ and define

$$x_t^i = \begin{cases} x_{t-1} + r & \text{if} \quad x_{t-1} + r \leqslant n \\ x_{t-1} - l & \text{if} \quad x_{t-1} + r > n \end{cases}$$

where $t = 1, 2, \ldots, s$. Then $v_{x_1^i}, v_{x_2^i}, \ldots, v_{x_s^i}, v_{x_1^i}$ form a cycle in $G(P_n, D)$. These k cycles are the only components of $G(P_n, D)$, since each of these cycles are vertex disjoint and has $\frac{n}{k}$ vertices.

(ii): Let $g.c.d(r,l) = 1$. Let $x_0 = 0$ and

$$x_k = \begin{cases} x_{k-1} + r & if \quad x_{k-1} + r \leqslant n \\ x_{k-1} - l & if \quad x_{k-1} + r > n \end{cases}$$

for $k = 1, 2, \ldots, n$. Proceeding as in Subsubcase (2b)I(ii), we get $v_{x_1}, v_{x_2}, \ldots, v_{x_n}, v_{x_1}$ is a cycle in $G(P_n, D)$ with n distinct vertices. Hence $G(P_n, D)$ is a cycle.

Case (3): Let $|D| \geqslant 3$. Then $d_{G(P_n,D)}(v_1) \geqslant 3$, and so $G(P_n, D)$ is neither a path nor a cycle.

Combining all the above cases together completes the proof. □

THEOREM 2.2. *The graph $G(P_n, D)$ is regular if and only if $D = \{n_1, n - n_1, \ldots, n_r, n - n_r\}$ for some $r \geqslant 1$ and $n_1, \ldots, n_r \leqslant n/2$.*

PROOF. We divide the proof into several cases.

Case (1): Let $|D| = 1$.

Let $D = \{k\}$. Then v_1 and v_n are pendant vertices in $G(P_n, D)$.

Subcase (1a): Let n be even.

If $k < \frac{n}{2}$, then $d_{G(P_n,D)}(v_{\frac{n}{2}}) = 2$, and so $G(P_n, D)$ is not regular. If $k > \frac{n}{2}$, then $d_{G(P_n,D)}(v_{\frac{n}{2}}) = 0$, and so $G(P_n, D)$ is not regular. If $k = \frac{n}{2}$, then $G(P_n, D) \cong \frac{n}{2}K_2$, and so $G(P_n, D)$ is regular.

Subcase (1b): Let n be odd.

If $k < \lceil \frac{n}{2} \rceil$, then $d_{G(P_n,D)}(v_{\lceil \frac{n}{2} \rceil}) = 2$, and so $G(P_n, D)$ is not regular. If $k \geqslant \lceil \frac{n}{2} \rceil$, then $d_{G(P_n,D)}(v_{\lceil \frac{n}{2} \rceil}) = 0$, and so $G(P_n, D)$ is not regular.

Case (2): Let $|D| = 2$.

Let $D = \{r, l\}$. Then v_1 and v_n are of degree 2 in $G(P_n, D)$. So $G(P_n, D)$ is regular only when $d_{G(P_n,D)}(v_i) = 2$, for all $i = 1, 2, \ldots n$. By the Case (2) in the proof of Theorem 2.1, it follows that $G(P_n, D)$ is regular only when $r + l = n$.

Case (3): Let $|D| \geqslant 3$.

Subcase (3a): Let $D = \{n_1, n_2, \ldots, n_r \mid r \geqslant 3, n_i + n_j \neq n$ for all i, j and $i \neq j\}$.

We assume that $n_1 < n_2 < \cdots < n_r$. Clearly $d_{G(P_n,D)}(v_1) = r$. We have to consider the following cases:

(i): Let $n_i + n_j > n$ for all $i, j = 1, 2, \ldots, r$. Then $d_{G(P_n,D)}(v_{n-n_1}) = 2$, since v_{n-n_1} is adjacent to v_n and v_{n-2n_1} in $G(P_n, D)$, so $G(P_n, D)$ is not regular.

