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BI-CONDITIONAL DOMINATION RELATED PARAMETERS OF A GRAPH-I

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Abstract. In a graph $G = (V, E)$, a set $D \subseteq V$ is a dominating set of G. The Bi-conditional domination number $\gamma(G : \mathcal{P}_i)$ for $1 \leqslant i \leqslant 6$, is the minimum cardinality of a dominating set D such that induced subgraph $\langle D\rangle$ and $\langle V-D\rangle$ satisfy the following property:
$\mathcal{P}_1$: $\langle D\rangle$ and $\langle V-D\rangle$ are totally disconnected.
$\mathcal{P}_2$: $\langle D\rangle$ and $\langle V-D\rangle$ have no isolated vertices.
$\mathcal{P}_3$: $\langle D\rangle$ and $\langle V-D\rangle$ have a perfect matching.
$\mathcal{P}_4$: $\langle D\rangle$ and $\langle V-D\rangle$ are complete graphs.
$\mathcal{P}_5$: $\langle D\rangle$ and $\langle V-D\rangle$ are the union of vertex disjoint cycles.
$\mathcal{P}_6$: $\langle D\rangle$ and $\langle V-D\rangle$ are acyclic.
In this paper, we initiate a study of these new parameters and obtain some bounds and properties on these parameters.

1. Introduction

All graphs considered here are finite, nontrivial, undirected with no loops and multiple edges. As usual $p = |V|$ and $q = |E|$ denote the number of vertices and edges of a graph G, respectively. In general, we use $\langle X\rangle$ to denote the subgraph induced by the set of vertices X. $N(v)$ and $N[v]$ denote the open and closed neighborhoods of a vertex v, respectively. Let $deg(v)$ be the degree of a vertex v and as usual $\delta(G)$, the minimum degree and $\triangle(G)$, the maximum degree of a graph G. A vertex of degree one is called a leaf and its neighbor is a support vertex. Unless defined or mentioned otherwise, we refer to the reader to Harary [**8**] for standard terminology and notation in graph theory.

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A set $D \subseteq V$ is a dominating set of G if every vertex in $V - D$ is adjacent to some vertex in D. The minimum cardinality taken over all dominating sets in G is called the domination number and is denoted by $\gamma(G)$. The concept of domination has existed and studied for a long time. Books on domination [**9**], [**10**] and [**18**] have stimulated sufficient inspiration leading to the expansive growth of this field.

Let $D \subseteq V$ be a dominating set of G. Then

$\mathcal{P}_1$: $\langle D\rangle$ and $\langle V - D\rangle$ are totally disconnected.
$\mathcal{P}_2$: $\langle D\rangle$ and $\langle V - D\rangle$ have no isolated vertices.
$\mathcal{P}_3$: $\langle D\rangle$ and $\langle V - D\rangle$ have a perfect matching.
$\mathcal{P}_4$: $\langle D\rangle$ and $\langle V - D\rangle$ are complete graphs.
$\mathcal{P}_5$: $\langle D\rangle$ and $\langle V - D\rangle$ are the union of vertex disjoint cycles.
$\mathcal{P}_6$: $\langle D\rangle$ and $\langle V - D\rangle$ are acyclic.

A dominating set D_i of G is called a bi-conditional dominating set if D_i satisfies the property $\mathcal{P}_i$, $1 \leqslant i \leqslant 6$. The Bi-conditional domination number $\gamma(G : \mathcal{P}_i)$ for $1 \leqslant i \leqslant 6$, is the minimum cardinality of a dominating set D_i of G. A graph G is called a $\mathcal{P}_i$-graph if it has a bi-conditional dominating set D with respect to P_i for $1 \leqslant i \leqslant 6$. For more details on Bi-conditional domination related parameters on connected domination due to Cyman et al. [**7**] and other domination related parameters, refer [**2**], [**3**] and [**16**].

2. Bi-independent Domination

A set $D \subseteq V$ is a bi-independent dominating (BID) set of G, if it satisfies the property $\mathcal{P}_1$. The minimum cardinality taken over all BID-sets is called the bi-independent domination number and is denoted by $\gamma(G : \mathcal{P}_1)$. For more details, refer [**1**], [**14**] and [**17**].

First, we start with couple of Propositions, which are starightforward.

