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LEFT BI-QUASI IDEALS OF SEMIRINGS

Marapureddy Murali Krishna Rao

Abstract. In this paper, we introduce the notion of left (right) bi-quasi ideal and bi-quasi ideal of semiring which are generalizations of bi-ideal and quasi ideal of semiring. Also we study the properties of bi-quasi ideals, left bi-quasi ideals and characterize the left bi-quasi simple semring and regular semiring.

1. Introduction

Semiring, the algebraic structure which is a common generalization of rings and distributive lattices, was first introduced by American mathematician Vandiver [**15**] in 1934, but non-trivial examples of semirings had appeared in the earlier studies on the theory of commutative ideals of rings by German mathematician Richard Dedekind in 19th century. Semiring is a universal algebra with two binary operations called addition and multiplication where one of them is distributive over the other. Bounded distributive lattices are commutative semirings which are both additively and multiplicatively idempotent.A natural example of semiring which is not a ring, is the set of all natural numbers under usual addition and multiplication of numbers. The theory of rings and semigroups has considerable impact on the development of the theory of semirings. In algebraic structure, semirings lie between semigroups and rings. Additive and multiplicative structures of a semiring play important roles in determining the structure of a semiring. Semiring, as the basic algebraic structure was used in the areas of theoretical computer science as

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This paper is dedicated to my teacher Prof. K.L.N. Swamy (Former Rector, Andhra University, Visakhapatnam, Andhra Pradesh, INDIA.) on the occasion of his 75th birth day, .

well as in the solutions of graph theory, optimization theory. Semirings for studying automata, coding theory and formal languages. Semiring theory has many applications in other branches of mathematics.

The notion of ideals introduced by Dedekind for the theory of algebraic numbers was generalized by E. Noether for associative rings. The one and two sided ideals introduced by her,are still central concepts in ring theory.The notion of the bi-ideal in semigroups is a special case of (m,n) ideals introduced by Lajos [**9, 10**]. In 1952, the concept of bi-ideals for semigroup was introduced by Good and Hughes [**3**] and the notion of bi-ideals in associative rings was introduced by Lazos and Szasz [**11**]. Henriksen [**4**] studied ideals in semirings.Jagatap et al. cite7 investigated quasi ideals in Γ semirings, Ali et al. [**2**] studied properties of quasi ideals in semirings and Lajos et al. [**11**] investigated bi ideals in rings. An interesting particular case of the bi-ideal is the notion of quasi-ideals that was first introduced for semigroup and then for rings by Steinfeld [**13, 14**] in 1956. Iseki [**5, 6, 7**] studied ideals in semirings. The notion of quasi ideals is a generalization of left and right ideals whereas the bi-ideals are generalization of quasi-ideals. Marapureddy Murali Krishna Rao [**12**] introduced bi-quasi-ideals and fuzzy bi-quasi–ideals of $\Gamma-$semigroups.In this paper, we introduce the notion of bi-quasi ideal and left (right)bi-quasi ideal of semiring, which are generalizations of bi-ideal and quasi-ideal of semiring. We study the properties of left bi-quasi ideal and characterize the left bi-quasi simple semiring and regular semiring.

2. Preliminaries

In this section we recall some of the fundamental concepts, results and definitions which are necessary for this paper.

Definition 2.1. *An algebraic structure $(S, \cdot)$ consisting of a non-empty set S together with an associative binary operation $\cdot$ is called a semigroup.*

Definition 2.2. *An element a of semigroup S is called a regular element if there exists an element b of S such that $a = aba$.*

Definition 2.3. *A semigroup S is called a regular semigroup if every element of S is a regular element.*

Let A, B be subsets of a semigroup S. The product AB is defined as $AB = \{ab \mid a \in A \text{ and } b \in B\}$.

Definition 2.4. *Let S be a semigroup and T be a non-empty subset of S. Then*

(i) *T is called a subsemigroup if $TT \subseteq T$,*
(ii) *T is called a bi-ideal if $TT \subseteq T$ and $TST \subseteq T$,*
(iii) *T is called a quasi ideal if $ST \cap TS \subseteq T$,*
(iv) *T is called a left (right) ideal if $ST \subseteq T(TS \subseteq T)$,*
(v) *T is called an ideal if $ST \subseteq T$ and $TS \subseteq T$,*
(vi) *A subsemigroup T of S is called an interior ideal of S if $STS \subseteq T$.*

Definition 2.5. *An algebraic structure $(S, +, \cdot)$ is called semiring if and only if*

(i) $(S,+)$ *is a semigroup,*
(ii) $(S,\cdot)$ *is a semigroup,*
(iii) $a(b+c)=ab+ac; (a+b)c=ac+bc$, *for all* $a,b,c\in S$.

