BULLETIN OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4874, ISSN (o) 2303-4955
www.imvibl.org /JOURNALS / BULLETIN
Vol. 8(2018), 113-122
DOI: 10.7251/BIMVI1801113N

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

ON YANG MEANS III

Edward Neuman

Abstract. Optimal inequalities involving the p-Yang means are established. Bounding quantities are either the arithmetic or geometric or the harmonic combinations of the p-geometric and the p-quadratic means.

1. Introduction

Recently Z. -H Yang [**24**] introduced two bivariate means denoted in the sequel by V and U. For the sake of presentation we include below explicit formulas for these means.

Throughout the sequel the letters a and b will stand for two positive and unequal numbers. The Yang means of a and b are defined as follows:

$$V(a,b) = \frac{a-b}{\sqrt{2}\sinh^{-1}\left(\frac{a-b}{\sqrt{2ab}}\right)} \tag{1.1}$$

and

$$U(a,b) = \frac{a-b}{\sqrt{2}\tan^{-1}\left(\frac{a-b}{\sqrt{2ab}}\right)}. \tag{1.2}$$

These means have been studied extensively in [**24, 25**] and recently in [**16, 17**].

This paper is a continuation of a research initiated in [**16, 17**] and is organized as follows. Definitions of other bivariate means utilized in this work are given in Section 2. List of those means include two Seiffert means, logarithmic mean, Neuman-Sándor mean and the Schwab-Borchardt mean SB. The latter plays a crucial role in our presentation. Concept of the p-mean is recalled in Section 3.

2010 *Mathematics Subject Classification.* Primary: 26E60; Secondary: 26D07.
Key words and phrases. Yang means, Schwab-Borchardt mean, bivariate means, inequalities.

Therein we include some facts about the p-means. Optimal inequalities involving the p-means U_p and V_p which are derived, respectively, from the Yang means V and U established in Section 4. The lower and upper bounds are either the convex arithmetic combination or the convex geometric combination or the convex harmonic combination of the p-means derived from the geometric and the quadratic means of a and b.

2. Definitions and preliminaries

Recall that the unweighted arithmetic mean of a and b is defined as

$$A = \frac{a+b}{2}.$$

Other unweighted bivariate means used in this paper are the harmonic mean H, geometric mean G, root-square mean (quadratic mean) Q and the contra-harmonic mean C which are defined as follows (cf. [**2**])

$$H = \frac{2ab}{a+b}, \quad G = \sqrt{ab}, \quad Q = \sqrt{\frac{a^2+b^2}{2}}, \quad C = \frac{a^2+b^2}{a+b}. \tag{2.1}$$

Let

$$v = \frac{a-b}{a+b}. \tag{2.2}$$

Clearly $0 < |v| < 1$. One can easily verify that the means defined in (2.1) all can be expressed in terms of A and v. We have

$$\begin{aligned} H &= A(1-v^2), \quad G = A\sqrt{1-v^2}, \\ Q &= A\sqrt{1+v^2}, \quad C = A(1+v^2). \end{aligned} \tag{2.3}$$

Other bivariate means utilized in this paper include the first and the second Seiffert means, denoted by P and T, respectively, the Neuman-Sándor mean M, and the logarithmic mean L. Recall that

$$\begin{aligned} P &= A\frac{v}{\sin^{-1} v}, \quad T = A\frac{v}{\tan^{-1} v}, \\ M &= A\frac{v}{\sinh^{-1} v}, \quad L = A\frac{v}{\tanh^{-1} v} \end{aligned} \tag{2.4}$$

(see [**21**], [**22**], [**19**]). All the means mentioned above are comparable. It is known that

$$H < G < L < P < A < M < T < Q < C \tag{2.5}$$

(see, e.g., [**19**]).

The four means listed in (2.4) are special cases of the Schwab-Borchardt mean SB which is defined as follows

$$SB(a,b) \equiv SB = \begin{cases} \dfrac{\sqrt{b^2-a^2}}{\cos^{-1}(a/b)} & \text{if } \ a<b, \\[2ex] \dfrac{\sqrt{a^2-b^2}}{\cosh^{-1}(a/b)} & \text{if } \ b<a. \end{cases}$$

(see, e.g., [**1**], [**3**]). This mean has been studied extensively in [**11**], [**19**], and [**20**]. It is well known that the mean SB is strict, nonsymmetric and homogeneous of degree one in its variables.

