BULLETIN OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4874, ISSN (o) 2303-4955
www.imvibl.org /JOURNALS / BULLETIN
Vol. 8(2018), 279-285
DOI: 10.7251/BIMVI1802279U

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

FIXED POINT THEOREM FOR INTEGRAL TYPE CONTRACTION QUASI b - METRIC SPACE

Mujeeb Ur Rahman and Muhammad Sarwar*

ABSTRACT. In this paper, we introduce contractive conditions of integral type in the setting of dislocated quasi b-metric spaces. Using contractive conditions of integral type, we have presented a fixed point theorem in the framework of dislocated quasi b-metric spaces. Our established result generalize and extend various fixed point theorems of the literature in the context of dislocated quasi b-metric spaces. An example is given in the support of our main results.

1. Introduction

The first important theorem on fixed point for contraction mapping in complete metric space was published by Banach [**5**] in 1922. After this classical result this principle has been generalized by various researchers in different types of distance spaces for different type of contraction conditions (see [**3**], [**11**], [**7**], [**8**], [**9**], [**2**], [**10**]) etc.

The aim of this work is to analyze the existence of fixed point for a mapping satisfying general type of contractive condition of integral type in complete dislocated quasi b-metric spaces. Our main results extend and generalize some existing fixed point results. At the end of the paper some remarks and an example concerning such a type of contractive conditions are given.

2. Preliminaries

Throughout this paper $\mathbb{R}^+$ represent the set of non-negative real numbers.

We need the following definitions which may be found in [**7**].

2010 *Mathematics Subject Classification.* Primary 47H10, Secondary 54H25.

Key words and phrases. Complete dislocated quasi b-metric space, self-mapping, Cauchy sequence, fixed point.

$*$ Corresponding Author.

DEFINITION 2.1. Let X be a nonempty set and $k \geqslant 1$ be a real number then a mapping $d : X \times X \to [0, \infty)$ is called dislocated quasi b-metric if $\forall\ x, y, z \in X$

(d_1) $d(x, y) = d(y, x) = 0$ implies that $x = y$;

(d_2) $d(x, y) \leqslant k[d(x, z) + d(z, y)]$.

The pair (X, d) is called dislocated quasi-b-metric space or shortly (dq b-metric) space.

REMARK 2.1. In the definition of dislocated quasi-b-metric space if $k = 1$ then it becomes (usual) dislocated quasi-metric space. Therefore every dislocated quasi metric space is dislocated quasi b-metric space and every b-metric space is dislocated quasi-b-metric space with same coefficient k and zero self distance. However, the converse is not true as clear from the following example.

EXAMPLE 2.1. Let $X = \mathbb{R}$ and suppose

$$d(x, y) = |2x - y|^2 + |2x + y|^2.$$

Then (X, d) is a dislocated quasi-b-metric space with the coefficient $k = 2$. But it is not dislocated quasi-metric space nor b-metric space.

REMARK 2.2. Like dislocated quasi-metric space in dislocated quasi-b-metric space the distance between similar points need not to be zero necessarily as clear from the above example.

DEFINITION 2.2. A sequence $\{x_n\}$ is called dq-$b-$convergent in (X, d) if for $n \in N$ we have $\lim\limits_{n\to\infty} d(x_n, x) = 0$. Then x is called the dq-b-limit of the sequence $\{x_n\}$.

DEFINITION 2.3. A sequence $\{x_n\}$ in dq-b-metric space (X, d) is called Cauchy sequence if for $\epsilon > 0$ there exists $n_0 \in N$, such that for $m, n \geqslant n_0$ we have $d(x_m, x_n) < \epsilon$ (OR) $\lim\limits_{m,n\to\infty} d(x_m, x_n) = 0$.

DEFINITION 2.4. A dq-b-metric space (X, d) is said to be complete if every Cauchy sequence in X converges to a point of X.

DEFINITION 2.5. Let (X, d_1) and (Y, d_2) be two dq-b-metric spaces. A mapping $T : X \to Y$ is said to be continuous if for each $\{x_n\}$ which is dq-b convergent to x_0 in X, the sequence $\{Tx_n\}$ is dq-b convergent to Tx_0 in Y.

