COMMON BEST PROXIMITY POINTS
IN COMPLEX VALUED METRIC SPACES

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Abstract. In this paper, we obtain the existence and the uniqueness of common best proximity point theorems for non-self mappings between two subsets of a complex valued metric space satisfying certain contractive conditions. Our results supported by some examples.

1. Introduction and Preliminaries

Fixed point theory focuses on solving the equation $Tx = x$, where $T$ is a self-mapping defined on a subset of a metric space or other suitable space. If it is assumed that, $T$ is not a self-mapping then the equation $Tx = x$ is likely to have no solution. Consequently, the significant aim is determining an element $x$ that is in close proximity to $Tx$ in some sense. Eventually, the target is finding an element $x$ in a metric space, that satisfy in the following condition, $d(x, Tx) = d(A, B)$ and $d(x, Sx) = d(A, B)$ which $d$ is a metric function and $d(A, B) := \inf\{d(x, y) : x \in A, y \in B\}$. Now, if $T, S : A \to B$ are two non-self mappings, then the equations $Sx = x$ and $Tx = x$ are likely to have no solution, the solution known as a common fixed point of the mappings $S$ and $T$ (see, [1, 7, 9, 12, 8, 15]). So, the purpose is finding an element $x$ in $A$ such that $d(x, Sx) = d(A, B)$ and $d(x, Tx) = d(A, B)$ which $x$ is called the common best proximity point of mappings $S$ and $T$ in a metric space (see, [2, 13, 14]). In 2011, Azam et al. [3] introduced the notion of complex valued metric space, which is a generalization of the classical metric space and established the existence of common fixed point theorems for mappings satisfying contraction condition (see [3], Theorem 4). The purpose of this article is generalizing some well-known results about common best proximity points that

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were established in the classic metric space (see, [2, 13]), in the complex valued metric space by some new definitions and presenting a type of contractive condition and developing a common best proximity point theorem for non-self mappings which satisfy in this contractive condition, in the complex valued metric space.

Let \( \mathbb{C} \) be the set of complex numbers and \( z_1, z_2 \in \mathbb{C} \). Define a partial order \( \preceq \) on \( \mathbb{C} \) as follows:

\[
z_1 \preceq z_2 \text{ if and only if } \Re(z_1) \leq \Re(z_2), \; \Im(z_1) \leq \Im(z_2).
\]

It follows that \( z_1 \preceq z_2 \) if and only if one of the following conditions is satisfied:

(i) \( \Re(z_1) = \Re(z_2), \; \Im(z_1) < \Im(z_2) \),
(ii) \( \Re(z_1) < \Re(z_2), \; \Im(z_1) = \Im(z_2) \),
(iii) \( \Re(z_1) < \Re(z_2), \; \Im(z_1) < \Im(z_2) \),
(iv) \( \Re(z_1) = \Re(z_2), \; \Im(z_1) = \Im(z_2) \).

In particular, we will write \( z_1 \npreceq z_2 \) if \( z_1 \neq z_2 \) and one of (i), (ii), and (iii) is satisfied where we denote \( z_1 \npreceq z_2 \) if only (iii) is satisfied. Note that

\[
0 \preceq z_1 \npreceq z_2 \implies |z_1| < |z_2|,
\]

\[
z_1 \npreceq z_2, z_2 \npreceq z_3 \implies z_1 \npreceq z_3.
\]

**Definition 1.1.** [3] Let \( X \) be a nonempty set. Suppose that the mapping \( d : X \times X \to \mathbb{C} \), satisfies:

(a) \( 0 \preceq d(x, y) \), for all \( x, y \in X \) and \( d(x, y) = 0 \) if and only if \( x = y \);
(b) \( d(x, y) = d(y, x) \) for all \( x, y \in X \);
(c) \( d(x, z) \preceq d(x, y) + d(y, z) \), for all \( x, y, z \in X \).

Then \( d \) is called a complex valued metric on \( X \), and \( (X, d) \) is called a complex valued metric space.

**Example 1.1.** Let \( X = \mathbb{C} \). Define the mapping \( d : X \times X \to \mathbb{C} \) for all \( x, y \in X \), by

\[
d(x, y) = i|x - y|.
\]

Clearly, the pair \( (X, d) \) is a complex valued metric space.