It has been proven in [**19**] that

$$\begin{aligned} P &= SB(G,A), \quad T = SB(A,Q), \\ M &= SB(Q,A), \quad L = SB(A,G). \end{aligned} \tag{2.6}$$

Yang means can also be represented in terms of the Schwab-Borchardt mean. We have

$$V = SB(Q,G) \qquad \text{and} \qquad U = SB(G,Q) \tag{2.7}$$

(see [**16**]).

The following chain of inequalities

$$L < V < P < U < M < T \tag{2.8}$$

is known (see [**16**]).

For the sake of presentation we include new formulas for means SB. We have [**18**]

$$SB(x,y) \equiv SB = \begin{cases} y\dfrac{\sin r}{r} = x\dfrac{\tan r}{r} & \text{if } \ 0 \leqslant x < y, \\[2ex] y\dfrac{\sinh s}{s} = x\dfrac{\tanh s}{s} & \text{if } \ y < x, \end{cases} \tag{2.9}$$

where

$$\cos r = x/y \qquad \text{if} \qquad x<y \qquad \text{and} \qquad \cosh s = x/y \qquad \text{if} \qquad x>y. \tag{2.10}$$

Clearly

$$0 < r \leqslant r_0, \qquad \text{where} \qquad r_0 = \max\{\cos^{-1}(x/y) : 0 \leqslant x < y\}$$

and

$$0 < s \leqslant s_0, \qquad \text{where} \qquad s_0 = \max\{\cosh^{-1}(x/y) : x > y > 0\}.$$

3. Definition and basic properties of the p - means

We begin this section with a simple construction of a family of bivariate means which depend on the parameter p which satisfies $|p| \leqslant 1$. This idea has been introduced in author's paper [**13**].

First two nonnegative numbers w_1 and w_2 are defined as follows:

$$w_1 = \frac{1+p}{2}, \quad w_2 = \frac{1-p}{2}. \tag{3.1}$$

Clearly $w_1 + w_2 = 1$. We associate with the pair (a, b) a pair of positive numbers (x, y), where

$$x = w_1 a + w_2 b, \quad y = w_1 b + w_2 a. \tag{3.2}$$

Thus x and y are the convex combinations of a and b. One can easily verify that $a < x < y < b$ if $a < b$ or $b < y < x < a$ if $b < a$.

For the sake of presentation let N stand for a bivariate symmetric mean. We define a mean $N_p(a, b) \equiv N_p$ as follows

$$N_p(a, b) = N(x, y). \tag{3.3}$$

We call the mean N_p the p-mean or the p-mean generated by N.

For the reader's convenience we present now some elementary properties of the p-means. It follows (3.3), (3.1), and (3.2) we see that

$$N_{-p}(a, b) = N(y, x) = N(x, y) = N_p(a, b).$$

Thus the function $p \to N_p$ is an even function. To this end we will assume that $0 \leqslant p \leqslant 1$. It follows from (3.1) and (3.2) that

$$N_0 = A, \quad N_1 = N. \tag{3.4}$$

Moreover, the function $p \to N_p$ is strictly decreasing if $N < A$, i.e.,

$$N_1 \leqslant N_p \leqslant N_0 \tag{3.5}$$

or is strictly increasing if $N > A$, i.e.,

$$N_0 \leqslant N_p \leqslant N_1. \tag{3.6}$$

We now present formulas for the p-means. Let us begin with the case when $N = A$. We have

$$A_p = A_p(a, b) = A(x, y) = A.$$

Thus we shall always write A instead of A_p when no confusion would arise. To obtain the p-versions of the four means listed in (2.3) let us introduce a quantity u, where

$$u = \frac{x-y}{x+y}. \tag{3.7}$$

Using (3.2) and (2.2) we obtain

$$u = pv. \tag{3.8}$$

Since $0 < |v| < 1$, $0 < |u| < p \leqslant 1$

Formulas for the p-means derived from means listed in (2.3) read as follows

$$
\begin{aligned}
H_p &= A(1-u^2), \quad G_p = A\sqrt{1-u^2}, \\
Q_p &= A\sqrt{1+u^2}, \quad C_p = A(1+u^2).
\end{aligned} \tag{3.9}
$$