LEMMA 2.1. *Let (X, d) be a dq b-metric space and $\{x_n\}$ be a sequence in dq b-metric space such that*

$$\int\limits_0^{d(x_n,x_{n+1})} \rho(t)dt \leqslant \alpha \int\limits_0^{d(x_{n-1},x_n)} \rho(t)dt \tag{2.1}$$

for $n = 1, 2, 3, \ldots$ and $0 \leqslant \alpha < 1$, with $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesque integrable mapping which is summable on each compact subset of $\mathbb{R}^+$, non-negative and such that for any $s > 0$ $\int\limits_0^s \rho(t)dt > 0$. Then $\{x_n\}$ is a Cauchy sequence in X.

Proof. Let for $n, m > 0$ and $m > n$. Taking into account condition (d_2) of the Definition 2.1 and property of Lebesgue integration we have

$$\begin{aligned}\int_0^{d(x_n,x_m)} \rho(t)dt &\leqslant \int_0^{k[d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})]} \rho(t)dt \cdots \\ &\leqslant \int_0^{kd(x_n,x_{n+1})} \rho(t)dt + \int_0^{k^2[d(x_{n+1},x_{n+2})+d(x_{n+2},x_{n+3})]} \rho(t)dt \cdots \\ &\leqslant \int_0^{kd(x_n,x_{n+1})} \rho(t)dt + \int_0^{k^2 d(x_{n+1},x_{n+2})} \rho(t)dt + \int_0^{k^3 d(x_{n+2},x_{n+3})} \rho(t)dt \cdots\end{aligned}$$

Now using (2.1) we get the following

$$\begin{aligned}\int_0^{d(x_n,x_m)} \rho(t)dt &\leqslant \alpha^n \int_0^{kd(x_0,x_1)} \rho(t)dt + \alpha^{n+1} \int_0^{k^2 d(x_0,x_1)} \rho(t)dt + \alpha^{n+3} \int_0^{k^3 d(x_0,x_1)} \rho(t)dt + \cdots \\ &\leqslant \alpha^n(1+\alpha+\alpha^2+\cdots) \int_0^{(1+k+k^2+\cdots)kd(x_0,x_1)} \rho(t)dt. \\ &= \alpha^n \frac{1}{1-\alpha} \int_0^{\left(\frac{k}{1-k}\right)d(x_0,x_1)} \rho(t)dt.\end{aligned}$$

Taking limit $m, n \to \infty$ we have

$$\lim_{m,n\to\infty} d(x_n, x_m) = 0.$$

Hence $\{x_n\}$ is a Cauchy sequence in dislocated quasi b-metric space X. □

The following simple but important results can be seen in [**7**].

Lemma 2.2. *Limit in dq b-metric space is unique.*

Theorem 2.1. *Let (X, d) be a complete dq b-metric space $T : X \to X$ be a contraction. Then T has a unique fixed point.*

Branciari [**2**] proved the following theorem in complete metric spaces.

THEOREM 2.2. *Let (X,d) be a complete metric space for $\alpha \in (0,1)$. Let $T : X \to X$ be a mapping such that for all $x,y \in X$ satisfying*

$$\int_0^{d(Tx,Ty)} \rho(t)dt \leqslant \alpha \cdot \int_0^{d(x,y)} \rho(t)dt.$$

Where $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesque integrable mapping which is summable on each compact subset of $\mathbb{R}^+$, non-negative and such that for any $s > 0$ $\int_0^s \rho(t)dt > 0$. Then T has a unique fixed point in X.

3. Main Results

THEOREM 3.1. *Let (X,d) be a complete dislocated quasi b-metric space, for $a,b,c,e,f \geqslant 0$ with $\frac{a+b}{1-(c+e+f)} < \frac{1}{k}$, where $k \geqslant 1$ and let $T : X \to X$ be a continuous self-mapping such that for all $x,y \in X$ satisfying the condition*

$$\int_0^{d(Tx,Ty)} \rho(t)dt \leqslant a \cdot \int_0^{d(x,y)} \rho(t)dt + b \cdot \int_0^{d(x,Tx)} \rho(t)dt + c \cdot \int_0^{d(y,Ty)} \rho(t)dt +$$

$$e \cdot \int_0^{\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}} \rho(t)dt + f \cdot \int_0^{\frac{d(x,Ty)d(y,Ty)}{k[d(x,y)+d(y,Ty)]}} \rho(t)$$

where $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesque integrable mapping which is summable on each compact subset of $\mathbb{R}^+$, non-negative and such that for any $s > 0$ $\int_0^s \rho(t)dt > 0$. Then T has a unique fixed point.