Proposition 2.1. *For any path P_p with $p \geqslant 2$ vertices,*

$$\gamma(P_p : \mathcal{P}_1) = \begin{cases} \frac{p}{2}, & \text{if } p \text{ is even} \\ \frac{p-1}{2}, & \text{if } p \text{ is odd.} \end{cases}$$

Proposition 2.2. *For any cycle C_p with $p = 2n; n \geqslant 2$ vertices,*

$$\gamma(C_p : \mathcal{P}_1) = \frac{p}{2}.$$

Proposition 2.3. *For a complete bipartite graph $K_{r,s}$ with $1 \leqslant r \leqslant s$ vertices,*

$$\gamma(K_{r,s} : \mathcal{P}_1) = r.$$

Theorem 2.1. *A nontrivial graph G is a $\mathcal{P}_1$-graph if and only if G is bipartite.*

Proof. Let G be a bipartite graph and let (V_1, V_2) be a bipartition of G with V_1 contains all the isolated vertices. It is clear that V_1 is an independent dominating set and $V_2 = V - V_1$ is also an independent set. Hence V_1 satisfy the property $\mathcal{P}_1$. Hence G is a $\mathcal{P}_1$-graph.

Conversely, suppose, the graph G is not a bipartite, then it contains an odd cycle. So we can not partition V into two independent vertex subsets. Hence, there exists no a BID-set, a contradiction to the fact that G is a $\mathcal{P}_1$-graph. Therefore, G is bipartite. □

By above theorem, we characterize an independent dominating set and BID-set of a graph G.

OBSERVATION 2.1. *If G is a $\mathcal{P}_1$- graph then $\gamma_i(G) = \gamma(G : \mathcal{P}_1)$, where $\gamma_i(G)$ is the independent domination number of G.*

PROPOSITION 2.4. *Let G be a $\mathcal{P}_1$-graph. Then the difference $\gamma(G : \mathcal{P}_1) - \gamma(G)$ can be arbitrary large.*

PROOF. Consider a complete bipartite graph $K_{r,s}$ with $1 \leqslant r \leqslant s$ vertices. By the definition of domination number, we have $\gamma(K_{r,s}) = 2$ and by Proposition 2.3, we have $\gamma(K_{r,s} : \mathcal{P}_1) = r$. Thus $\gamma(K_{r,s} : \mathcal{P}_1) - \gamma(K_{r,s}) = r - 2$ for $r \geqslant 3$ vertices. □

3. Bi-Total Domination

A set $D \subseteq V$ is a bi-total dominating (BTD) set of G, if it satisfies the property $\mathcal{P}_2$. The minimum cardinality taken over all BTD-sets is called the bi-total domination number and is denoted by $\gamma(G : \mathcal{P}_2)$. For more details, we refer to [**5**] and [**15**].

Bi-total domination is defined only for graphs without isolated vertices. In this section, we consider BTD- set D such that $\mid V - D \mid \neq \phi$, which is possible only for graphs of order at least four.

OBSERVATION 3.1. *For any graph G with no isolated vertices,*

$$\gamma(G) \leqslant \gamma_t(G) \leqslant \gamma(G : \mathcal{P}_2).$$

PROPOSITION 3.1. *For any complete graph K_p, fan graph $F_p = K_1 + P_{p-1}$, wheel $W_p = K_1 + C_{p-1}$ and complete bipartite graph $K_{m,n}$, with $p \geqslant 4$ and $2 \leqslant m \leqslant n$ vertices,*

$$\gamma(K_p : \mathcal{P}_2) = \gamma(W_p : \mathcal{P}_2) = \gamma(F_p : \mathcal{P}_2) = \gamma(K_{m,n} : \mathcal{P}_2) = 2.$$

PROPOSITION 3.2. *Let C_p be a cycle. If $p = 4m + k$ with $m \geqslant 1$ and $0 \leqslant k \leqslant 3$, then $\gamma(C_p : \mathcal{P}_2) = 2m + k$.*

PROOF. Let C_p be a cycle with labeled as $C_p : v_1, v_2, v_3, v_4, \ldots, v_p, v_1$. Now we construct a minimum BTD-set. Since, D is the BTD-set of C_p , and necessary to choose the adjacent vertices $v_1, v_2 \in D$ and $v_3, v_4 \in V - D$; $v_5, v_6 \in D$ and so on. To complete the formation of D, here the following cases arise.