Similarly, using (2.4) we obtain

$$
\begin{aligned}
P_p &= A\frac{u}{\sin^{-1} u}, \quad T_p = A\frac{u}{\tan^{-1} u}, \\
M_p &= A\frac{u}{\sinh^{-1} u}, \quad L_p = A\frac{u}{\tanh^{-1} u}.
\end{aligned} \tag{3.10}
$$

It is worth mentioning that the means P_p, T_p, M_p, and L_p can be represented as the Schwab-Borchardt means. Making use of (2.6) and (2.7) we obtain

$$
\begin{aligned}
P_p &= SB(G_p, A), \quad T_p = SB(A, Q_p), \\
M_p &= SB(Q_p, A), \quad L_p = SB(A, G_p), \\
V_p &= SB(Q_p, G_p), \quad U_p = SB(G_p, Q_p).
\end{aligned} \tag{3.11}
$$

For this reason we call (G_p, A), (A, Q_p), (Q_p, A), (A, G_p), (Q_p, G_p) and (G_p, Q_p) the pairs of generating means.

We close this section with the following remarks. The idea of using the p-means was motivated by a recent development in theory of means. Let R and S be bivariate symmetric means and let $0 \leqslant \lambda \leqslant 1$. Many researchers (see, e.g., [**4**], [**5**], [**6**], [**7**], [**8**], [**9**], [**10**], [**23**]) have studied problems of finding all values of λ for which inequality $R(\lambda a + (1-\lambda)b) < S(r, s)$ is satisfied for all positive numbers r and s. Let us note that with $\lambda = (1+p)/2 = w_1$ we have $1 - \lambda = (1-p)/2 = w_2$. Thus the inequality in question can be written as $R_p(r, s) < S(r, s)$. With the parameter λ used instead of p formula (3.8) should be changed $u = (2\lambda - 1)v$, which is a little bit more cumbersome in analytic computations than (3.8) is.

4. Main results

The goal of this section is to determine coefficients of six optimal convex combinations which form both lower and upper bounds for the p-means $U_p(a, b) \equiv U_p$ and $V_p(a, b) \equiv V_p$. Convex combinations employed here involve the p-means Q_p and G_p of positive and unequal numbers a and b. Our first result reads as follows:

Theorem 4.1. *The two-sided inequality*

$$\alpha_1 Q_p + (1-\alpha_1)G_p < U_p < \beta_1 Q_p + (1-\beta_1)G_p \tag{4.1}$$

holds true provided

$$\alpha_1 \leqslant \frac{2}{\pi} \qquad \textit{and} \qquad \beta_1 \geqslant \frac{2}{3}. \tag{4.2}$$

Proof. Taking into account that $G_p < Q_p$ one can rewrite (4.1) as

$$\alpha_1 < \frac{U_p/Q_p - G_p/Q_p}{1 - G_p/Q_p} < \beta_1. \tag{4.3}$$

Since $U_p = SB(G_p, Q_p)$ (see (3.11)) we get using (2.9) and (2.10)

$$\frac{U_p}{Q_p} = \frac{\sin r}{r} \qquad \text{and} \qquad \frac{G_p}{Q_p} = \cos r,$$

where $0 \leqslant r \leqslant \pi/2$. This in conjunction with (4.3) yields

$$\alpha_1 < \Phi_1(r) < \beta_1,$$

where

$$\Phi_1(r) = \frac{\sin r - r\cos r}{r(1-\cos r)}.$$

It follows from Theorem 3 in [**14**] that the function $\Phi_1(r)$ is strictly decreasing on its domain. Moreover,

$$\Phi_1(0^+) = \frac{2}{3} \qquad \text{and} \qquad \Phi_1(\frac{\pi}{2}^-).$$

Hence the assertion follows. $\square$

A counterpart of the last theorem for the mean V_p reads as follows:

THEOREM 4.2. *The two-sided inequality*

$$\gamma_1 Q_p + (1-\gamma_1)G_p < V_p < \delta_1 Q_p + (1-\delta_1)G_p \tag{4.4}$$

holds true provided

$$\gamma_1 \leqslant 0 \qquad \textit{and} \qquad \delta_1 \geqslant \frac{1}{3}. \tag{4.5}$$