PROOF. Let x_0 be arbitrary in X we define a sequence $\{x_n\}$ in X defined as follows

$$x_0, x_1 = Tx_0, \cdots, x_{n+1} = Tx_n.$$

To show that $\{x_n\}$ is a Cauchy sequence in X. Consider

$$\int_0^{d(x_n,x_{n+1})} \rho(t)dt = \int_0^{d(Tx_{n-1},Tx_n)} \rho(t)dt$$

By given condition in the theorem we have

$$\leqslant a \cdot \int_0^{d(x_{n-1},x_n)} \rho(t)dt + b \cdot \int_0^{d(x_{n-1},Tx_{n-1})} \rho(t)dt + c \cdot \int_0^{d(x_n,Tx_n)} \rho(t)dt +$$

$$e \cdot \int_0^{\frac{d(x_n,Tx_n)[1+d(x_{n-1},Tx_{n-1})]}{1+d(x_{n-1},x_n)}} \rho(t)dt + f \cdot \int_0^{\frac{d(x_{n-1},Tx_n)d(x_n,Tx_n)}{k[d(x_{n-1},x_n)+d(x_n,Tx_n)]}} \rho(t)dt.$$

Using the definition of the defined sequence we have

$$\leqslant a\cdot\int_0^{d(x_{n-1},x_n)}\rho(t)dt+b\cdot\int_0^{d(x_{n-1},x_n)}\rho(t)dt+c\cdot\int_0^{d(x_n,x_{n+1})}\rho(t)dt+$$
$$e\cdot\int_0^{\frac{d(x_n,x_{n+1})[1+d(x_{n-1},x_n)]}{1+d(x_{n-1},x_n)}}\rho(t)dt+f\cdot\int_0^{\frac{d(x_{n-1},x_{n+1})d(x_n,x_{n+1})}{k[d(x_{n-1},x_n)+d(x_n,x_{n+1})]}}\rho(t)dt.$$

Simplification yields

$$\leqslant a\cdot\int_0^{d(x_{n-1},x_n)}\rho(t)dt+b\cdot\int_0^{d(x_{n-1},x_n)}\rho(t)dt+$$
$$c\cdot\int_0^{d(x_n,x_{n+1})}\rho(t)dt+e\cdot\int_0^{d(x_n,x_{n+1})}\rho(t)dt+f\cdot\int_0^{d(x_n,x_{n+1})}\rho(t)dt$$
$$\int_0^{d(x_n,x_{n+1})}\rho(t)dt\leqslant\left(\frac{a+b}{1-(c+e+f)}\right)\cdot\int_0^{d(x_{n-1},x_n)}\rho(t)dt.$$

Let $h=\frac{a+b}{1-(c+e+f)}<\frac{1}{k}$, so the above inequality become

$$\int_0^{d(x_n,x_{n+1})}\rho(t)dt\leqslant h\cdot\int_0^{d(x_{n-1},x_n)}\rho(t)dt.$$

Hence by Lemma 2.1 $\{x_n\}$ is a Cauchy sequence in complete dq b-metric space. So there must exists $u\in X$ such that

$$\lim_{n\to\infty}x_n=u.$$

Since T is continuous so

$$Tu=\lim_{n\to\infty}Tx_n=\lim_{n\to\infty}x_{n+1}=u.$$

Thus u is the fixed point of T.

Uniqueness. If $u\in X$ is a fixed point of T. Then by given condition in the theorem we have

$$\int_0^{d(u,u)}\rho(t)dt=\int_0^{d(Tu,Tu)}\rho(t)dt$$
$$\int_0^{d(u,u)}\rho(t)dt\leqslant(a+b+c+e+f)\int_0^{d(u,u)}\rho(t)dt.$$