Case 1. If $p = 4m$, $m \geqslant 1$, it has to end up with a pair of vertices $v_{4m-1}, v_{4m} \in V - D$, and the resulting set D is a minimum BTD-set containing $2m$ vertices.

Case 2. If $p = 4m + 1$, $m \geqslant 1$, it has to ends up with a pair of adjacent vertices $v_{4m-1}, v_{4m} \in V - D$. The left out vertex $p = 4m + 1$ is dominated by a vertex v_1, but the vertex v_{4m} which we have already belongs in $V - D$ is not dominated by any of the vertices in D. Hence, it is necessary to choose the vertex $p = 4m + 1 \in D$.

Hence, the constructed set D is the minimum BTD-set containing $2m+1$ vertices.

Case 3. If $p = 4m + 2$, $m \geqslant 1$, it has to end up with a pair of adjacent vertices $v_{4m+1}, v_{4m+2} \in D$. Hence the constructed set D is the minimum BTD-set containing $2m + 2$ vertices.

Case 4. If $p = 4m + 3$, $m \geqslant 1$, it has to end up with a pair of adjacent vertices v_{4m+1}, v_{4m+2} in D. Now D contains $2m + 2$ vertices and $V - D$ contains $2m$ vertices. The left out vertex $4m+3$ is dominated by the vertex v_1 which is in D. Now, let v_{4m+3} in to $V - D$ that will give isolated vertex in $V - D$. Hence, it is necessary to choose $4m + 3$ also in D. Hence D is the γ_{bt}-set containing $2m + 2 + 1 = 2m + 3$ vertices. Hence the proof. □

THEOREM 3.1. *Let G be a r-regular graph. If $r \geqslant (p - 2)$ with $p \geqslant 4$ vertices, then $\gamma(G : \mathcal{P}_2) = 2$.*

PROOF. Let G be a regular graph with regularity at least $p-2$. First we prove, the set $D \subset V(G)$ consisting of two adjacent vertices forms a minimum BTD-set. Here D can not be minimize further, because D does not contain isolated vertex. Since, the degree of each vertex in $V-D$ is at least $p-2$ depending on the regularity of G, each vertex in $V - D$ is adjacent to at least one vertex of D. Hence, D is a minimum dominating set such that $\langle D\rangle$ has no isolated vertices.
Now we prove $\langle V - D\rangle$ also has no isolated vertices. The following two cases arise:

Case 1. Suppose G is a $(p - 1)$-regular graph. Then the graph G is a complete graph with at least four vertices. Clearly, $V - D$ contains at least two vertices. Hence, $\langle V - D\rangle$ has no isolated vertices.

Case 2. Suppose G is a $(p - 2)$-regular graph. On the contrary $\langle V - D\rangle$ contains an isolated vertex. If $p = 4$ then $G = C_4$. If $p \geqslant 5$ then each vertex of G has degree at least three. Let u be an isolated vertex in $\langle V - D\rangle$. Then even if u is adjacent to all the vertices in D we have $deg(u) = 2$, which is a contradiction. Hence $\langle V - D\rangle$ has no isolates. Thus the result follows. □

THEOREM 3.2. *Let G be a graph with $p \geqslant 4$ vertices. Then G has a BTD-set if and only if there exist at least two vertices $u, v \in V(G)$ such that $uv \in E(G)$, $deg(u) \geqslant 2$, $deg(v) \geqslant 2$, u and v are not the support vertices.*

PROOF. Suppose D is a BTD-set of G. On contrary, if there exist at least two adjacent vertices u and v in $V(G)$, does not satisfy the given condition, then it is necessary to take all the pendant vertices and their respective adjacent vertices in to the set D. The remaining vertices (if any) are dominated and form an independent set, but $V - D$ does not contain any of these remaining vertices, otherwise we have isolates in $V - D$. Hence, D contains all the vertices of G, which is a contradiction to the fact that $V - D$ is nonempty. This proves the necessity.
The sufficiency is straightforward. □

4. Bi-Paired Domination

A set $D \subseteq V$ is a bi-paired dominating (BPD) set of G, if it satisfies the property $\mathcal{P}_3$. The minimum cardinality taken over all BPD-sets is called the bi-paired domination number and is denoted by $\gamma(G : \mathcal{P}_3)$. For more details, we refer to [**11**].