**Definition 1.2.** [3] Let \( (X, d) \) be a complex valued metric space.

(a) A point \( x \in X \) is called interior point of a set \( A \subseteq X \) whenever there exists \( 0 \prec r \in \mathbb{C} \) such that \( B(x, r) = \{y \in X : d(x, y) \prec r\} \subseteq A \).
(b) A point \( x \in X \) is called a limit point of a subset \( A \subseteq X \) whenever for every \( 0 \prec r \in \mathbb{C}, B(x, r) \cap (A \setminus \{x\}) \neq \emptyset \).
(c) A subset \( A \subseteq X \) is called open whenever each element of \( A \) is an interior point of \( A \).
(d) A subset \( A \subseteq X \) is called closed whenever each limit point of \( A \) belongs to \( A \).
(e) The family \( F = \{B(x, r) : x \in X, \; 0 \prec r\} \) is a sub-basis for a Hausdorff topology \( \tau \) on \( X \).

**Definition 1.3.** [4] Let \( A \) be a subset of \( \mathbb{C} \). If there exists \( u \in \mathbb{C} \) such that \( z \preceq u \) for all \( z \in A \), then \( A \) is bounded above and \( u \) is an upper bound. Similarly,
if there exists \( l \in \mathbb{C} \) such that \( l \preceq z \), for all \( z \in A \), then \( A \) is bounded below and \( l \) is a lower bound.

**Definition 1.4.** [4] For a \( A \subseteq \mathbb{C} \) which is bounded above if there exists an upper bound \( s \) of \( A \) such that, for every upper bound \( u \) of \( A \), \( s \preceq u \), then the upper bound \( s \) is called \( \sup \) \( A \). Similarly, for a subset \( A \subseteq \mathbb{C} \) which is bounded below if there exists a lower bound \( t \) of \( A \) such that for every lower bound \( l \) of \( A \), \( l \preceq t \), then the lower bound \( t \) is called \( \inf \) \( A \).

Suppose that \( A \subseteq \mathbb{C} \) is bounded above. Then there exists \( q = u + iv \in \mathbb{C} \) such that \( z = x + iy \preceq q = u + iv \), for all \( z \in A \). It follows that \( x \preceq u \) and \( y \preceq v \), for all \( z = x + iy \in A \); that is, \( S = \{ x : z = x + iy \in A \} \) and \( T = \{ y : z = x + iy \in A \} \) are two sets of real numbers which are bounded above. Hence both \( \sup S \) and \( \inf T \) exist. Let \( \bar{x} = \sup S \) and \( \bar{y} = \sup T \). Then \( z = \bar{x} + i\bar{y} \) is \( \sup A \).

Similarly, if \( A \subseteq \mathbb{C} \) is bounded below, then \( z^* = x^* + iy^* \) is \( \inf A \), where \( x^* = \inf S = \inf \{ x : x + iy \in A \} \) and \( y^* = \inf T = \inf \{ y : x + iy \in A \} \).

Any subset \( A \subseteq \mathbb{C} \) which is bounded above has supremum. Equivalently, any subset \( A \subseteq \mathbb{C} \) which is bounded below has infimum.

**Definition 1.5.** [3] Let \((X, d)\) be a complex valued metric space. Let \( \{x_n\} \) be a sequence in \( X \) and \( x \in X \).

(i) If for every \( c \in \mathbb{C} \), with \( 0 < c \) there is \( n_0 \in \mathbb{N} \) such that \( d(x_n, x) < c \), for all \( n > n_0 \), then \( \{x_n\} \) is said to be convergent, \( \{x_n\} \) converges to \( x \), \( x \) is the limit point of \( \{x_n\} \). We denote this by \( \lim_{n \to \infty} x_n = x \) or \( x_n \to x \) as \( n \to \infty \).

(ii) If for every \( c \in \mathbb{C} \), with \( 0 < c \) there is \( N \in \mathbb{N} \) such that for all \( n > N \), \( d(x_n, x_{n+m}) < c \), where \( m \in \mathbb{N} \), then \( \{x_n\} \) is said to be Cauchy sequence.

(iii) If every Cauchy sequence is convergent in \((X, d)\), then \((X, d)\) is called a complete complex valued metric space.

**Lemma 1.1 ([3], Lemma 3).** Let \((X, d)\) be a complex valued metric space and let \( \{x_n\} \) be a sequence in \( X \). Then \( \{x_n\} \) is a Cauchy sequence if and only if \( |d(x_n, x_{n+m})| \to 0 \) as \( n \to \infty \).

