

ON DOMINATION OF A VERTEX AS EXPONENTIALLY

Aysun Aytaç and Betül Atay

ABSTRACT. The assessment of the stability and the vulnerability of complex networks is the important concept while studying of them. In the literature, a sort of measures have been given in order to assessment the stability of systems. And also, some graph-theoretic parameters have been formed to provide formulas which calculate the reliability of a network. In this paper, we study the vulnerability of some cycles and related graphs to the dominating strategy of a vertex, using a measure called *exponential domination number* which is a new characteristic for graph vulnerability.

1. Introduction

The domination in graphs is one of the concepts in graph theory which has attracted many researchers to work on it because of its many and varied applications in such fields as linear algebra and optimization, design and analysis of communication networks, and social sciences and military surveillance. Many variants of dominating models are available in the existing literature [10]. The present paper is focused on exponential domination number in graphs. There have been several domination parameters including domination number, edge domination number, independent domination number, strong-weak domination number, distance domination number etc. Domination in graphs is a well-studied concept in graph theory. Domination based parameters reveal an underlying efficient communication network in which a vertex can affect all its neighborhood vertices in some sense. In real life applications, we can encounter that a vertex can affect both its neighborhood vertices and all vertices within a given distance. Distance domination is a kind of this situation. There has been no framework yet in which the effect of a vertex broadens beyond its neighborhood while decreasing by distance. It has

2010 *Mathematics Subject Classification.* 05C40; 05C69; 68M10; 68R10 .

Key words and phrases. Graph vulnerability, Network design and communication, Domination, Exponential domination number, Stability, Communication network, Cycles.

been suggested [8] that exponential domination is a model for the reliability of the spread of information or gossip. In this model, the dominating strategy of a vertex decreases exponentially with a distance, by the factor $1/2$. Therefore, it is possible that a vertex v is dominated by one of its neighbors or by some vertices that are closer to v . The assumption is that gossip heard directly from a source is totally reliable, while gossip passed from person to person loses half its credibility with each individual in the chain. Finding the exponential domination number in this application amounts to determining the minimum number of sources needed so that each person gets fully reliable information.

We begin with simple, finite, connected, and undirected graph

$$G = (V(G), E(G))$$

of order n . The set $S \subseteq V(G)$ of vertices in a graph G is called a dominating set if every vertices $v \in V(G)$ are either an element of S or are adjacent to an element of S . A dominating set S is a minimal dominating set (or MDS) if no proper subset $S' \subset S$ is a dominating set.

The minimum cardinality of a dominating set of G is called the domination number of G which is denoted by $\gamma(G)$, and the corresponding dominating set is called a γ -set of G [13].

The degree of vertex v in graph G is the number of edges which are incident to v . The distance $d(u, v)$ between two vertices u and v in G is the length of a shortest path between them. If u and v are not connected, then $d(u, v) = \infty$, and for $u = v$, $d(u, v) = 0$. The diameter of G , denoted by $diam(G)$ is the largest distance between two vertices in $V(G)$ [4, 5].

In this article, we study the exponential domination number of a graph which is a new characteristic for graph vulnerability introduced in [8]. This parameter is a variation of distance domination in which, as described in the motivation already given, the dominating power radiating from a vertex declines exponentially with distance [14]. Let G be a graph and $S \subseteq V(G)$. We denote by $\langle S \rangle$ the subgraph of G induced by S . For each vertex $u \in S$ and for each $v \in V(G) - S$, we define $\bar{d}(u, v) = \bar{d}(v, u)$ to be the length of a shortest path in $\langle V(G) - (S - \{u\}) \rangle$ if such a path exists, and ∞ otherwise. Let $v \in V(G)$. The definition is

$$w_s(v) = \begin{cases} \sum_{u \in S} 1/2^{\bar{d}(u,v)-1}, & \text{if } v \notin S \\ 2, & \text{if } v \in S \end{cases}$$

We refer to $w_s(v)$ as the weight of S at v . If, for each, $v \in V(G)$, we have $w_s(v) \geq 1$, then S is an exponential dominating set. The smallest cardinality of an exponential dominating set is the exponential domination number, $\gamma_e(G)$, and such a set is a minimum exponential dominating set, or γ_e -set for short. If $u \in S$ and $v \in V(G) - S$ and

$$\frac{1}{2^{\bar{d}(u,v)-1}} > 1,$$

then we say that u exponentially dominates v . Note that if S is an exponential dominating set, then every vertex of $V(G) - S$ is exponentially dominated, but the converse is not true [8, 7].