Observation 4.1. *If G is a $\mathcal{P}_3$-graph, then G contains even number of vertices and $|V(G)| \geqslant 4$.*

Observation 4.2. *For any nontrivial graph G,*

$$\gamma(G) \leqslant \gamma_p(G) \leqslant \gamma(G : \mathcal{P}_3).$$

Theorem 4.1. *If a graph G is a $\mathcal{P}_3$-graph then G contains no support vertex which supports at least two vertices.*

Proof. Suppose u be a support vertex of v and w. Let D be any BPD-set of G. If v or $w \in V - D$ then $\langle V - D\rangle$ has no perfect matching. Hence $v, w \in D$. Let F be a matching in $\langle D\rangle$. If $uv \in F$ then there is no edge in F to cover w. If $uv \in F$ then there is no edge in F to cover v. Hence F is not a perfect matching of $\langle D\rangle$. Thus D is not a BID-set. This proves the necessity.

The sufficiency is obvious. □

By above theorem we conclude that not all trees are $\mathcal{P}_3$ - graphs. In the following results we construct different classes of trees which are $\mathcal{P}_3$-graphs.

Theorem 4.2. *A path P_p is a $\mathcal{P}_3$ - graph if and only if $p = 4n + 2, n \geqslant 1$.*

Proof. Let P_p be a $\mathcal{P}_3$ - graph. Then there exist a BPD-set D of a graph G. Clearly, D contains pair of consecutive vertices and $V - D$ contains the remaining pairs of consecutive vertices and hence both D and $V - D$ contains even number of vertices, whose induced subgraph contains perfect matching, respectively. Thus the number of vertices in P_p is $|D \cup (V - D)|$. This implies that $p = |D \cup (V - D)| = 2 + 2n + 2n = 4n + 2$.

Conversely, let P_p be a path on $p = 4n+2$, $n \geqslant 1$ vertices. The set D containing first pair of consecutive vertices and every alternating pairs of consecutive vertices form a BPD-set D. Hence, P_p is a $\mathcal{P}_3$ - graph. □

Proposition 4.1. *If G is a path P_p with $p = 4n + 2$, $n \geqslant 1$ vertices, then*

$$\gamma(P_p : \mathcal{P}_3) = 2(n + 1).$$

Now we give a class of trees other than paths which are $\mathcal{P}_3$-graphs.

Theorem 4.3. *Let T be a tree with $p = 4n + 6$, $n \geqslant 1$, vertices. Then T is a $\mathcal{P}_3$-graph.*

Proof. Since every BPD-set D of a tree T is formed by taking both end vertices of all pendant edges into th set D and other vertices into the set $V - D$. Clearly the graph $\langle D\rangle$ has a perfect matching. The construction of tree T by using Theorem 4.2, we have $\langle V - D\rangle$ is also a perfect matching. Further, every vertex in

$V - D$ is adjacent to some vertex in D, hence D is a dominating set such that both $\langle D\rangle$ and $\langle V - D\rangle$ have perfect matchings. Hence the result follows. □

OBSERVATION 4.3. *A BPD-set D consists of pendant vertices and their respective support vertices.*

Consider the graph C_6 is not a $\mathcal{P}_3$-graph. For this instance, in our next result, we characterize cycles which are $\mathcal{P}_3$-graphs.

THEOREM 4.4. *A cycle C_p is a $\mathcal{P}_3$-graph if and only if $p = 4n$, $n \geqslant 1$.*

PROOF. Let a cycle C_p be a $\mathcal{P}_3$ - graph. Then C_p contains a BPD-set D such that both $\langle D\rangle$ and $\langle V - D\rangle$ contain a perfect matching. The number of edges in $\langle D\rangle$ is same as the number of edges in $\langle V - D\rangle$, otherwise $\langle D\rangle$ or $\langle V - D\rangle$ does not contain a perfect matching. Also $\langle D\rangle$ and $\langle V - D\rangle$ both consist of only independent edges, implies $|D| = 2n$ and $|V - D| = 2n$. Hence $|V| = |D| + |V - D| = 2n + 2n = 4n$.