(ii): Let $n_i + n_j < n$ for all $i, j = 1, 2, \ldots, r$. Then $d_{G(P_n,D)}(v_{n-n_1}) = r + 1$, since v_{n-n_1} is adjacent to $v_n, v_{n-2n}, v_{n_1+n_2}, \ldots v_{n_1+n_r}$ in $G(P_n, D)$, this implies that $G(P_n, D)$ is not regular.

(iii): Let $n_i + n_j < n$ for some $i, j \in \{1, 2, \ldots, r\}$ and $n_s + n_t > n$ for some $s, t \in \{1, 2, \ldots, r\}$. Then $d_{G(P_n,D)}(v_{n-n_1}) = k + 1$, where $k = |\{n_s, n_t \in D \mid n_s + n_t < n\}|$, and so $G(P_n, D)$ is not regular.

Subcase (3b): $D = \{n_1, n - n_1, n_2, n - n_2, \ldots, n_r, n - n_r \mid r \geqslant 1\}$.

Then $G(P_n, D) = \bigcup_{i=1}^{r} G(P_n, \{n_i, n - n_i\})$. By Subcase of $2(c)$ in the proof of Theorem 2.1, for each $i = 1, 2, 3, \ldots, r$, $G(P_n, \{n_i, n - n_i\})$ is the disjoint union of cycles. It follows that $G(P_n, D)$ is regular of degree $2r$.

Subcase (3c): Let $D = \{n_1, n-n_1, n_2, n-n_2, \dots, n_r, n-n_r, a_1, a_2, \dots, a_k \mid r \geqslant 1, k \geqslant 2, a_i + a_j \neq n$ for all i, j and $i \neq j; a_i \neq n_j, n-n_j$ for all $i = 1, \dots, k, j = 1, \dots, r\}$.

If n is even, further we assume that $n_i \neq \frac{n}{2}$ for all $i = 1, \dots, r$. Now let us assume that $a_1 < a_2 < \cdots < a_k$. Then $d_{G(P_n,D)}(v_1) = 2r + k$. We have to consider the following cases:

(i): Let $a_i + a_j > n$ for all $i, j = 1, 2, \dots, k$. Then $d_{G(P_n,D)}(v_{n-a_1}) = 2r + 1$, and so $G(P_n, D)$ is not regular.

(ii): Let $a_i + a_j < n$ for all $i, j = 1, 2, \dots, k$. Then $d_{G(P_n,D)}(v_{n-n_1}) = 2r + k + 1$, and so $G(P_n, D)$ is not regular.

(iii): Let $a_i + a_j > n$ for some $i, j \in \{1, 2, \dots, k\}$ and $a_s + a_t < n$ for some $s, t \in \{1, 2, \dots, k\}$. Then $d_{G(P_n,D)}(v_{n-n_1}) = 2r + m + 1$, where $m = |\{a_s, a_t \in D \mid a_s + a_t < n\}|$, and so $G(P_n, D)$ is not regular.

Subcase (3d): $D = \{n_1, n-n_1, n_2, n-n_2, \dots, n_r, n-n_r, l \mid r \geqslant 1, l \neq n_i, n-n_i$ for all $i\}$.

Clearly $d_{G(P_n,D)}(v_1) = 2r + 1$.

(i): Let n be even. If $l < \frac{n}{2}$, then $d_{G(P_n,D)}(v_{\frac{n}{2}}) = 2r + 2$, and so $G(P_n, D)$ is not regular. If $l > \frac{n}{2}$, then $d_{G(P_n,D)}(v_{\frac{n}{2}}) = 2r$, and so $G(P_n, D)$ is not regular. If $l = \frac{n}{2}$, then $G(P_n, D)$ is the union of edge disjoint cycles and $\frac{n}{2}K_2$, so it is a regular graph of degree $2r + 1$.

(ii): Let n be odd. If $l < \lceil \frac{n}{2} \rceil$, then $d_{G(P_n,D)}(v_{\lceil \frac{n}{2} \rceil}) = 2r + 2$, and so $G(P_n, D)$ is not regular. If $l \geqslant \lceil \frac{n}{2} \rceil$, then $d_{G(P_n,D)}(v_{\lceil \frac{n}{2} \rceil}) = 2r$, and so $G(P_n, D)$ is not regular.

Proof follows by combining all the above cases together. $\square$

Theorem 2.3. *$G(P_n, D)$ is a wheel if and only if $n = 4$ and $D = \{1, 2, 3\}$.*

Proof. We consider the following cases.