**Lemma 1.2 ([3], Lemma 2).** Let \((X, d)\) be a complex valued metric space and let \( \{x_n\} \) be a sequence in \( X \). Then \( \{x_n\} \) converges to \( x \) if and only if \( |d(x_n, x)| \to 0 \) as \( n \to \infty \).

Given nonempty subsets \( A \) and \( B \) of complex valued metric space \((X, d)\). Then \( \{d(x, y) : x \in A, y \in B\} \subseteq \mathbb{C} \) is always bounded below by \( z_0 = 0 + i0 \) and hence \( \inf \{d(x, y) : x \in A, y \in B\} \) exists. Here we define

\[d(A, B) = \inf \{d(x, y) : x \in A \text{ and } y \in B\},\]
\[A_0 = \{ x \in A : d(x, y) = d(A, B) \text{ for some } y \in B\},\]
\[B_0 = \{ y \in B : d(x, y) = d(A, B) \text{ for some } x \in A\}.

From the above definition, it is clear that for every \( x \in A_0 \) there exists \( y \in B_0 \) such that \( d(x, y) = d(A, B) \) and conversely, for every \( y \in B_0 \) there exists \( x \in A_0 \) such that \( d(x, y) = d(A, B) \).
DEFINITION 1.6. Given non-self mapping \( S : A \to B \) and \( T : A \to B \), an element \( x \in X \) is called a common best proximity point of the mappings if they satisfy the condition that
\[
d(x, Sx) = d(x, Tx) = d(A, B).
\]

DEFINITION 1.7. Let \( (A, B) \) be a pair of nonempty subsets of a complex valued metric space \((X, d)\) with \( A_0 \neq \emptyset \). Then that pair \((A, B)\) is said to have the weak \( P \)-property if and only if
\[
\begin{align*}
\left\{ \begin{array}{ll}
\frac{d(x_1, y_1)}{d(x_2, y_2)} = \frac{d(A, B)}{d(A, B)} \Rightarrow d(x_1, x_2) \leq d(y_1, y_2),
\end{array} \right.
\end{align*}
\]
where \( x_1, x_2 \in A_0 \) and \( y_1, y_2 \in B_0 \).

DEFINITION 1.8. The mappings \( S : A \to B \) and \( T : A \to B \) are said to be commute proximally if there exists a non-negative real number \( \alpha \) such that for all \( u \in A \):
\[
\frac{d(Sx, Sy)}{d(Sx, Sy)} \leq \alpha \frac{d(Tx, Ty)}{d(Tx, Ty)} + \alpha_2 \frac{d(Tx, Sx)}{d(Tx, Sx)} + \alpha_3 \frac{d(Ty, Sy)}{d(Ty, Sy)} + \alpha_4 \left[ d(Ty, Sx) + d(Sy, Tx) \right].
\]

DEFINITION 1.9. Let \( S \) and \( T \) be two non-empty subsets of a complex valued metric space \((X, d)\). Non-self mappings \( S, T : A \to B \) are said to satisfy a \( L \)-contractive condition if there exist non-negative numbers \( \alpha_i \) where \( i = 1, \ldots, 4 \) and \( \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 < 1 \), then for each \( x, y \in A \),
\[
d(Sx, Sy) \leq \alpha_1 d(Tx, Ty) + \alpha_2 d(Tx, Sx) + \alpha_3 d(Ty, Sy) + \alpha_4 \left[ d(Ty, Sx) + d(Sy, Tx) \right].
\]

DEFINITION 1.10. A mapping \( T : A \to B \) is said to dominate a mapping \( S : A \to B \) proximally if there exists a non-negative real number \( \alpha < 1 \) such that for all \( u_1, u_2, v_1, v_2, x_1, x_2 \) in \( A \),
\[
d(u_1, Sx_1) = d(u_2, Sx_2) = d(A, B) = d(v_1, Tx_1) = d(v_2, Tx_2)
\Rightarrow d(u_1, u_2) \leq \alpha d(v_1, v_2)
\]