Definition 2.6. *Let* M *be a semiring. If there exists* $1\in M$ *such that* $a\cdot 1=1\cdot a=a$, *for all* $a\in M$, *called unity element of* M *then* M *is said to be semiring with unity.*

Definition 2.7. *A nonempty subset* A *of semiring* M *is called*

(i) *a subsemiring of* M *if* A *is an additive subsemigroup of* M *and* $AA\subseteq A$,
(ii) *a nonempty subset* A *of* M *is called a left(right) ideal of* M *if* A *is an additive subsemigroup of* M *and* $MA\subseteq A(AM\subseteq A)$,
(iii) *an ideal if* A *is an additive subsemigroup of* M, $MA\subseteq A$ *and* $AM\subseteq A$,
(iv) *a quasi-ideal of* M *if* A *is an additive subsemigroup of* M *and* $AM\cap MA\subseteq A$,
(v) *a bi ideal of* M *if* A *is an additive subsemigroup of* M *and* $AMA\subseteq A$,
(vi) *an interior ideal of* M *if* A *is an additive subsemigroup of* M *and* $MAM\subseteq A$.

Definition 2.8. *An element* a *of a semiring* S *is called a regular element if there exists an element* b *of* S *such that* $a=aba$.

Definition 2.9. *A semiring* S *is called a regular semiring if every element of* S *is a regular element.*

Definition 2.10. *An element* a *of a semiring* S *is called a multiplicatively idempotent (an additively idempotent) element if* $aa=a(a+a=a)$.

3. Left bi-quasi Ideals of Semirings

In this section, we introduce the notion of left (right) bi- quasi ideal of semiring, bi- quasi ideal, left bi- quasi simple semiring and study some of their properties, Throghout this paper, M is a semiring with unity.

Definition 3.1. *Let* M *be a semiring.A non-empty subset* L *of* M *is said to be left bi-quasi (right bi-quasi) ideal of* M *if* L *is a subsemigroup of* $(M,+)$ *and* $ML\cap LML\subseteq L(LM\cap LML\subseteq L)$.

Definition 3.2. *Let* M *be a semiring.A non-empty subset* L *of* M *is said to be bi-quasi ideal of* M *if* L *is a subsemigroup of* $(M,+), ML\cap LML\subseteq L$ *and* $LM\cap LML\subseteq L$.

Definition 3.3. *A semiring* M *is called a left bi-quasi simple semiring* M *has no left bi-quasi ideals other than itself*

Example 3.1. Let Q be the set of all rational numbers.

If $M=\left\{\begin{pmatrix} a & b\\ 0 & c\end{pmatrix} \mid a,b,c\in Q, c\neq 0.\right\}$ then M is a semiring with respect to binary operations, usual addition and multiplication of matrices.

If $A = \left\{ \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} \mid a, c \in Q, c \neq 0. \right\}$ then A is a left bi-quasi ideal of semiring M and A is not a bi-ideal of semiring M.

Let

$$\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}, \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}, \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \in A \text{ and } \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, \begin{pmatrix} r & s \\ 0 & t \end{pmatrix} \in M.$$

Then $\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} = \begin{pmatrix} axp & bxq \\ 0 & cyq \end{pmatrix}$. Therefore A is not a bi-ideal of semiring M.

Suppose

$$\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \begin{pmatrix} r & s \\ 0 & t \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} = \begin{pmatrix} l & 0 \\ 0 & m \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} ur & us \\ 0 & vt \end{pmatrix} = \begin{pmatrix} axp & bxq \\ 0 & cyq \end{pmatrix} = \begin{pmatrix} l & 0 \\ 0 & m \end{pmatrix}$$

$$\begin{aligned} \Rightarrow & ur = axp = l \\ & us = bxq = 0 \\ & vt = cyq = m \neq 0. \end{aligned}$$

These equations have solutions. Therefore $AM \cap AMA \neq \phi$ and $AM \cap AMA \subseteq A$. Hence A is a bi-quasi ideal of M.

