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A NOTE ON DEGREE DISTANCE INDEX

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Abstract. In this note, we give an upper bound for degree distance index of a graph in terms of vertex Padmakar-Ivan index, first Zagreb index, diameter and number of triangles. Also, we give a lower bound for degree distance index of a graph in terms of vertex Padmakar-Ivan index and number of triangles.

1. Introduction

All graphs considered in this paper are simple, connected and finite. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The length of a shortest path between two vertices u and v in G is known as the distance between the vertices u and v. It is denoted by $d(u,v)$. The maximum of all distances between any pair of vertices of G is known as the diameter of G and we denote the diameter of a graph G by D. The neighborhood set of a vertex u, denoted by $N(u)$ is a set consisting of all vertices of G that are adjacent with u in G. The cardinality of the neighborhood set of a vertex u is known as the degree of u in G and is denoted by $d(u)$. In 1972, Gutman and Trinajstić [**8**] introduced a graph invariant called the first Zagreb index M_1, which is defined as follows:

$$M_1 = \sum_{uv \in E(G)} [d(u) + d(v)].$$

The papers [**7**] and [**12**] marked the 30th anniversary of the first Zagreb index. Summarized mathematical and chemical properties of the first Zagreb index can be found in these papers. The status of a vertex or the total distance of a vertex $u \in G$ is denoted by $\sigma(u)$, i.e., $\sigma(u) = \sum_{v \in V(G)} d(u,v)$. For $e = uv \in E(G)$, $n_e(u)$ denotes the number of vertices in G, whose distance from u is smaller than the

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distance from v. The vertex Padmakar-Ivan index [**10**] of a graph G is denoted by PI and is defined as

$$PI = \sum_{e=uv\in E(G)} [n_e(u) + n_e(v)].$$

The degree distance index of a graph G, denoted by $DD(G)$, is defined as

$$DD(G) = \sum_{\{u,v\}\subset V(G)} [d(u) + d(v)]d(u, v).$$

The degree distance index of a graph G was introduced independently by Dobrynin, Kochetova [**5**] and Gutman [**6**]. In [**5**], it was conjectured that for a graph G on n vertices, $DD(G) \leqslant \frac{n^4}{32} + \mathrm{O}(n^3)$. Later in [**13**], Tomescu disproved this conjecture, in fact he showed the existence of graphs on n vertices having $\frac{n^4}{27} + \mathrm{O}(n^3)$ as its degree distance and also conjectured that $DD(G) \leqslant \frac{n^4}{27} + \mathrm{O}(n^3)$. In the same paper he confirmed the conjecture on a lower bound for the degree distance made by Dobrynin and Kochetova in [**5**]. Ten years later, Tomescu's conjecture was settled, see [**4, 11**]. In literature, several bounds for degree distance in terms of various graph theoretical parameters like order, minimum degree, diameter, edge-connectivity, Zagreb indices were obtained, see [**1, 2, 3, 9, 14**]. Motivated by these, in this note, we give an upper bound for degree distance index of a graph in terms of vertex Padmakar-Ivan index, first Zagreb index, diameter and number of triangles. Also, we give a lower bound for degree distance index of a graph in terms of vertex Padmakar-Ivan index and number of triangles.

2. Main Results

In the following theorem, we give a new upper bound for degree distance index. We denote by t, the number of triangles in G.

THEOREM 2.1. *Let G be a graph with n vertices and m edges. If $D \geqslant 2$, then*

$$DD(G) \leqslant PI - 2(D-1)[M_1 - 3t] - m(D^2 - (2n+5)D + 10).$$

Equality holds if and only if $D = 2$.

PROOF. From the definition of degree distance index, we have

$$\begin{aligned} DD(G) &= \sum_{\{u,v\}\subset V(G)} [d(u) + d(v)]d(u, v) \\ &= \sum_{u\in V(G)} d(u)\sigma(u) \\ &= \sum_{uv\in E(G)} [\sigma(u) + \sigma(v)]. \qquad (2.1) \end{aligned}$$

For $e = uv \in E(G)$, we have

$$\begin{aligned}\sigma(u) + \sigma(v) &= 2\sigma(u) + \sigma(v) - \sigma(u)\\ &= 2\sigma(u) + n_e(u) - n_e(v)\\ &\leqslant 2\sigma(u) - 2d(v) + n_e(u) + n_e(v) + 2|N(u) \cap N(v)|, \end{aligned} \tag{2.2}$$

since $n_e(v) \geqslant d(v) - |N(u) \cap N(v)|$.