Case (1): If $|D| = 1$, then $d_{G(P_n,D)}(v_1) = 1 = d_{G(P_n,D)}(v_n)$, so $G(P_n, D)$ is not a wheel.

Case (2): If $|D| = 2$, then $d_{G(P_n,D)}(v_1) = 2 = d_{G(P_n,D)}(v_n)$, so $G(P_n, D)$ is not a wheel.

Case (3): Let $|D| = 3$.

Suppose $G(P_n, D)$ is a wheel. Then the degree of its central vertex is at most 6, since $|D| = 3$. Thus $n \leqslant 7$. Since the central vertex in $G(P_n, D)$ is adjacent to all the remaining vertices, it follows that $D = \{1, 2, 3\}$. Now it is easy to see that $G(P_n, D)$ is a wheel only when $n = 4$.

Case(4): Let $|D| \geqslant 4$, then $d_{G(P_n,D)}(v_1) \geqslant 4$, and $d_{G(P_n,D)}(v_n) \geqslant 4$, so $G(P_n, D)$ is not a wheel.

The proof follows by combining all the above cases together. $\square$

Theorem 2.4. *Let $n \geqslant 2$ be an odd integer. Then $G(P_n, D)$ contains isolated vertices if and only if $k \in D$ for some k, $\lceil \frac{n}{2} \rceil + 1 \leqslant k \leqslant n - 1$.*

Proof. We prove only the case where n is even as The case n is odd is similar. So, assume n is an even positive integer. Now let n be an even positive integer.

Case (1): Let $k \in D$ for some k, $1 \leqslant k \leqslant \frac{n}{2}$.
Then each vertex v_i is adjacent to either v_{i+k} or v_{i-k}, in $G(P_n, D)$, and so $G(P_n, D)$ has no isolated vertices.
Case (2): Let $k \in D$ for some k, $\frac{n}{2} + 1 \leqslant k \leqslant n-1$.
Then $v_{\frac{n}{2}+1}$ is an isolated vertex in $G(P_n, D)$, since $d_{P_n}(v_{\frac{n}{2}+1}, v_i) < \frac{n}{2}$ for all $i = 1, 2, \dots, n-1$.

The above two cases completes the proof. □

THEOREM 2.5. *$G(P_n, D)$ is C_r-free ($r \geqslant 3$) if and only if D does not contain the elements $a_1, a_2, \dots, a_{r-1}$ (a_i's not necessarily distinct), such that $\sum_{i=1}^{r-1} a_i \in D$, and for each $k = 2, 3, \dots, r-2$, $\sum_{s=1}^{k} a_{i_s} \notin D$, $i_s \in \{1, 2, \dots, r-1\}$.*

PROOF. Suppose D contains the elements $a_1, a_2, \dots, a_{r-1}$, such that $\sum_{i=1}^{r-1} a_i \in D$, and for each $k = 2, 3, \dots, r-2$, $\sum_{s=1}^{k} a_{i_s} \notin D$, $i_s \in \{1, 2, \dots, r-1\}$. Then

$$v_1, v_{a_1+1}, v_{a_1+a_2+1}, \dots,\ v_{a_1+a_2+\cdots+a_r+1}, v_1$$

is an induced cycle in $G(P_n, D)$. Conversely, suppose that $G(P_n, D)$ contains C_r as an induced subgraph, let it be $C : v_{i_1}, v_{i_2}, \dots, v_{i_r}, v_{i_1}$. Take

$$a_t := d_{G(P_n,D)}(v_{i_t}, v_{i_{t+1}}), t = 1, 2, \dots, r-1$$

and

$$a_r := d_{G(P_n,D)}(v_{i_k}, v_{i_1}).$$

Then D contains the elements $a_1, a_2, a_3, \dots, a_r$. Also they satisfies the conditions $\sum_{i=1}^{r-1} a_i = a_r \in D$, and for each $k = 2, 3, \dots, r-2$, $\sum_{s=1}^{k} a_{j_s} \notin D$, $j_s \in \{1, 2, \dots, r-1\}$, for otherwise, we get an induced cycle

$$v_{i_1}, v_{a_{j_1}+i_1}, v_{a_{j_1}+a_{j_2}+i_1}, \dots, v_{a_{j_1}+a_{j_2}+\cdots+a_{j_k}+i_1}, v_{i_1}$$

for some k, $3 \leqslant k \leqslant r-2$, as a subgraph of C, which contradicts to our assumption that C is an induced cycle. Hence the proof. □

By a similar argument used in the proof of Theorem 2.5, we get the following result.