THEOREM 3.1. *If L is a left bi-quasi ideal of semiring M then L is a subsemiring of M.*

PROOF. Suppose L is a left bi-quasi ideal of M. Then $ML \cap LML \subseteq L$. Let $x \in LL$. Then $x = yz$, where $y, z \in L$. Now

$$\begin{aligned} x = yz = y1z \in LML \text{ and } x \in LL \subseteq ML. \\ \Rightarrow x \in ML, \\ \Rightarrow x \in ML \cap LML \subseteq L. \end{aligned}$$

Hence $LL \subseteq L$. Hence the theorem. □

THEOREM 3.2. *Every left ideal of semiring M is a bi-quasi ideal of M.*

PROOF. Let L be the left ideal of semiring M. Then $ML \subseteq L$. Therefore $LM \cap LML \subseteq LML \subseteq LL \subseteq L$ and $ML \cap LML \subseteq ML \subseteq L$. Hence L is a bi-quasi ideal of M. □

THEOREM 3.3. *Every right ideal of semiring M is a bi-quasi ideal of M.*

PROOF. Let L be the right ideal of semiring M. Then $LM \subseteq L$. Therefore $LML \subseteq LL \subseteq L, ML \cap LML \subseteq LML \subseteq L$ and $LM \cap LML \subseteq LML \subseteq L$. Hence L is a bi-quasi ideal of M. □

COROLLARY 3.1. *Every ideal of semiring is a bi-quasi ideal.*

THEOREM 3.4. *Every quasi ideal of semiring M is a bi- quasi ideal of M.*

Proof. Let L be a quasi ideal of semiring M. Then $LM \cap ML \subseteq L$. We have $LML \subseteq LM$, since $ML \subseteq M$. Therefore $ML \cap LML \subseteq LML \cap LM \subseteq L$. Similarly we can prove, L is a right bi-quasi ideal of M. Hence the theorem. □

Theorem 3.5. *Every bi-ideal of semiring M is a bi-quasi ideal of M.*

Proof. Let L be a bi-ideal of semiring M. Then $LML \subseteq L$. Therefore $ML \cap LML \subseteq LML \subseteq L$ and $LM \cap LML \subseteq LML \subseteq L$. Hence every bi-ideal of semiring M is a bi-quasi ideal of M. □

Theorem 3.6. *Arbitrary intersection of left bi-quasi ideals of semiring M is either empty or a left bi-quasi ideal of M.*

Proof. Let $\phi \neq L = \bigcap_{i \in I} L_i$, where L_i is a left bi-quasi ideal of semiring M. We have $ML_i \cap L_iML_i \subseteq L_i$, for all $i \in I$. Then $ML \cap LML \subseteq ML_i \cap L_iML_i \subseteq L_i$, for all $i \in I$. Hence L is a left bi-quasi ideal of semiring M. □

Corollary 3.2. *Let M be a semiring, L be a left ideal of M and R be a right ideal of M. Then $L \cap R$ is a bi-quasi ideal of M.*

For the convenience of the reader the generalization of various ideals in an algebraic structure is shown in figure 1.

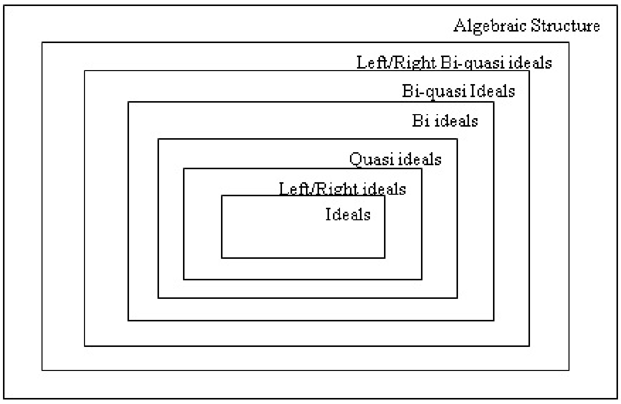

Figure 1. The generalizations of various ideals in an algebraic structure

Theorem 3.7. *Let M be a semiring. If $M = Ma$, for all $a \in M$ then every left bi-quasi of semiring is a quasi ideal of semiring M.*