Also

$$\begin{aligned}\sigma(u) \leqslant\ & d(u) + 2\,[d(v) - |N(u) \cap N(v)| - 1] + 3 + 4 + \ldots + D - 1\\ & + D\,[n - D - d(u) - d(v) + |N(u) \cap N(v)| + 3]\\ =\ & |N(u) \cap N(v)|(D-2) - d(u)(D-1) - d(v)(D-2)\\ & - \frac{1}{2}(D^2 - (2n+5)D + 10). \end{aligned} \tag{2.3}$$

Using (2.2) and (2.3) in (2.1), we get

$$\begin{aligned} DD(G) \leqslant & \sum_{e=uv\in E(G)} \{2(D-1)[|N(u)\cap N(v)| - d(u) - d(v)]\\ & - (D^2 - (2n+5)D + 10) + n_e(u) + n_e(v)\}\\ & = 2(D-1)\left\{ \sum_{e=uv\in E(G)} |N(u)\cap N(v)| - \sum_{e=uv\in E(G)} [d(u)+d(v)] \right\}\\ & + \sum_{e=uv\in E(G)} [n_e(u) + n_e(v)] - m(D^2 - (2n+5)D + 10). \end{aligned}$$

Therefore,

$$DD(G) \leqslant PI - 2(D-1)[M_1 - 3t] - m(D^2 - (2n+5)D + 10).$$

Moreover, equality holds if and only if the equalities in (2.2) and (2.3) holds. Thus, for equality it is necessary that if $uv \in E(G)$ and $w \notin N(v)$, then either $d(u,w) = d(v,w)$ or $d(v,w) = d(u,w) + 1$. If $D \geqslant 3$, for equality it is also necessary that $d(u,w) \geqslant 3$ whenever $w \notin N(v)$ and $w \notin N(u)$. Now, if $D \geqslant 3$ and $w_1 w_2 \ldots, w_{D+1}$ is a diametrical path in G, then for $u = w_1$ and $v = w_2$, we have $d(v, w_{D+1}) = D-1$ and for $u = w_2$ and $v = w_1$, we have $w_4 \notin N(v), N(u)$ and $d(u, w_4) = 2$. Thus, for equality one should have $D \leqslant 2$. Suppose $D = 2$, then it is easy to see that the equality in (2.3) holds and for $uv \in E(G)$ and $w \notin N(v)$, we have either $d(u,w) = 1$ or $d(u,w) = d(v,w) = 2$. This completes the proof. □

The following corollary follows immediately from the above theorem and the fact that PI index of a bipartite graph with n vertices and m edges is nm.

Corollary 2.1. *Let G be a bipartite graph with n vertices and m edges. Suppose $D \geqslant 2$, then*

$$DD(G) \leqslant nm - 2(D-1)M_1(G) - m(D^2 - (2n+5)D + 10).$$

Equality holds if and only if $D = 2$.

Now, we give a lower bound for degree distance index of a graph.

Theorem 2.2. *Let G be a graph. Then*

$$DD(G) \geqslant 4m(n-1) - PI - 6t.$$

Equality holds if and only if $D \leqslant 2$.

Proof. For $uv \in E(G)$, we have

$$d(u,w) - 1 \leqslant d(v,w) \text{ for all } w \in V(G).$$

Thus, for $uv \in E(G)$,

$$\sigma(u) + \sigma(v) \geqslant 2\sigma(u) + 2(d(u) - |N(u) \cap N(v)|) - (n_e(u) + n_e(v)) \tag{2.4}$$

and

$$\sigma(u) \geqslant 2(n-1) - d(u). \tag{2.5}$$

Using (2.4) and (2.5) in (2.1), we obtain

$$\begin{aligned} DD(G) &\geqslant \sum_{uv \in E(G)} \{4(n-1) - 2|N(u) \cap N(v)| - (n_e(u) + n_e(v))\} \\ &= -\sum_{uv \in E(G)} [n_e(u) + n_e(v)] - 2\sum_{uv \in E(G)} |N(u) \cap N(v)| + 4m(n-1) \\ &= -PI - 6t + 4m(n-1). \end{aligned}$$

Moreover, the equality holds if and only if $D \leqslant 2$ and equality in (2.4) holds. For $D \leqslant 2$, it is easy to check that the equality in (2.4) holds. This completes the proof. □

The following corollary follows immediately from the above theorem.

Corollary 2.2. *Let G be a bipartite graph with n vertices and m edges. Then*

$$DD(G) \geqslant m(3n-4).$$

Equality holds if and only if $D \leqslant 2$.

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