Conversely, let C_p be a cycle on $p = 4n$, $n \geqslant 1$ vertices. Choosing the vertices v_{4m-3} and v_{4m-2}, where $1 \leqslant m \leqslant n$, into the set D and the other vertices into the set $V - D$, we get a dominating set D such that both $\langle D\rangle$ and $\langle V - D\rangle$ have a perfect matching. Therefore, D is a BPD-set in C_p. Hence, C_p is a $\mathcal{P}_3$-graph. □

COROLLARY 4.1. *For any positive integer $l \geqslant 1$, there exists a $\mathcal{P}_3$ - graph such that $\gamma(G : \mathcal{P}_3) = 2l$.*

OBSERVATION 4.4. *Theorem 4.4 shows the existence of graphs other than trees which are $\mathcal{P}_3$-graphs.*

DEFINITION 4.1. *A $\mathcal{P}_3$-graph G is said to be a $\mathcal{P}_3^{'}$-graph if both $E(\langle D\rangle)$ and $E(\langle V - D\rangle)$ are perfect matchings in $\langle D\rangle$ and $\langle V - D\rangle$ respectively.*

REMARK 4.1. *In a $\mathcal{P}_3$-graph, $\langle D\rangle$ and $\langle V - D\rangle$ may contain more than one perfect matching, That is $|M| \leqslant |E(\langle D\rangle)|$ and $|M^{'}| \leqslant |E(\langle V - D\rangle)|$ where M and $M^{'}$ are perfect matchings in $\langle D\rangle$ and $\langle V - D\rangle$ respectively. In $\mathcal{P}_3^{'}$-graph $M = E(\langle D\rangle)$ and $M^{'} = E(\langle V - D\rangle)$ where M and $M^{'}$ are perfect matchings in $\langle D\rangle$ and $\langle V - D\rangle$ respectively.*

THEOREM 4.5. *If G is a $\mathcal{P}_3^{'}$-graph with $\gamma(G : \mathcal{P}_3) = k$, then*

$$\frac{3p - 2k}{2} \leqslant q \leqslant \frac{p + 2k(p - k)}{2}$$

PROOF. Let G be a $\mathcal{P}_3^{'}$-graph of order p. If D is a BPD-set of G, then the number of edges in $\langle D\rangle \cup \langle V - D\rangle$ is $\frac{p}{2}$. Since D is a dominating set of G and each vertex in $V - D$ should have at least one vertex in D adjacent to it. Therefore the number of edges between D and $V - D$ is at least $p - k$. Hence the number of edges in any $\mathcal{P}_3^{'}$-graph is at least $\frac{p}{2} + p - k = \dfrac{3p - 2k}{2}$. Thus the lower bound follows. Since every vertex in $V - D$ can be adjacent to at most k vertices in D, the number of edges between D and $V - D$ is at most $k(p - k)$. Then the number of edges in a $\mathcal{P}_3^{'}$-graph is at most $\dfrac{p + 2k(p - k)}{2}$. Thus the upper bound follows. □

THEOREM 4.6. *If G is a $\mathcal{P}_3'$-graph with $\delta(G) \geqslant 2$, then*

$$\gamma(G : \mathcal{P}_3) \leqslant \frac{p}{2}.$$

PROOF. Let G be a $\mathcal{P}_3'$-graph with $\delta(G) \geqslant 2$. Let D be a BPD-set of G. Then $\langle D \rangle$ and $\langle V - D \rangle$ are perfect matching. Since $\delta(G) \geqslant 2$, every vertex in D is adjacent to some vertex in $V - D$ and every vertex in $V - D$ is adjacent to some vertex in D. Hence, both D and $V - D$ are dominating sets. If D is the smallest among D and $V - D$, we have $|D| \leqslant |V - D|$, otherwise renaming $V - D$ as D and D as $V - D$, we get $|D| \leqslant |V - D|$. Thus the result follows. □

THEOREM 4.7. *If G is a $\mathcal{P}_3'$-graph with $\gamma(G : \mathcal{P}_3) = k$ and $\delta(G) \geqslant 2$, then $deg(v) \leqslant p - k + 1$, for all $v \in V(G)$.*