Proof. Suppose $M = Ma$, for all $a \in M$ and L be a left bi-quasi ideal of semiring M. Then $ML \cap LML \subseteq L \Rightarrow ML \cap LM \subseteq L$, since $M = Ma$, for all $a \in M$. Hence L is a quasi ideal of semiring M. □

Theorem 3.8. *Let M be a semiring and T be a non-empty subset of M. Then every subsemigroup of T containing $TMT \cup TM$ is a left bi-quasi ideal of M.*

Proof. Let B be a subsemigroup of T containing $TMT \cup TM$. Then $BMB \subseteq TMT \subseteq TMT \cup TM \subseteq B$ and $MB \subseteq MT \subseteq MT \cup TM \subseteq B$. Therefore $MB \cap BMB \subseteq B$. Hence B is a left bi-quasi ideal of M. □

Theorem 3.9. *Let M be a semiring and T be an additive subsemigroup of M. Then every additive subsemigroup of T containing $TM \cap TMT$ is a left bi-quasi ideal of M.*

Proof. Let C be an additive subsemigroup of T of M containing $TM \cap MT$. Then $CM \cap CMC \subseteq TM \cap TMT \subseteq C$. Hence C is a left bi-quasi ideal of semiring M. □

Theorem 3.10. *Let L be a left ideal of semiring M and e be a multiplicative idempotent of M. Then eL is a left bi-quasi ideal of M.*

Proof. Let L be a left bi-quasi ideal of semiring M and e be a multiplicative idempotent of M. Suppose $x \in L \cap eM$. Then $x \in L$ and $x \in eM$. Then

$$x = ey, y \in M$$
$$\Rightarrow x = eey = ex \in eL$$
$$\Rightarrow L \cap eM \subseteq eL.$$

We have $eL \subseteq L$. and $eL \subseteq eM$. Therefore $eL \subseteq L \cap eM$. Hence eL is the intersection of a left bi-quasi ideal L and a right ideal eM. By Theorem 3.6, eL is a bi quasi ideal of M. □

Corollary 3.3. *Let R be a right bi-quasi ideal of semiring M and e be a multiplicative idempotent of M. Then Re is a bi-quasi ideal of M.*

Theorem 3.11. *Let e and f be multiplicative idempotents of semiring M. Then eMf is a left bi-quasi ideal of M.*

Proof. Let e and f be multiplicative idempotents of semiring M. Then eM and Mf are right ideal and left ideal of M respectively. We have $eMf \subseteq eM$ and $eMf \subseteq Mf$. Therefore $eMf \subseteq eM \cap Mf$. Let $a \in eM \cap Mf$. Then $a = ec$ and $a = df, c, d \in M$.
$a = ec = eec = ea = edf \in eMf$. Therefore $a \in eMf$. Hence $eMf = eM \cap Mf$. Thus by Theorem 3.6, eMf is a left bi quasi ideal of M. □

Theorem 3.12. *Let M be semiring. If L is a left bi-quasi ideal of M and T is a non-empty subset of M such that LT is an additive subsemigroup of M. Then LT is a left bi-quasi ideal of M.*

Proof. Supposr L is a left bi-quasi ideal of M and T is a non-empty subset of M such that LT is an additive subsemigroup of M.Then

$$ML \cap LML \subseteq LLT$$
$$\Rightarrow (ML \cap LML)T \subseteq LT$$
$$\Rightarrow MLT \cap LMLT \subseteq LT$$
$$\Rightarrow MLT \cap LTMLT \subseteq MLT \cap LMLT \subseteq LT.$$

Therefore LT is a left bi-quasi ideal of M. □

COROLLARY 3.4. *Let L_1 and L_2 be left bi-quasi ideals of semiring M such that L_1L_2 and L_2L_1 be additive subsemigroups of M. Then L_1L_2 and L_2L_1 are left bi-quasi ideals semiring M.*

THEOREM 3.13. *Let M be a semiring and T be a non-empty subset of M. Then every additive sub semigroup of T containing $MT \cap TMT$ is a left bi-quasi ideal of M.*

PROOF. Let D be an additive subsemigroup of T containing $MT \cap TMT$, where T is a non-empty subset of M. Since $D \subseteq T$, we have $MD \subseteq MT$ and $DMD \subseteq TMT$. Then $MD \cap DMD \subseteq MT \cap TMT \subseteq D$. Hence an additive subsemigroup D of T is a left bi-quasi ideal of M. □

The foiiowing theorem is a standard and welknown result in semirings.