Throughout this article, the largest integer not larger than x is denoted by $\lfloor x \rfloor$ and the smallest integer not smaller than x is denoted by $\lceil x \rceil$.

The paper proceeds as follows. In Section 2, the some known results are given. There are different classes of cycle related graphs that have been studied for variety of purposes. Fans, k -pyramid graphs, n -gon books and shadow graphs, claw-free graphs are among such graphs. The exponential domination number values for cycles and various classes of cycle related graphs are developed in Section 3, 4, 5 and 6. Finally, concluding remarks of this paper are given in Section 7.

2. Some Elementary Results

THEOREM 2.1 ([8]). For every positive integer n , $\gamma_e(P_n) = \lceil (n+1)/4 \rceil$.

THEOREM 2.2. ([8]) For every positive integer n ,

$$\gamma_e(C_n) = \begin{cases} 2, & \text{if } n = 4 \\ \lceil n/4 \rceil, & \text{if } n \neq 4 \end{cases}.$$

THEOREM 2.3 ([8]). If G is a connected graph of diameter d , then $\gamma_e(G) \geq \lceil (d+2)/4 \rceil$.

THEOREM 2.4 ([8]). If G is a connected graph of order n , then $\gamma_e(G) \leq \frac{2}{5}(n+2)$.

THEOREM 2.5 ([8]). Let G be a connected graph of order n and T a spanning tree of G . Then $\gamma_e(G) \leq \gamma_e(T)$.

THEOREM 2.6 ([8]). For every graph G , $\gamma_e(G) \leq \gamma(G)$. Also, $\gamma_e(G) = 1$ if and only if $\gamma(G) = 1$.

THEOREM 2.7. ([1], [3]) Let G_1 and G_2 be any two graphs. Let $(G_1 \circ G_2)$ and $(G_1 + G_2)$ be corona and join operations of G_1 and G_2 , respectively.

a) For any two graphs G_1 and G_2 , $\gamma_e(G_1 \circ G_2) \geq \lceil \frac{\text{diam}(G_1 \circ G_2)}{2} \rceil$.

b) Let G_1 and G_2 be any two graphs. If $\text{diam}(G_1) < \text{diam}(G_2)$, then $\gamma_e(G_1 + G_2) = \gamma_e(G_1)$.

THEOREM 2.8. ([2], [3]) Let G be any connected graph of order n and diameter 2. If G has not a vertex with degree $n-1$, then $\gamma_e(G) = 2$.

THEOREM 2.9. ([2], [3]) Let G be any connected graph of order n . If G has a vertex with degree $n-1$, then $\gamma_e(G) = 1$.

The aim of this paper is to give in the some of the cycle related graphs types results depending on exponential domination number as the vulnerability parameter.

There are different classes of cycle related graphs. Fans, k -pyramid graphs including bipyramids and wheels, n -gon books, shadow graphs and claw-free graphs are among such graphs. The exponential domination values for various classes of cycle related graphs are developed in the following section.

3. Claw-Free Graphs

Our aim in this section is to consider exponential domination number of connected claw-free graph. We will need the following family of graphs. For, $b \geq 2$ let $ecor(K_b)$ denote the graph obtained from $Cor(K_b)$. Given a graph G , the corona $Cor(G)$ is formed by adding for each vertex $v \in V(G)$, one extension vertex v' , and joining it to v . Then for $t \geq 1$ let $G_{b,t}$ be claw-free graph obtained from taking t copies of $ecor(K_b)$ and adding edges such that the extension vertices from a clique [15].