PROOF. We follow the lines used in the proof of the last theorem. Firstly, we rewrite (4.4) as follows

$$\gamma_1 < \frac{V_p/Q_p - 1}{Q_p/G_p - 1} < \delta_1. \tag{4.6}$$

Since $V_p = SB(Q_p, G_p)$ (see (3.11)) we get using (2.9) and (2.10)

$$\frac{V_p}{G_p} = \frac{\sinh s}{s} \qquad \text{and} \qquad \frac{Q_p}{G_p} = \cosh s,$$

where $0 \leqslant s < \infty$. This in conjunction with (4.6) yields

$$\gamma_1 < \Psi_1(s) < \delta_1,$$

where

$$\Psi_1(s) = \frac{\sinh s - s}{s(\cosh s - 1)}.$$

It follows from [**12**] that the function $\Psi_1(s)$ is strictly decreasing on its domain. Moreover,

$$\Psi_1(0^+) = \frac{1}{3} \qquad \text{and} \qquad \Psi_1(\infty^-) = 0.$$

Hence the assertion follows. $\square$

In the next two theorems we will deal with optimal bounds for U_p and V_p where now bounding quantities are the geometric convex combinations of Q_p and G_p.

THEOREM 4.3. *The following inequality*

$$G_p^{\alpha_2} Q_p^{1-\alpha_2} < U_p < G_p^{\beta_2} Q_p^{1-\beta_2} \tag{4.7}$$

is valid if

$$\alpha_2 \geqslant \frac{1}{3} \qquad \textit{and} \qquad \beta_2 \leqslant 0. \tag{4.8}$$

PROOF. First we rewrite (4.7) as follows

$$(G_p/Q_p)^{\alpha_2} < U_p/Q_p < (G_p/Q_p)^{\beta_2}. \tag{4.9}$$

Since $U_p = SB(G_p, Q_p)$ (2.9) and (2.10) yield

$$G_p/Q_p = \cos r \qquad \text{and} \qquad U_p/Q_p = \frac{\sin r}{r},$$

where $0 \leqslant r \leqslant \pi/2$. This in conjunction with (4.9) yields

$$(\cos r)^{\alpha_2} < \frac{\sin r}{r} < (\cos r)^{\alpha_2}.$$

Taking logarithms we can write the last two-sided inequality as

$$\beta_2 < \Phi_2(r) < \alpha_2, \tag{4.10}$$

where

$$\Phi_2(r) = \frac{\ln(\frac{\sin r}{r})}{\ln(\cos r)}.$$

It follows from Lemma 2 in [**15**] that the function $\Phi_2(r)$ is strictly decreasing on its domain. This in conjunction with

$$\Phi_2(0^+) = \frac{1}{3} \qquad \text{and} \qquad \Phi_2(\frac{\pi}{2}^-) = 0$$

yields the asserted result. The proof is complete. $\square$

A result for V_p, which is similar to that in Theorem 4.3, reads as follows:

THEOREM 4.4. *The following inequality*

$$Q_p^{\gamma_2} G_p^{1-\gamma_2} < V_p < Q_p^{\delta_2} G_p^{1-\delta_2} \tag{4.11}$$

is valid if

$$\gamma_2 \leqslant \frac{1}{3} \qquad \textit{and} \qquad \delta_2 \geqslant 1. \tag{4.12}$$

PROOF. Dividing each member of (4.11) by G_p and next taking logarithms and using formulas (3.11), (2.9) and (2.10) we obtain, after a little algebra

$$\gamma_2 < \Psi_2(s) < \delta_2, \tag{4.13}$$

where

$$\Psi_2(s) = \frac{\ln(\dfrac{\sinh s}{s})}{\ln(\cosh s)}$$

$(0 < s < \infty)$. It is known (cf. [**26**] and [**15**]) that the function $\Psi_2(s)$ is strictly increasing on its domain and also that

$$\Psi_2(0^+) = 1/3 \qquad \text{and} \qquad \Psi_2(\infty^-) = 1.$$

This in conjunction with (4.13) gives the desired result. $\square$

The remaining two results deal with the optimal bounds for the reciprocals of two means U_p and V_p. Bounding expressions have a structure of the reciprocals of the harmonic means of Q_p and G_p. We shall establish now the following