Since $a+b+c+e+f<1$, so the above inequality is possible if $d(u,u)=0$ similarly if $v\in X$ is the fixed point of T. Then we can show that $d(v,v)=0$. Now consider that u, v are two distinct fixed points of T then again by given condition in the theorem we have

$$\int_0^{d(u,v)} \rho(t)dt = \int_0^{d(Tu,Tv)} \rho(t)dt$$

$$\leqslant a \cdot \int_0^{d(u,v)} \rho(t)dt + b \cdot \int_0^{d(u,Tu)} \rho(t)dt + c \cdot \int_0^{d(v,Tv)} \rho(t)dt +$$

$$e \cdot \int_0^{\frac{d(v,Tv)[1+d(u,Tu)]}{1+d(u,v)}} \rho(t)dt + f \cdot \int_0^{\frac{d(u,Tv)d(v,Tv)}{d(u,v)+d(v,Tv)}} \rho(t)dt.$$

Now using the fact that u, v are fixed points of T and then simplifying We get the following inequality

$$\int_0^{d(u,v)} \rho(t)dt \leqslant a. \int_0^{d(u,v)} \rho(t)dt.$$

Since $a < 1$ so the a above inequality is possible only if $d(u, v) = 0$ similarly we can show that $d(v, u) = 0$ which implies that $u = v$. Hence fixed point of T is unique. □

Theorem (3.1) yields the following corollaries.

COROLLARY 3.1. . *Let (X, d) be a complete dislocated quasi b-metric space, with $a \in [0, 1)$ and $ak < 1$ where $k \geqslant 1$. Let $T : X \to X$ be a continuous self-mapping such that for all $x, y \in X$ satisfying the condition*

$$\int_0^{d(Tx,Ty)} \rho(t)dt \leqslant a \cdot \int_0^{d(x,y)} \rho(t)dt.$$

Where $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesque integrable mapping which is summable on each compact subset of $\mathbb{R}^+$, non-negative and such that for any $s > 0$ $\int_0^s \rho(t)dt > 0$. Then T has a unique fixed point.

COROLLARY 3.2. *Let (X, d) be a complete dislocated quasi b-metric space, for $a, b, c \geqslant 0$, with $ka + kb + c < 1$ and $k \geqslant 1$. Let $T : X \to X$ be a continuous self-mapping such that for all $x, y \in X$ satisfying the condition*

$$\int_0^{d(Tx,Ty)} \rho(t)dt \leqslant a \cdot \int_0^{d(x,y)} \rho(t)dt + b \cdot \int_0^{d(x,Tx)} \rho(t)dt + c \cdot \int_0^{d(y,Ty)} \rho(t)dt.$$

Where $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesque integrable mapping which is summable on each compact subset of $\mathbb{R}^+$, non-negative and such that for any $s > 0$ $\int_0^s \rho(t)dt > 0$. Then T has a unique fixed point.

Corollary 3.3. *Let (X,d) be a complete dislocated quasi b-metric space, for $a,c,e \geqslant 0$, with $ka+c+e<1$ and $k \geqslant 1$. Let $T: X \to X$ be a continuous self-mapping such that for all $x,y \in X$ satisfying the condition*

$$\int_0^{d(Tx,Ty)} \rho(t)dt \leqslant a \cdot \int_0^{d(x,y)} \rho(t)dt + c \cdot \int_0^{d(y,Ty)} \rho(t)dt + e \cdot \int_0^{\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}} \rho(t)dt$$

Where $\rho: \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesque integrable mapping which is summable on each compact subset of $\mathbb{R}^+$, non-negative and such that for any $s>0$ $\int_0^s \rho(t)dt > 0$. Then T has a unique fixed point.

Corollary 3.4. . *Let (X,d) be a complete dislocated quasi b-metric space, for $a,f \geqslant 0$, with $ka+f<1$ and $k \geqslant 1$. Let $T: X \to X$ be a continuous self-mapping such that for all $x,y \in X$ satisfying the condition*

$$\int_0^{d(Tx,Ty)} \rho(t)dt \leqslant a \cdot \int_0^{d(x,y)} \rho(t)dt + f \cdot \int_0^{\frac{d(x,Ty)d(y,Ty)}{k[d(x,y)+d(y,Ty)]}} \rho(t)dt.$$

Where $\rho: \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesque integrable mapping which is summable on each compact subset of $\mathbb{R}^+$, non-negative and such that for any $s>0$ $\int_0^s \rho(t)dt > 0$. Then T has a unique fixed point.

Remark 3.1. We have the following remarks from the above corollaries.