THEOREM 3.1. *Let $G_{b,t}$ be a connected claw-free graph with $t(2b + 1)$ vertices. Then, for $t \geq 8$ is $\gamma_e(G_{b,t}) = 8$.*

PROOF. The degree set of $G_{b,t}$ is $\{1, b + 1, b + t - 1\}$. Vertex set of $G_{b,t}$ can be partitioned into three vertex sets such as $V(G_{b,t}) = V_1^j(G_{b,t}) \cup V_2^j(G_{b,t}) \cup V_3^j(G_{b,t})$, where $j \in \{1, 2, \dots, t\}$ and $V_1^j(G_{b,t})$ is the vertex set of K_b in the copy j th, $V_2^j(G_{b,t})$ is the vertex set of $Cor(K_b)$ in the copy j th and the degree of every is one and $V_3^j(G_{b,t})$ is the vertex set of $ecor(K_b)$ in the copy j th and every vertex is extension vertex. These vertices constitute a complete graph K_t with t vertices.

It is clear that $\gamma_e(K_b) = 1$. Let S be γ_e -set of $G_{b,t}$. S must include the vertices of $V_1^j(G_{b,t})$. That is, $S = \{v | v \in V_1^j(G_{b,t})\}$. We have three cases depending on the shortest path between vertices of $G_{b,t}$ and vertices of S .

Case 1: We consider the distance between any vertex in S and any vertex in $V_1^j(G_{b,t})$. For u in $V_1^x(G_{b,t})$ and v in $V_1^y(G_{b,t})$, if $x = y$, then $d(u, v) = 1$, otherwise $d(u, v) = 3$.

Case 2: We consider the distance between any vertex in S and any vertex in $V_2^j(G_{b,t})$. For u in $V_1^x(G_{b,t})$ and v in $V_2^y(G_{b,t})$, if $x = y$, then $d(u, v) \leq 2$ otherwise $d(u, v) = 4$.

Case 3: We consider the distance between any vertex in S and any vertex in $V_3^j(G_{b,t})$: For u in $V_1^x(G_{b,t})$ and v in $V_3^y(G_{b,t})$, if $x = y$, then $d(u, v) = 1$ otherwise $d(u, v) = 2$.

If S has only one vertex from $V_1^j(G_{b,t})$, then it is easily to see that the condition $w_s(v) \geq 1$ for v in $G_{b,t}$ is not satisfied according to Case 1, 2 and 3 above. The minimum weight of v at S is at least $1/2^3$. Thus, this requires at least eight vertices from different copies of $V_1^j(G_{b,t})$ in S .

By the definition of the exponential domination number of a graph, it is trivial that $\gamma_e(G_{b,t}) = 8$.

The values of $\gamma_e(G_{b,t})$ for $2 < t \leq 7$ are in the following table.

TABLE 1. The values of $\gamma_e(G_{b,t})$ for $2 < t \leq 7$

t	2	3	4	5	6	7
$\gamma_e(G_{b,t})$	4	5	5	5	6	7

The proof is completed. □

4. n -gon books

When k copies of C_n ($n \geq 3$) share a common edge, it will form an n -gon book of k pages and is denoted by $B(n, k)$. The degree set of $B(n, k)$ is $\{2, k+1\}$. Therefore, the vertices of $B(n, k)$ are of two kinds: vertices of degree 2 will be referred to as minor vertices and vertices of degree $k+1$ to as major vertices [6].

THEOREM 4.1. *Let $B_{n,k}$ is an n -gon book of k pages with $k(n-2)+2$ vertices. Then, for $n \geq 5$*

$$\gamma_e(B_{n,k}) = \begin{cases} k(\lceil n/4 \rceil - 2) + 2, & \text{if } n \equiv_4 1 \\ k(\lceil n/4 \rceil - 1) + 1, & \text{otherwise} \end{cases}.$$

PROOF. There are k copies C_n in $B_{n,k}$. We know that $\gamma_e(C_n) = \lceil n/4 \rceil$ from the Theorem 2.2. Therefore, we must consider the minimum exponential dominating set of every cycle graph.