DEFINITION 1.11. A mapping \( T : A \to B \) is said to weakly dominate a mapping \( S : A \to B \) proximally if there exists a non-negative real number \( \alpha < 1 \) such that for all \( u_1, u_2, v_1, v_2, x_1, x_2 \) in \( A \),
\[
d(u_1, Sx_1) = d(u_2, Sx_2) = d(A, B) = d(v_1, Tx_1) = d(v_2, Tx_2)
\Rightarrow d(u_1, u_2) \leq \alpha \omega_{u_1, u_2, v_1, v_2},
\]
where \( \omega_{u_1, u_2, v_1, v_2} = \Re \omega_{u_1, u_2, v_1, v_2} + i \Im \omega_{u_1, u_2, v_1, v_2} \) and
\[
\begin{align*}
\Re \omega_{u_1, u_2, v_1, v_2} &= \max \{ \Re d(v_1, v_2), \Re d(v_1, u_1), \Re d(v_2, u_2) \}, \\
\Im \omega_{u_1, u_2, v_1, v_2} &= \max \{ \Im d(v_1, v_2), \Im d(v_1, u_1), \Im d(v_2, u_2) \}.
\end{align*}
\]

If \( T \) dominates \( S \) then \( T \) weakly dominates \( S \). But the converse is not true.
Example 1.2. Let us consider the complex valued metric space \((X, d)\) where \(X = \mathbb{C}\) and let \(d : X \times X \to \mathbb{C}\) be given as
\[
d(z_1, z_2) = |x_1 - x_2| + |y_1 - y_2|,\]
where \(z_1 = x_1 + iy_1\) and \(z_2 = x_2 + iy_2\). Let \(A\) and \(B\) be two subsets of \(X\) given by
\[
A = \{ z \in \mathbb{C} : \text{Re}(z) = -1, 0 \leq \text{Im}(z) \leq 1 \},
\]
\[
B = \{ z \in \mathbb{C} : \text{Re}(z) = 1, 0 \leq \text{Im}(z) \leq 1 \}.
\]
So we have that \(A_0 = A\), \(B_0 = B\) and \(d(A, B) = 2 + 0i\). Let \(T, S : A \to B\) be defined as
\[
Tz = -x + iy \quad \text{for each } z = x + iy \in A
\]
and
\[
S_z = \begin{cases} 
1 + i & 0 \leq y < 1 \\
1 + i & y = 1 
\end{cases}
\]
for each \(z = x + iy \in A\). If we suppose that \(v_1 = x_1 = -1 + \frac{12}{13}i, v_2 = x_2 = -1 + i, u_1 = -1 + \frac{1}{2}i, u_2 = -1 + \frac{1}{2}i\), it implies that
\[
d(u_1, Sx_1) = d(u_2, Sx_2) = d(A, B) = d(v_1, Tx_1) = d(v_2, Tx_2).
\]
Clearly, \(0 + \frac{1}{2}i = d(u_1, u_2) \leq \alpha d(v_1, v_2) = \alpha(0 + \frac{1}{13}i)\) for each non-negative real number \(\alpha < 1\). But obviously, we have that for \(\alpha = \frac{1}{5}\), \(T\) weakly dominates \(S\) proximally.

2. Common Best Proximity Point by Weakly Dominate Proximally Property

Theorem 2.1. Let \((X, d)\) be a complete complex valued metric space, \(A\) and \(B\) be two non-empty subsets of \(X\). Assume that \(A_0\) and \(B_0\) are nonempty and \(A_0\) is closed. Let \(S : A \to B\) and \(T : A \to B\) be two non-self mappings that satisfy the following conditions:

(a) \(T\) weakly dominates \(S\) proximally
(b) \(S\) and \(T\) commute proximally
(c) \(S\) and \(T\) are continuous
(d) \(S(A_0) \subseteq B_0\)
(e) \(S(A_0) \subseteq T(A_0)\)

Then there exists a unique element \(x \in A\) such that
\[
d(x, Tx) = d(A, B) \quad \text{and} \quad d(x, Sx) = d(A, B).
\]