COROLLARY 2.1. *$G(P_n, D)$ is acyclic if and only if D does not contain the elements $a_1, a_2, \dots, a_r$ (a_i's are not necessarily distinct) such that $\sum_{i=1}^{r} a_i \in D$.*

THEOREM 2.6. *$G(P_n, D)$ is $K_{1,r}$-free ($r \geqslant 2$) if and only if D does not contain the elements $a_1, a_2, \dots, a_r$ such that $|a_i - a_j| \notin D$ for all $i, j \in \{1, 2, \dots, r\}$.*

PROOF. Suppose D contains the elements $a_1, a_2, a_3, \ldots, a_r$, such that $|a_i - a_j| \notin D$ for all $i, j \in \{1, 2, \ldots, r\}$. Without loss of generality, we may take $a_1 < a_2 < \cdots < a_r$. Then the vertex v_1 is adjacent to the vertices $v_{a_1+1}, v_{a_2+1}, \ldots, v_{a_r+1}$ in $G(P_n, D)$. Also the distance between any two of these vertices does not belong to D, since $|a_i - a_j| \notin D$ for all i, j. It follows that $G(P_n, D)$ contains $K_{1,r}$ as an induced subgraph. Conversely, assume that $G(P_n, D)$ contains $K_{1,r}$ as an induced subgraph. Let v be its central vertex, and $v_{i_1}, v_{i_2}, \ldots, v_{i_r}$ be its remaining vertices. Without loss of generality, we may assume that $i_1 < i_2 < \cdots < i_r$. For each t, $1 \leqslant t \leqslant r$, let $a_t := d_{P_n}(v, v_{i_t})$. Then a_t, $1 \leqslant t \leqslant r$ are all elements of D, and $|a_i - a_j| \notin D$ for all $i, j \in \{1, 2, \ldots, r\}$. Hence the proof. $\square$

THEOREM 2.7. *$G(P_n, D)$ is bipartite if and only if D does not contain the elements $a_1, a_2, \ldots, a_{2r}$, ($1 \leqslant r \leqslant \lceil \frac{n-1}{2} \rceil$) such that $\sum_{i=1}^{2r} a_i \notin D$*

PROOF. Suppose D contains elements $a_1, a_2, \ldots, a_{2r}$ ($1 \leqslant r \leqslant \lceil \frac{n-1}{2} \rceil$), such that $\sum_{i=1}^{2r} a_i \in D$. Then the cycle $v_1, v_{a_1+1}, v_{a_1+a_2+1}, \ldots, v_{a_1+a_2+\cdots+a_{2r}+1}, v_1$, is of odd length in $G(P_n, D)$, since $\sum_{i=1}^{2r} a_i \in D$, and P_n is a path, so this implies the existence of the vertex $v_{a_1+a_2+\cdots+a_{2r}+1}$ in P_n. Hence $G(P_n, D)$ is not bipartite. Conversely, suppose $G(P_n, D)$ is not bipartite. Then it has an odd cycle, let it be $C : v_{i_1}, v_{i_2}, \ldots, v_{i_k}, v_{i_1}$. Take $a_t := d_{G(P_n,D)}(v_{i_t}, v_{i_{t+1}}), t = 1, 2, \ldots, k-1$ and $a_k := d_{G(P_n,D)}(v_{i_k}, v_{i_1})$. Then $a_i \in D$ for all $i = 1, 2, \ldots, k$ and since P_n is a path, $\sum_{t=1}^{k-1} a_{i_t} = a_k \in D$. Note that, since C is an odd cycle, we have $k-1$ is even. This completes the proof. $\square$

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Department of Mathematics, The Gandhigram Rural Institute–Deemed University, Gandhigram, Tamilnadu, India
E-mail address: `rrajmaths@yahoo.co.in`
E-mail address: `celineprabha.s@gmail.com`