Let S be γ_e -set of $B_{n,k}$. Let u, v be vertices of S on C_n such that $[u, v] \cap S = \{u, v\}$, where $[u, v]$ is the set of all vertices of $B_{n,k}$ that lie on at least one $u-v$ shortest path. This implies that $d(u, v) \leq 4$. Hence, S must be constructed according to this rule. Let v_1, v_2 be vertices in $V(B_{n,k})$. Since the degree of them is $k+1$, these vertices are called the major vertices. If S must include vertices of only one cycle graph C_n , then we have three cases depending on whether or not the major vertices are in S .

Case 1: Let the major vertices v_1, v_2 be in S . The condition $w_s(v) \geq 1$ for v in $B_{n,k}$ is not satisfied. Therefore, we must add some vertices to S . Hence, the cardinality of S is

$$|S| = \begin{cases} \lceil n/4 \rceil, & \text{if } n \equiv_4 1 \\ \lceil n/4 \rceil + 1, & \text{otherwise} \end{cases}.$$

Case 2: Let S include only one major vertex. The condition $w_s(v) \geq 1$ for v in $B_{n,k}$ is not satisfied. Therefore, we must add some vertices to S . Hence, the cardinality of S is

$$|S| = \begin{cases} \lceil n/4 \rceil, & \text{if } n \equiv_4 1 \\ \lceil n/4 \rceil, & \text{otherwise} \end{cases}.$$

Case 3. Let S not include any major vertex. The condition $w_s(v) \geq 1$ for v in $B_{n,k}$ is not satisfied. Therefore, we must add some vertices to S . Hence, the cardinality of S is

$$|S| = \begin{cases} \lceil n/4 \rceil, & \text{if } n \equiv_4 1 \\ \lceil n/4 \rceil, & \text{otherwise} \end{cases}.$$

The calculations made in Case 1, 2 and 3 are valid only to one copy of C_n . Since $B_{n,k}$ includes k copies C_n , for $n \equiv_4 1$, we obtain $\gamma_e(B_{n,k}) = k(\lceil n/4 \rceil - 2) + 2$, $k(\lceil n/4 \rceil - 1) + 1$ and $k(\lceil n/4 \rceil)$, respectively.

By the definition of exponential domination number, we must determine the minimum value of S by Case 1, 2 and 3. Thus, we have

$$\begin{aligned} \gamma_e(B_{n,k}) &= \min\{k(\lceil n/4 \rceil - 2) + 2, k(\lceil n/4 \rceil - 1) + 1, k(\lceil n/4 \rceil)\} \\ &= k(\lceil n/4 \rceil - 2) + 2. \end{aligned}$$

Similarly for $n \not\equiv_4 1$, we obtain $\gamma_e(B_{n,k}) = k(\lceil n/4 \rceil - 1) + 2, k(\lceil n/4 \rceil - 1) + 1$ and $k(\lceil n/4 \rceil)$, respectively. By the definition of exponential domination number, we have

$$\begin{aligned} \gamma_e(B_{n,k}) &= \min\{k(\lceil n/4 \rceil - 1) + 2, k(\lceil n/4 \rceil - 1) + 1, k\lceil n/4 \rceil\} \\ &= k(\lceil n/4 \rceil - 1) + 1. \end{aligned}$$

Consequently, we obtain

$$\gamma_e(B_{n,k}) = \begin{cases} k(\lceil n/4 \rceil - 2) + 2, & \text{if } n \equiv_4 1 \\ k(\lceil n/4 \rceil - 1) + 1, & \text{otherwise} \end{cases}.$$

The proof is completed. □

5. Fan Graph and k -pyramid Graph

DEFINITION 5.1. A fan graph, denoted by $F_{m,n}$, is defined as the graph join, $\overline{K_m} + P_n$ where $\overline{K_m}$ is the isolated graph on m vertices and P_n is the path graph on n vertices [6, 9].