THEOREM 3.14. *Let M be a regular semiring if and only if $AB = A \cap B$ for any right ideal A and left ideal B of semiring M.*

THEOREM 3.15. *Let M be a semiring. If M is a regular semiring if and only if $B = MB \cap BMB$ for every left bi-quasi ideal B of M.*

PROOF. Suppose M is a regular semiring, B is a left bi-quasi ideal of M and $x \in B$. Since B is a left bi-quasi ideal of M, we have $MB \cap BMB \subseteq B$. As M is a regular semiring, there exists $y \in M$ such that $x = xyx \in BMB$ and $x \in MB$. Therefore $x \in MB \cap BMB$. Hence $MB \cap BMB = B$.

Conversely, suppose that $B = MB \cap BMB$, for a left bi quasi ideal B of M. Let R and L be a right ideal and a left ideal of M respectively. Therefore by Corollary 3.2, $R \cap L$ is a left bi-quasi ideal of M. Therefore $R \cap L = M[R \cap L] \cap [R \cap L]M[R \cap L] \Rightarrow R \cap L \subseteq ML \cap RML \subseteq RL$. We have $RL \subseteq L$ and $RL \subseteq R$. Therefore $RL \subseteq R \cap L$. Hence $RL = R \cap L$, by Theorem 3.14, M is a regular semiring. □

THEOREM 3.16. *Let M be a semiring. Then the following statements are equivalent.*

(i) *M is a left bi-quasi simple,*
(ii) *$Ma = M$, for all $a \in M$,*
(iii) *Let (a) be the smallest left bi-quasi ideal containing a. Then $(a) = M$, for all $a \in M$.*

PROOF. Let M be a semiring.

$(i) \Rightarrow (ii)$: Suppose M is a left bi-quasi simple semiring and $a \in M$. Let $L = Ma$. Then L is a left ideal. By Theorem 3.2, L is a left bi-quasi ideal of M. Therefore $Ma = M$.

$(ii) \Rightarrow (iii)$: Suppose $Ma = M$, for all $a \in M$ and (a) is the smallest left bi-quasi ideal containing a. Then $Ma \subseteq (a) \Rightarrow M \subseteq (a)$. Therefore $M = (a)$.

$(iii) \Rightarrow (i)$: Let A be a left bi-quasi ideal of M and $a \in A$.
Then $(a) \subseteq A \subseteq M. \Rightarrow A = M$. Therefore M is a left bi-quasi simple semiring.

□

Theorem 3.17. *Let M be a semiring. Then M is a left bi-quasi simple semiring if and only if $Ma \cap aMa = M$, for all $a \in M$.*

Proof. Let M be a left bi- quasi simple semiring and $a \in M$. By Theorem 3.6, $Ma \cap aMa$ is a left bi-quasi ideal of semiring. Therefore $Ma \cap aMa = M$, for all $a \in M$.

Conversely suppose that $Ma \cap aMa = M$, for all $a \in M$. Let T be a left bi-quasi ideal of M and $a \in T$. Then $M = Ma \cap aMa \subseteq MT \cap TMT \subseteq T \subseteq M$. Therefore $M = T$.

Hence M is a left bi-quasi simple semiring. □

Theorem 3.18. *Let M be a regular semiring. Then every left bi-quasi ideal of semiring M is an ideal of semiring.*

Proof. Let L be a left bi-quasi ideal of a regularsemiring M. We know that LM and ML are right and left ideals of semiring M respectively. By Theorem 3.14, we have $LM \cap ML = LMML \subseteq ML$ and $LM \cap ML = LMML \subseteq LML$. Therefore $LM \cap ML \subseteq ML \cap LML \subseteq L$. Hence L is a quasi ideal of semiring M. Therefore L is an ideal of semiring M, since every quasi ideal of regular semiring is ideal. □

4. Conclusion

We introduced the notion of bi-quasi ideal and left (right) bi-quasi ideal of semiring as a generalization of bi-ideal and quasi-ideal of semiring and studied some of their properties. We introduced the notion of bi-quasi simple semiring and characterized the bi-quasi simple semiring. In continuity of this paper, we study prime bi-quasi ideals and maximal and minimal bi-quasi ideals of semiring.

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Department of Mathematics, GIT, GITAM University, Visakhapatnam- 530 045, A.P., India.

E-mail address: mmarapureddy@gmail.com