THEOREM 4.5. *The two-sided inequality*

$$\frac{\alpha_3}{G_p} + \frac{1-\alpha_3}{Q_p} < \frac{1}{U_p} < \frac{\beta_3}{G_p} + \frac{1-\beta_3}{Q_p} \tag{4.14}$$

holds true provided

$$\alpha_3 \leqslant 0 \qquad \textit{and} \qquad \beta_3 \geqslant \frac{1}{3}. \tag{4.15}$$

PROOF. First we rewrite (4.14) as follows

$$\alpha_3 < \left(\frac{G_p}{U_p}\right)\frac{1 - U_p/Q_p}{1 - G_p/Q_p} < \beta_3. \tag{4.16}$$

Making use of (3.11), (2.9) and (2.10) we obtain

$$\frac{G_p}{U_p} = \frac{r}{\tan r}, \qquad \frac{U_p}{Q_p} = \frac{\sin r}{r} \qquad \text{and} \qquad \frac{G_p}{Q_p} = \cos r.$$

Applying these formulas to (4.16) we obtain

$$\alpha_3 < \Phi_3(r) < \beta_3, \tag{4.17}$$

where

$$\Phi_3(r) = \frac{r - \sin r}{\tan r - \sin r}$$

$(0 < r < \pi/2)$. It follows from Lemma 2 in [**15**] that the function $\Phi_3(r)$ is strictly decreasing on its domain and also that

$$\Phi_3(0^+) = \frac{1}{3} \qquad \text{and} \qquad \Phi_3(\frac{\pi}{2}^-) = 0.$$

This in conjunction with (4.17) gives the desired result. $\square$

We close this section with the following

THEOREM 4.6. *The following double inequality*

$$\frac{\gamma_3}{G_p} + \frac{1-\gamma_3}{Q_p} < \frac{1}{V_p} < \frac{\delta_3}{G_p} + \frac{1-\delta_3}{Q_p} \tag{4.18}$$

holds true provided

$$\gamma_3 \leqslant 0 \qquad \textit{and} \qquad \delta_3 \geqslant \frac{2}{3}. \tag{4.19}$$

PROOF. Its easy to see that the two-sided inequality (4.19) is equivalent to the following one

$$\gamma_3 < \Big(\frac{G_p}{V_p}\Big)\frac{1 - V_p/Q_p}{1 - G_p/Q_p} < \delta_3. \tag{4.20}$$

Making use of (3.11), (2.9) and (2.10) we obtain

$$\frac{G_p}{V_p} = \frac{s}{\sinh s}, \qquad \frac{V_p}{Q_p} = \frac{\tanh s}{s} \qquad \text{and} \qquad \frac{G_p}{Q_p} = \frac{1}{\cosh s}.$$

Applying these formulas to (4.20) we obtain

$$\gamma_3 < \Psi_3(s) < \delta_3, \tag{4.21}$$

where

$$\Psi_3(s) = \frac{s - \tanh s}{\sinh s - \tanh s}$$

$(0 < s < \infty)$. It follows from Lemma 3 in [**15**] that the function $\Psi_3(s)$ is strictly decreasing on its domain and also that

$$\Psi_3(0^+) = \frac{2}{3} \qquad \text{and} \qquad \Phi_3(\infty^-) = 0.$$

This in conjunction with (4.21) yields the asserted result. ☐

References

[1] J.M. Borwein, P.B. Borwein. *Pi and the AGM: A Study in Analytic Number Theory and Computational Complexity* John Wiley and Sons, New York, 1987.

[2] P.S. Bullen, *A Directory of Inequalities*, Pitman Monographs and Surveys in Pure and Applied Mathematics, vol. 97, Addison Wesley Longman Limited, Longman, Harlow, 1998.

[3] B.C. Carlson. Algorithms involving arithmetic and geometric means. *Amer. Math. Monthly*, **78**(5)(1971), 496–505.