Proof. Let \(x_0\) be a fixed element in \(A_0\). Since \(S(A_0) \subseteq T(A_0)\), then there exists an element \(x_1 \in A_0\) such that \(Sx_0 = Tx_1\). Then by continuing this process we can choose \(x_n \in A_0\) such that there exists \(x_{n+1} \in A_0\) satisfying
\[
Sx_n = Tx_{n+1} \quad \text{for each } n \in N
\]
since \(S(A_0) \subseteq B_0\), there exists an element \(u_n \in A\) such that
\[
d(Sx_n, u_n) = d(A, B) \quad \text{for each } n \in N.
\]
By choosing $x_n$ and $u_n$ it follows that
\begin{equation}
\tag{2.2}
d(Sx_n, u_n) = d(Sx_{n+1}, u_{n+1})
\end{equation}
and
\[ d(A, B) = d(Tx_n, u_{n-1}) = d(Tx_{n+1}, u_n). \]
Since $T$ weakly dominates $S$ proximally then we have
\[ d(u_n, u_{n+1}) \leq \alpha \omega_{u_n,u_{n+1},u_{n-1},u_n}, \]
where $\alpha < 1$ and
\[ Re \omega_{u_n,u_{n+1},u_{n-1},u_n} = \alpha \max\{ Re \ d(u_{n-1}, u_n), Re \ d(u_{n-2}, u_{n-1}) \}, \]
and
\[ Im \omega_{u_n,u_{n+1},u_{n-1},u_n} = \alpha \max\{ Im \ d(u_{n-1}, u_n), Im \ d(u_{n-2}, u_{n-1}) \}. \]
We focus on $Re \ d(u_n, u_{n+1})$ and conclude for $Im \ d(u_n, u_{n+1})$ and finally for $d(u_n, u_{n+1})$,
\[ Re \ d(u_n, u_{n+1}) \leq \alpha \max\{ Re \ d(u_{n-1}, u_n), Re \ d(u_{n-2}, u_{n-1}) \} \]
\[ \leq \alpha \max\{ Re \ d(u_{n-1}, u_n), Re \ d(u_{n-2}, u_{n-1}) \}. \]
We will prove that $\{ u_n \}$ is a Cauchy sequence. We distinguish two cases.

**Case I.** Suppose that
\[ Re \ d(u_n, u_{n+1}) \leq \alpha Re \ d(u_{n-1}, u_n), \]
so we get that
\[ Re \ d(u_n, u_{n+1}) \leq \alpha^n Re \ d(u_0, u_1), \]
Therefore for any $m > n$ we have
\[ Re \ d(u_n, u_m) \leq Re \ d(u_n, u_{n+1}) + Re \ d(u_{n+1}, u_{n+2}) + \ldots + Re \ d(u_{m-1}, u_m) \]
\[ \leq \alpha^n Re \ d(u_0, u_1) + \alpha^{n+1} Re \ d(u_0, u_1) + \ldots + \alpha^{m-1} Re \ d(u_0, u_1) \]
\[ \leq \left( \frac{\alpha^n}{1-\alpha} \right) Re \ d(u_0, u_1) \rightarrow 0 \quad \text{as} \quad m, n \rightarrow \infty. \]

**Case II.** Assume that
\[ Re \ d(u_n, u_{n+1}) \leq \alpha \frac{Re \ d(u_{n-1}, u_n) + Re \ d(u_n, u_{n+1})}{2} \]
\[ \leq \alpha \frac{\alpha/2}{1-\alpha/2} Re \ d(u_{n-1}, u_n). \]
Put $h = \frac{\alpha/2}{1-\alpha/2} < 1$, so we have that
\[ Re \ d(u_n, u_{n+1}) \leq h^n Re \ d(u_0, u_1). \]
It follows that for any $m > n$,
\[ \Re d(u_n, u_m) \leq \left( \frac{h^n}{1 - h} \right) \Re d(u_0, u_1) \to 0 \quad \text{as } m, n \to \infty. \]

Similarly we can conclude that for any $m > n$,
\[ \Im d(u_n, u_m) \leq \left( \frac{a^n}{1 - a} \right) \Im d(u_0, u_1) \to 0 \quad \text{as } m, n \to \infty, \]
or
\[ \Im d(u_n, u_m) \leq \left( \frac{h^n}{1 - h} \right) \Im d(u_0, u_1) \to 0 \quad \text{as } m, n \to \infty. \]

This implies that for any $m > n$,
\[ d(u_n, u_m) \to 0 \quad \text{as } m, n \to \infty. \]

Then $\{u_n\}$ is a Cauchy sequence and since $X$ is complete and $A_0$ is closed, there exists $u \in A_0$ such that $u_n \to u$. By hypothesis, mappings $S$ and $T$ are commuting proximally and by (2.2) we have that
\[ Tu_n = Su_{n-1}, \quad \text{for every } n \in N. \]

Since $T$ and $S$ are continuous it implies that
\[ Tu = \lim_{n \to \infty} Tu_n = \lim_{n \to \infty} Su_{n-1} = Su. \]

As $Su \in S(A_0) \subseteq B_0$, there exists an $x \in A_0$ such that
\[ d(x, Su) = d(A, B) = d(x, Tu). \tag{2.3} \]

Since $S$ and $T$ commute proximally, $Sx = Tx$. Also, $Sx \in S(A_0) \subseteq B_0$, there exists a $z \in A_0$ such that
\[ d(z, Sx) = d(A, B) = d(z, Tx). \tag{2.4} \]