PROOF. Let G be a $\mathcal{P}_3'$-graph with $\gamma(G : \mathcal{P}_3) = k$ and $\delta(G) \geqslant 2$. Then there exists a minimum BPD - set D with $|D| = k$ and $|V - D| = p - k$. By Theorem 4.6, we have $|D| \leqslant |V - D|$ and each vertex $v \in D$ is adjacent to at most $p - k$ vertices of $V - D$ and exactly one vertex in D. Hence $deg(v) \leqslant p - k + 1$, for all $v \in D$. Also, each vertex $v \in V - D$ is adjacent to at most k vertices of D and exactly one vertex of $V - D$. Hence, $deg(v) \leqslant k + 1 \leqslant p - k + 1$ for all $v \in V - D$. Hence $deg(v) \leqslant p - k + 1$ for all $v \in V(G)$. □

5. Bi-Clique Domination

A set $D \subseteq V$ is a bi-clique dominating (BCLD) set of G, if it satisfies the property $\mathcal{P}_4$. The minimum cardinality taken over all $BCLD$-sets is called the bi-clique domination number and is denoted by $\gamma(G : \mathcal{P}_4)$. For more details, we refer to [**6**].

THEOREM 5.1. *For any graph G, $\gamma(G : \mathcal{P}_4) = 1$ if and only if $G = K_p$.*

PROOF. Let $\gamma(G : \mathcal{P}_4) = 1$ and $D = \{u\}$, where u is any vertex in G. Suppose, G is not a complete graph then there exist at least two non adjacent vertices, say v, w other than u in $V - D$, which is a contradiction to the fact that D is a minimum $BCLD$-set of a graph G. Hence, G must be a complete graph.

Conversely, suppose G is a complete graph, then any singleton subset of $V(G)$ forms a $BCLD$-set of a graph G. Hence the result follows. □

To prove our next result we make use of the definition.

DEFINITION 5.1. *The sequential join of the graphs $G_1, G_2, G_3, \ldots, G_k$, $k \geqslant 3$ is $G_1 + G_2 + G_3 + \ldots + G_k = (G_1 + G_2) \cup (G_2 + G_3) \cup \ldots \cup (G_{k-1} + G_k)$.*

In order to prove the next result for finding the minimum $BCLD$-set, we consider the sequential join graph with $k = 3$.

THEOREM 5.2. *For any sequential join graph,*

$$\gamma(K_m + K_1 + K_n : \mathcal{P}_4) = \begin{cases} m + 1, & \text{if } m \leqslant n \\ n + 1, & \text{if } n \leqslant m. \end{cases}$$

PROOF. Let G be a sequential join graph. If, we consider $G_1 = K_m$, $G_2 = K_1$ and $G_3 = K_n$, then the following cases arise.

Case 1. If $m = n$, then a $BCLD$-set, $D = V(K_m) \cup \{u\}$ forms a minimum $BCLD$-set of G. Hence, $\gamma(K_m + K_1 + K_n : \mathcal{P}_4) = m + 1 = n + 1$.

Case 2. If $m < n$, then a $BCLD$-set, $D = V(K_m) \cup \{u\}$ forms a minimum $BCLD$-set of G. Hence, $\gamma(K_m + K_1 + K_n : \mathcal{P}_4) = m + 1$.

Case 3. If $m > n$, then a $BCLD$-set, $D = V(K_n) \cup \{u\}$ forms a minimum $BCLD$-set of G. Hence, $\gamma(K_m + K_1 + K_n : \mathcal{P}_4) = n + 1$. □

THEOREM 5.3. *Let G be a $\mathcal{P}_4$-graph. Then*

$$max\{diam(G), \beta_0(G)\} \leqslant 2.$$

PROOF. Let D be a $BCLD$-set of a graph G. If the eccentricity of a vertex in D is less than or equal to two, then the following cases arise.

Case 1. If a vertex $u \in D$ is adjacent to every vertex in $V - D$, then $e(u) = 1$, since $\langle D \rangle$ is a complete subgraph.

Case 2. If $V - D$ contains a vertex, say v, not adjacent to u, then $d(u, v) = 2$, since there exists a vertex say $w \in D$ adjacent to v.

Hence, the eccentricity of every vertex in D is less than or equal to 2.