DEFINITION 5.2. The join of graphs G and H , denoted by $G \vee H$, is obtained from the disjoint union $G + H$ by adding the edges $\{xy|x \in V(G), y \in V(H)\}$. The join graph $C_n \vee N_k$ $n \geq 3, k \geq 1$, where N_k is the null graph of order k , is called k -pyramid and is denoted by $kP(n)$. The 2-pyramid graph $C_n \vee N_2$ is called bipyramid graph and is denoted by $BP(n)$. The 1-pyramid graph $C_n \vee N_1$ is the wheel graph W_n [6].

The following two theorems can be easily seen by Theorem 2.8 and Theorem 2.9.

THEOREM 5.1. *Let $kP(n)$ be a k -pyramid graph with $k + n$ vertices. Then for $k \geq 2, \gamma_e(kP(n)) = 2$.*

THEOREM 5.2. *Let $F_{m,n}$ be a fan graph with $m + n$ vertices and $n(m + 1) - 1$ edges. Then for $n \geq 4, \gamma_e(F_{m,n}) = 2$.*

The values of $\gamma_e(F_{m,n})$ for $n < 4$ are in the following table.

TABLE 2. The values of $\gamma_e(F_{m,n})$ for $n < 4$

n	2	3
$\gamma_e(F_{m,n})$	1	1

6. Shadow Graphs

The shadow graph of a connected graph G is constructed by taking two copies of G , say G' and G'' . Join each vertex u' in G' to the neighbors of the corresponding vertex u'' in G'' . Then the graph obtained is denoted as $D_2(G)$ [11].

THEOREM 6.1. *Let $D_2(P_n)$ be a shadow graph of P_n with $2n$ vertices. Then,*

$$\gamma_e(D_2(P_n)) = \begin{cases} 2\lfloor n/6 \rfloor + 1, & \text{if } n \equiv_6 0, 1 \\ 2\lceil n/6 \rceil, & \text{otherwise} \end{cases} .$$

PROOF. Let v'_i be vertices of P'_n and v''_i be vertices of P''_n corresponding to all vertices v_i , where $i \in \{1, 2, \dots, n\}$. $deg(v'_1) = deg(v''_1) = deg(v'_n) = deg(v''_n) = 2$ and $deg(v'_i) = deg(v''_i) = 4$, $i \in \{2, 3, \dots, n - 1\}$. Let S be γ_e -set. Because of degrees of vertices, S must include some vertices in P'_n and also corresponding vertices in P''_n . Let u', v' in P'_n and u'', v'' in P''_n be vertices of S on $D_2(P_n)$ such that $[u', v'] \cap S = \{u', v'\}$, $[u'', v''] \cap S = \{u'', v''\}$ and $u' = x'_0, x'_1, \dots, x'_k = v'$ the $u' - v'$ path in P'_n and $u'' = x''_0, x''_1, \dots, x''_k = v''$ the $u'' - v''$ path in P''_n . Thus, the vertices u', v', u'', v'' in S contribute $1/2^2$ to $w_s(x'_k)$ or $w_s(x''_k)$. This implies that $d(u', v') \leq 6$ and $d(u'', v'') \leq 6$. Hence, if $n \equiv_6 0, 1$, then the cardinality of S is $2\lfloor n/6 \rfloor$. The condition $w_s(v) \geq 1$ is not satisfied for every vertex v in $D_2(P_n)$. The number of undominated vertices on $D_2(P_n)$ is three or four. Hence, S must contain one more vertex. Therefore, we have $\gamma_e(D_2(P_n)) = 2\lfloor n/6 \rfloor + 1$. If $n \not\equiv_6 0, 1$, then the cardinality of S is $2\lceil n/6 \rceil$. Thus, every vertex v in $D_2(P_n)$ is exponentially dominated by S . Consequently, we have

$$\gamma_e(D_2(P_n)) = \begin{cases} 2\lfloor n/6 \rfloor + 1, & \text{if } n \equiv_6 0, 1 \\ 2\lceil n/6 \rceil, & \text{otherwise} \end{cases} .$$