Since $T$ weakly dominates $S$ then from (2.3) and (2.4) we can conclude that
\[ d(x, z) \geq \alpha \omega_{x,z,x,z} = \alpha (\Re d(x, z) + i\Im d(x, z)) = \alpha d(x, z). \]

It follows that $x = z$, therefore we have that
\[ d(x, Sx) = d(A, B) = d(x, Tx). \tag{2.5} \]

We now show that $S$ and $T$ have unique common best proximity point. For this, assume that $x^*$ in $A$ is a second common best proximity point of $S$ and $T$, then
\[ d(x^*, Sx^*) = d(A, B) = d(x^*, Tx^*). \tag{2.6} \]

Since $T$ weakly dominate $S$ proximally then from (2.5) and (2.6), we have
\[ d(x, x^*) \leq \alpha d(x, x^*). \]

Consequently, $x = x^*$ and $S$ and $T$ have a unique common best proximity point. \qed
Example 2.1. Let us consider the complex valued metric space \((X, d)\) where \(X = \mathbb{C}\) and let \(d : X \times X \to \mathbb{C}\) be given as
\[
d(z_1, z_2) = |x_1 - x_2| + i|y_1 - y_2|,
\]
where \(z_1 = x_1 + iy_1\) and \(z_2 = x_2 + iy_2\). Let \(A\) and \(B\) be two subsets of \(X\) given by
\[
A = \{ z \in \mathbb{C} : \text{Re}(z) = -1, \ 0 \leq \text{Im}(z) \leq 1 \} \\
\cup \{ z \in \mathbb{C} : \text{Re}(z) = 1, \ 0 \leq \text{Im}(z) \leq 1 \},
\]
\[
B = \{ z \in \mathbb{C} : \text{Re}(z) = -2, \ 0 \leq \text{Im}(z) \leq 1 \} \\
\cup \{ z \in \mathbb{C} : \text{Re}(z) = 2, \ 0 \leq \text{Im}(z) \leq 1 \}.
\]
Then \(A\) and \(B\) are closed and bounded subsets of \(X\) such that
\[
d(A, B) = 1; \quad A_0 = A, \quad B_0 = B.
\]

Let \(T, S : A \to B\) be defined as
\[
Tz = 2|x| + iy \text{ for each } z = x + iy \in A
\]
and
\[
Sz = 2|x| + i\frac{y}{2} \text{ for each } z = x + iy \in A.
\]
Therefore \(T\) and \(S\) satisfy the properties mentioned in Theorem 2.1. Hence the conditions of Theorem 2.1 are satisfied and \(1 + 0i\) is the unique common best proximity point of \(S\) and \(T\).

By Theorem 2.1 we obtain the following results in the fixed point theorem.

Corollary 2.1. Let \((X, d)\) be a complex valued metric space. Let \(T : X \to X\) be a continuous mapping and \(S\) be any self-mapping on \(X\) that commutes with \(T\). Further let \(S\) and \(T\) satisfy \(S(X) \subseteq T(X)\) and there exists a constant \(\alpha \in [0, 1)\) such that for every \(x, y \in X\)
\[
d(Sx, Sy) \leq \alpha \omega_{Sx,Sy,Tx,Ty},
\]
where
\[
\text{Re} \omega_{Sx,Sy,Tx,Ty} = \max\{ \text{Re} \ d(Tx,Ty), \text{Re} \ d(Tx,Sx), \text{Re} \ d(Ty,Sy), \frac{\text{Re} \ d(Tx,Sy) + \text{Re} \ d(Ty,Sx)}{2} \},
\]
and
\[
\text{Im} \omega_{Sx,Sy,Tx,Ty} = \max\{ \text{Im} \ d(Tx,Ty), \text{Im} \ d(Tx,Sx), \text{Im} \ d(Ty,Sy), \frac{\text{Im} \ d(Tx,Sy) + \text{Im} \ d(Ty,Sx)}{2} \}.
\]
Then \(S\) and \(T\) have a unique common fixed point.

If \(T\) is assumed to be identity mapping in Corollary 2.1, then we have the following result.