[4] Y.-M. Chu and S.-W. Hou. Sharp bounds for Seiffert mean in terms of contra-harmonic mean. *Abstr. Appl. Anal.*, **2012** (2012), Article ID 425175.

[5] Y.-M. Chu, Y.-F. Qiu, M.-K. Wang and G.-D. Wang. The optimal convex combination bounds of arithmetic and harmonic means for the Seiffert mean. *J. Inequal. Appl.*, **2010**(2010), Article ID 436457.

[6] Y.-M. Chu, M.-K. Wang and W.-M. Gong. Two sharp double inequalities for Seiffert mean. *J. Inequal. Appl.*, **2011**(2011): 44, 7 pages.

[7] Y.-M. Chu, M.-K. Wang and Z.-K. Wang. A best possible double inequality between Seiffert and harmonic means. *J. Inequal. Appl.*, **2011**(2011): 94, 7 pages.

[8] Y.-M. Chu, C. Zong and G.-D. Wang. Optimal convex combination bounds of Seiffert and geometric means for the arithmetic mean. *J. Math. Inequal.*, **5**(3)(2011), 429–434.

[9] G. Shaoqin, G. Hongya and S. Wenying. Optimal convex combination bounds of the centroidal and harmonic means for the Seiffert mean. *Int. J. Pure Appl. Math.*, **70**(5)(2011), 701–709.

[10] H. Liu and X.-J. Meng. The optimal convex combination bounds for Seiffert's mean. *J. Inequal. Appl.*, **2011**(2011), Article ID 686834.

[11] E. Neuman. Inequalities for the Schwab - Borchardt mean and their applications. *J. Math. Inequal.*, **5**(4)(2011), 601–609.

[12] E. Neuman. A note on a certain bivariate mean. *J. Math. Inequal.*, **6**(4)(2012), 637–643.
[13] E. Neuman. A one-parameter family of bivariate means. *J. Math. Inequal.*, **7**(3)(2013), 399–412.
[14] E. Neuman. On some means derived from the Schwab-Borchardt mean. *J. Math. Inequal.*, **8**(1)(2014), 171–183.
[15] E. Neuman. On some means derived from the Schwab-Borchardt mean II. *J. Math. Inequal.*, **8**(2)(2014), 359–368.
[16] E. Neuman. On Yang means. *Adv. Inequal. Appl.*, **2017**(2017): 8.
[17] E. Neuman. On Yang means II. *Adv. Inequal. Appl.*, **2017**(2017): 11.
[18] E. Neuman. On two bivariate elliptic means. *J. Math. Inequal.*, **11**(2)(2017), 345–354.
[19] E. Neuman and J.Sándor, On the Schwab - Borchardt mean. *Math. Pannon.*, **14**(2)(2003), 253–266.
[20] E. Neuman and J.Sándor. On the Schwab - Borchardt mean II. Math. Pannon. **17**(1)(2006), 49–59.
[21] H.-J. Seiffert. Problem 887. *Nieuw. Arch. Wisk.*, **11**(1993), 176.
[22] H.-J. Seiffert. Aufgabe 16. *Würzel,* **29** (1995), 87.
[23] M.-K. Wang, Y.-M. Chu and B.-Y. Liu. Sharp inequalities for the Neuman-Sándor mean in terms of arithmetic and contra-harmonic means. preperint, arXiv: 1209.5825 vol 1 [math CA] 26 Sep. 2012.
[24] Z.-H. Yang. Three families of two-parameter means constructed by trigonometric functions. *J. Inequal. Appl.*, **2013**(2013), Article ID 541, 27 pages.
[25] Z-H Yang, L-M Wu and Y-M Chu. Optimal power mean bounds for Yang mean. *J. Inequal. Appl.*, **2014**(2014), Article ID 401, 10 pages.
[26] T. -H. Zhao, Y.-M. Chu and B.-Y. Liu. Optimal bounds for the Neuman-Sándor mean in terms of arithmetic and contra-harmonic means. *Abstr. Appl. Anal.*, **2012**(2012). Article ID 302635, 9 pages.

Receibed by editors 01.06.2017; Available online 07.08.2017.

Southern Illinois University, Institute of Mathematical Research, 144 Hawthorn Hollow, Carbondale, IL 62903, USA
E-mail address: edneuman76@gmail.com