The proof is completed. □

THEOREM 6.2. *Let $D_2(C_n)$ be a shadow graph of C_n with $2n$ vertices. Then,*

$$\gamma_e(D_2(C_n)) = \begin{cases} 2\lfloor n/6 \rfloor + 1, & \text{if } n \equiv_6 1, 2 \\ 2\lceil n/6 \rceil, & \text{otherwise} \end{cases} .$$

PROOF. The proof is similar to the proof of Theorem 6.1. □

THEOREM 6.3. *For any graph G , let $diam(G) \leq 2$ and $D_2(G)$ be a shadow graph of G . Then, $\gamma_e(D_2(G)) = 2$.*

PROOF. If G is a wheel, star, complete and complete bipartite graph, then $diam(G) \leq 2$ and $diam(D_2(G)) = 2$. The proof can be easily seen by Theorem 2.8 and Theorem 2.9. □

7. Conclusion

Here, we investigate some results about cycles and related graphs corresponding to the dominating strategy of a vertex. We study a new concept of the exponential domination number. Analogous work can be carried out for other graph families.

References

- [1] A. Aytaç and B. Atay. On Exponential Domination of Some Graphs. *Nonlinear Dynamics and Systems Theory*, **16**(1)(2016), 12–19.
- [2] A. Aytaç and B. Atay. Dominating Power in Some Graphs. Manuscript (to be submitted), 2017.
- [3] B. Atay. On the vulnerability concept of some graph structures. PhD thesis, Ege University, 2016.
- [4] G. Chartrand and L. Lesniak. *Graphs and Digraphs*, Second Edition, Wadsworth. Monterey, 1986.
- [5] F. Harary. *Graph Theory*, Addison-Wesley Publishing Company, 1972.
- [6] J. A. Gallian. A dynamic survey of graph labeling. *Elect. Jour. Combin.*, **15**(2016) DS6.
- [7] M. Anderson, R. C. Brigham, J. R. Carrington, R. P. Vitray and J. Yellen. On exponential domination of $C_m \times C_n$. *AKCE J. Graphs. Combin.*, **6**(3)(2009), 341–351.
- [8] P. Dankemann, D. Day, D. Erwin, S. Mukwembi and H. Swart. Domination with exponential decay. *Discrete Mathematics*, **309**(19)(2009), 5877–5883.
- [9] S. Intaja and T. Sitthiwiratham. Some graph parameters of fan graph. *Int. J. Pure and Appl. Math.*, **80**(2)(2012), 217–223.
- [10] S. T. Hedetniemi and R. C. Laskar. Bibliography on domination in graphs and some basic definitions of domination parameters. *Discrete Mathematics*, **86**(1-3)(1990), 257–277.
- [11] S. K. Vaidya and K. K. Kanani. Some cycle related product cordial graphs. *International Journal of Algorithms, Computing and Mathematics*, **3**(1)(2010), 109–116.
- [12] T.W. Haynes, S.T. Hedetniemi and P.J. Slater. *Domination in graphs, Advanced Topic*. Marcel Dekker, Inc, New York, 1998.
- [13] T.W. Haynes, S.T. Hedetniemi and P.J. Slater. *Fundamentals of Domination in Graphs*. Marcel Dekker, Inc, New York, 1998.
- [14] V. Aytaç and T. Turac. Exponential domination and bondage numbers in some graceful cyclic structure. *Nonlinear Dynamics and Systems Theory*, **17**(2)(2017), 139–149.
- [15] W. Goddard, M. A. Henning and C. A. McPillan. The disjunctive domination number of a graph. *Quaestiones Mathematicae*, **37**(4)(2014), 547-561.

Received by editors 25.04.2017; Revised version 20.11.2017; Available online 27.11.2017.

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, EGE UNIVERSITY, IZMIR, TURKEY
E-mail address: aysun.aytac@ege.edu.tr

DEPARTMENT OF COMPUTER AND INSTRUCTIONAL SCIENCES TECHNOLOGY, FACULTY OF EDUCATION, AĞRI İBRAHİM ÇEÇEN UNIVERSITY, AĞRI, TURKEY
E-mail address: batay@agri.edu.tr