Corollary 2.2. Let \((X, d)\) be a complex valued metric space. Let \(S\) be a self-mapping on \(X\) and there exists a constant \(\alpha \in [0, 1)\) such that for every \(x, y \in X\)
\[
d(Sx, Sy) \leq \alpha \omega_{Sx,Sy,x,y},
\]
where \( \text{Re} \omega_{Sx,Sy,x,y} \)
\[ = \max\{\text{Re} \, d(x, y), \text{Re} \, d(x, Sx), \text{Re} \, d(y, Sy), \frac{\text{Re} \, d(x, Sy) + \text{Re} \, d(y, Sx)}{2}\}, \]
and
\[ \text{Im} \omega_{Sx,Sy,x,y} \]
\[ = \max\{\text{Im} \, d(x, y), \text{Im} \, d(x, Sx), \text{Im} \, d(y, Sy), \frac{\text{Im} \, d(x, Sy) + \text{Im} \, d(y, Sx)}{2}\}. \]

Then \( S \) has a fixed point.

3. Common Best Proximity Point for \( L \)-contractive Condition

Mappings

**Theorem 3.1.** Let \((X,d)\) be a complex valued metric space, \( A \) and \( B \) be two non-empty closed subsets of \( X \) and the pair \((A,B)\) satisfies the weak \( P \)-property. Let \( A_0 \) and \( B_0 \) are non-empty. Assume also that \( S, T : A \rightarrow B \) are two non-self mappings satisfying the following conditions:

- \( S \) and \( T \) commute proximally;
- \( S \) and \( T \) are continuous;
- \( S(A_0) \subseteq B_0 \) and \( S(A_0) \subseteq T(A_0) \);
- \( S \) and \( T \) satisfy \( L \)-contractive condition.

Then, there exists a unique point \( x \in A \) such that
\[ d(x, Tx) = d(A, B) = d(x, Sx). \]

**Proof.** Let \( x_0 \) be a fixed element in \( A_0 \). Since \( S(A_0) \subseteq T(A_0) \), then there exists an element \( x_1 \in A_0 \) such that \( Sx_0 = Tx_1 \). Then by continuing this process we can choose \( x_n \in A_0 \) such that there exists \( x_{n+1} \in A_0 \) satisfying
\[ Sx_n = Tx_{n+1} \quad \text{for each } n \in \mathbb{N}. \]

Since \( S(A_0) \subseteq B_0 \) there exists an element \( u_n \in A_0 \) such that
\[ d(Sx_n, u_n) = d(A, B) \quad \text{for each } n \in \mathbb{N}. \]

Further, it follows from the choice \( x_n \) and \( u_n \) that
\[ d(Sx_n, u_n) = d(A, B) = d(Sx_{n+1}, u_{n+1}). \]

By using the weak \( P \)-property and \( L \)-contractive condition, we have
\[ d(u_n, u_{n+1}) \leq d(Sx_n, Sx_{n+1}) \]
\[ \leq \alpha_1 d(Tx_n, Tx_{n+1}) + \alpha_2 d(Tx_n, x_n) + \alpha_3 d(Tx_{n+1}, x_{n+1}) + \alpha_4 [d(Tx_{n+1}, x_n) + d(Sx_{n+1}, Tx_n)] \]
\[ \leq \alpha_1 d(Sx_{n-1}, Sx_n) + \alpha_2 d(Sx_{n-1}, x_n) + \alpha_3 d(Sx_n, Sx_{n+1}) + \alpha_4 d(Sx_n, x_{n+1}) \]
\[ + \alpha_4 d(Sx_{n-1}, Sx_n) + \alpha_4 d(Sx_n, Sx_{n+1}). \]