- Theorem 3.1 generalize the result of Mujeeb and Sarwar [**9**] in complete dislocated quasi b-metric space.
- In Corollary 3.1 if $\rho(t) = I$. Then we get the result of Mujeeb and Sarwar [**7**].
- In Corollary 3.2 and 3.3 if $\rho(t) = I$. Then our established results generalize the result of Aage and Salunke [**1**], Muraliraj and Hussain [**6**] and Kohli et al. [**4**] respectively in dislocated quasi b-metric space.

Remark 3.2. We have used the idea of contractive mappings of integral type to generalize the result of [**7**]. But in similar manner we can generalize other results for a single and a pair of mappings related to contractive condition of same kind, such is contained in ([**7**], [**1**], [**6**], [**4**]).

Example 3.1. Let $X = R$ and the complete dq b-metric defined on X is given by $d(x,y) = |2x-y|^2 + |2x+y|^2$ with self-mapping defined on X is $Tx = \frac{x}{2}$ and $\rho(t) = \frac{t}{2}$. Then

$$\int_0^{d(Tx,Ty)} \rho(t)dt = \int_0^{|x-\frac{y}{2}|^2+|x+\frac{y}{2}|^2} \frac{t}{2}dt = \int_0^{\frac{1}{4}(|2x-y|^2+|2x+y|^2)} \frac{t}{2}dt.$$

Integrating with respect to t and applying limits we have

$$\int_0^{d(Tx,Ty)} \rho(t)dt = \frac{1}{64}\left(|2x-y|^2 + |2x+y|^2\right)^2 \leqslant \frac{1}{4}\int_0^{d(x,y)} \rho(t)dt.$$

Satisfy all the conditions of the Corollary 3.1 for $a \in [\frac{1}{4}, 1)$ having $x = 0$ is the unique fixed point.

Acknowledgement. The author is grateful to the editor in chief and anonymous referees for there valuable comments and suggestions which improve this article.

References

[1] C. T. Aage and J. N. Salunke. The results of fixed points in dislocated and dislocated quasi-metric space. *Appl. Math. Sci.*, **2**(59)(2008), 2941-2948.

[2] A. Branciari. A fixed point theorem for mappings satisfying general contractive condition of integral type. *Int. J. Math. Math. Sci.*, **29**(9)(2002), 531-536.

[3] P. Hitzler. *Generalized metrics and topology in logic programming semantics*. Ph.D Thesis, National Univeristy of Ireland, University College Cork, 2001.

[4] M. Kohli, R. Shrivastava and A.D. Singh. Some results on fixed point theorems in dislocated quasi-metric space. *International Journal of Theoretical and Applied Sciences*, **2**(1)(2010), 27 - 28.

[5] E. Kresizing. *Introductory to functional analysis with application*. John Willey and sons,1978 .

[6] A. Muraliraj and R. J. Hussain. New results on fixed point theorems in dislocated quasi-metric spaces. *Int. J. Math. Sci. Appl.*, **3**(1)(2013), 321-326.

[7] M. U. Rahman and M. Sarwar. Dislocated quasi b-metric space and fixed pont theorems. *Elec. J. Math. Anal. Appl.*, **4**(2)(2016), 16 - 24.

[8] M. U. Rahman and M. Sarwar. Fixed point theorems in generalized types of b-dislocated metric spaces. *Elec. J. Math. Anal. Appl.*, **5**(2)(2017), 10 - 17.

[9] M. U. Rahman and M. Sarwar. Some fixed point theorems satisfying contractive conditions of integral type in dislocated quasi-metric space. *J. Anal. Num. Theory*, **3**(2)(2015), 97 - 102.

[10] M. Sarwar, M.U. Rahman and G. Ali. Some fixed point results in dislocated quasi-metric (dq-metric) spaces. *J. Ineq. Appl.*, **2014**, 2014:278.

[11] F. M. Zeyada, G. H. Hassan and M. A. Ahmed. A generalization of fixed point theorem due to Hitzler and Seda in dislocated quasi-metric space. *The Arabian Journal for Science and Engineering A*, , **31**(1A)(2006), 111-114.

Receibed by editors 29.03.2017; Revised version 24.04.2017: Second revised version 15.09.2017; Available online 04.12.2017.

Department of Mathematics, Government PG Jehanzeb College, Saidu Sharief, Swat, Pakistan.
E-mail address: mujeeb846@yahoo.com

Department of Mathematics, University of Malakand, Chakdara Dir(L), Pakistan.
E-mail address: sarwarswati@gmail.com