Consequently, it implies that
\[ d(u_n, u_{n+1}) \leq h d(Sx_{n-1}, Sx_n) \leq \ldots \leq h^n d(Sx_0, Sx_1), \]
where \( h = \frac{\alpha_1 + \alpha_2 + \alpha_4}{1 - (\alpha_3 + \alpha_4)} < 1 \). Therefore, \( \{u_n\} \) is a Cauchy sequence and since \((X, d)\) is a complete complex valued metric space and \( A \) is closed, then there exists \( u \in A \) such that \( u_n \to u \) as \( n \to \infty \). Also, we have that
\[
d(Sx_n, u_n) = d(A, B) = d(Tx_n, u_{n-1}),
\]
Since \( S \) and \( T \) commute proximally we get that
\[
Tu_n = S^1u_{n-1}.
\]
Thus, it follows that \( Tu = Su \), because \( S \) and \( T \) are continuous. Since \( \{Sx_n\} \) is also a Cauchy sequence, \( X \) is complete and \( B \) is closed we can easily prove that \( Su \in S(A_0) \subseteq B_0 \). Therefore, there exists \( x \in A_0 \) such that
\[
d(x, Su) = d(A, B) = d(x, Tu).
\]
Therefore, \( Tx = Sx \), because \( S \) and \( T \) commute proximally. Since \( Sx \in S(A_0) \subseteq B_0 \), there exists \( z \in A_0 \), it implies that
\[
d(z, Sx) = d(A, B) = d(z, Tx).
\]
By L-contractive condition, we get that
\[
d(Su, Sx) \leq \alpha_1 d(Tu, Tx) + \alpha_2 d(Su, Tu) + \alpha_3 d(Sx, Tx)
\]
\[
+ \alpha_4 [d(Su, Tx) + d(Sx, Tu)]
\]
(3.5)
Therefore, \( Su = Sx \). From (3.3) and (3.4) we have
\[
d(x, Su) = d(A, B) = d(z, Sx),
\]
the weak P-property of the pair \((A, B)\) implies
\[
d(x, z) \leq d(Sx, Su) = 0.
\]
So \( x = z \) and
(3.6)
\[
d(x, Sx) = d(A, B) = d(x, Tx).
\]
Suppose that \( x^* \) is another common best proximity point of the mappings \( S \) and \( T \) so that
(3.7)
\[
d(x^*, Sx^*) = d(A, B) = d(x^*, Tx^*).
\]
Since \( S \) and \( T \) commute proximally, then \( Sx = Tx \) and \( Sx^* = Tx^* \). So we have
\[
d(Sx, Sx^*) \leq \alpha_1 d(Tx, Tx^*) + \alpha_2 d(Tx, Sx) + \alpha_3 d(Tx^*, Sx^*)
\]
\[
+ \alpha_4 [d(Tx^*, Sx) + d(Tx, Sx^*)]
\]
\[
= (\alpha_1 + 2\alpha_4) d(Sx, Sx^*),
\]
Which implies that \( Sx = Sx^* \). Since the pair \((A, B)\) satisfies weak P-property, from (3.6) and (3.7) we have that
\[
d(x, x^*) \leq d(Sx, Sx^*).
\]
Eventually, we have that \( x = x^* \). Hence \( S \) and \( T \) have a unique common best proximity point. \( \square \)
Example 3.1. Let \((X,d)\) be a complex valued metric space defined as in Example 2.1 and \(A, B\) be two subsets of \(X\) given by
\[
A = \{ z \in \mathbb{C} : \text{Re}(z) = 0, 0 \leq \text{Im}(z) \leq 1 \}, \\
B = \{ z \in \mathbb{C} : \text{Re}(z) = 1, 0 \leq \text{Im}(z) \leq 1 \}.
\]
Let \(T, S : A \to B\) be defined as
\[
T(0 + iy) = 1 + iy \text{ for each } 0 \leq y \leq 1 \\
S(0 + iy) = 1 + i\frac{y}{4} \text{ for each } 0 \leq y \leq 1.
\]
Then \((A, B)\) is a pair of nonempty closed and bounded subsets of \(X\) such that \(A_0 = A, B_0 = B\) and \(d(A, B) = 1 + 0i\). It is verified that the \((A, B)\) satisfies the weak P-property. Also \(T\) and \(S\) satisfy the properties mentioned in Theorem 3.1. Hence the conditions of Theorem 3.1 are satisfied and it is seen that \(0 = 0 + i0\) is the unique common best proximity point of \(S\) and \(T\).

If we suppose that \(S\) and \(T\) are self-mappings, then Theorem 3.1 implies the following common fixed point theorem, that generalizes and complements the results of [5], [6], [10], [11] and others in complex valued metric spaces.

Corollary 3.1. Let \((X,d)\) be a complete complex valued metric space. Assume that \(S, T : X \to X\) are two self mappings satisfying the following conditions:

(a) there exist non-negative numbers \(\alpha_i\) where \(i = 1, ..., 4\) and \(\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 < 1\), such that for each \(x, y \in A\),
\[
d(Sx, Sy) \leq \alpha_1d(Tx, Ty) + \alpha_2d(Tx, Sx) + \alpha_3d(Ty, Sy) + \alpha_4[d(Ty, Sx) + d(Sy, Tx)].
\]

(b) \(S\) and \(T\) commute;
(c) \(T\) is continuous;
(d) \(S(X) \subseteq T(X)\);

Then \(S\) and \(T\) have a unique common fixed point.

References


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