Similarly, we can prove eccentricity of every vertex in $V - D$ is also less than or equal two. Hence, $diam(G) \leqslant 2$.

Now, we show $\beta_0(G) \leqslant 2$. Suppose on the contrary $\beta_0(G) \geqslant 3$ then every dominating set D or its complements contains at least two vertices of β_0-set of a graph G. Hence there does not exist a $BCLD$-set, which is a contradiction. Hence $\beta_0(G) \leqslant 2$. Thus the result follows. □

6. Bi-Cyclic Domination

A set $D \subseteq V$ is a bi-cyclic dominating (BCD) set of G, if it satisfy the property $\mathcal{P}_5$. The minimum cardinality taken over all BCD-sets is called the bi-cyclic domination number and is denoted by $\gamma(G : \mathcal{P}_5)$. For more details, we refer to [**16**].

OBSERVATION 6.1. *If G is a $\mathcal{P}_5$-graph, then $|V(G)| \geqslant 6$ and $|E(G)| \geqslant 9$.*

PROPOSITION 6.1. *For any graph G,*

$$\gamma(G) \leqslant \gamma_c(G) \leqslant \gamma(G : \mathcal{P}_5).$$

THEOREM 6.1. *Let G be a $\mathcal{P}_5$-graph with $p \geqslant 6$ vertices. If $\gamma(G : \mathcal{P}_5) = k$, where $k \geqslant 3$ is a positive integer, then*

$$2p - k \leqslant q \leqslant p + k(p - k).$$

PROOF. Let G be a $\mathcal{P}_5$-graph of order p and D be a BCD-set. The number of edges in $\langle D \rangle + \langle V - D \rangle = p$. Since D is a dominating set for each vertex in $V - D$ should have at least one neighbor in D, the number of edges between D and $V - D$ is at least $p - k$. Hence the number of edges in any $\mathcal{P}_5$-graph is at least $p + p - k = 2p - k$. That is $q \geqslant 2p - k$.

For upper bound each vertex in $V-D$ can be adjacent to at most k vertices in D. Then the number of edges between D and $V-D$ is at most $k(p-k)$. Hence, the number of edges in $\mathcal{P}_5$-graph is at most $p+k(p-k)$. Thus the result follows. $\square$

To prove our next result we make use of the following definition.

The *cartesian product* of two graphs G and H, denoted by $G\Box H$, is a graph with vertex set $V(G\Box H) = V(G)\times V(H)$, that is, the set $\{(g,h)/g \in G, h \in H\}$. The edge set of $G\Box H$ consists of all pairs $[(g_1,h_1),(g_2,h_2)]$ of vertices with $[g_1,g_2]\in E(G)$ and $h_1=h_2$, or $g_1=g_2$ and $[h_1,h_2] \in E(H)$. The *prism* of a graph G^* is defined as the cartesian product of $G\Box K_2$.

THEOREM 6.2. *Let $G^* = C_p\Box K_2$ be a prism graph. Then G^* satisfies the following conditions:*

(i) $\mathcal{P}_5$*-graph,*
(ii) $\gamma(G^*:\mathcal{P}_5)=p$.

PROOF. Let G^* be the prism of C_p. If $C_p^{'}$ and $C_p^{''}$ are two copies of C_p in the prism G^*, then the set of vertices of $C_p^{'}$ forms a dominating set such that both $V(C_p^{'})$ and $V(G^*)-V(C_p^{'})$ are cyclic. Hence, $V(C_p^{'})$ is a BCD- set of G^*. Therefore G^* is a $\mathcal{P}_5$-graph. Thus, we have $\gamma(G^*:\mathcal{P}_5)=p$.

To prove $\gamma(G^*:\mathcal{P}_5)\geqslant p$, we assert that $\gamma(G^*:\mathcal{P}_5)\leqslant p-1$. Let $S\subset V(G^*)$ be a set consisting of at most $p-1$ vertices. The the following cases arise.

Case 1. $S\subset V(C_p^{'})$.
Let $X=V(C_p^{'})-S$, then the set of vertices in $V(C_p^{''})$, which is the image set of T are not dominated. Hence S is not a dominating set.

Case 2. $S\subset V(C_p^{''})$.
Let $X=V(C_p^{''})-S$, then the set of vertices in $V(C_p^{'})$, which is the pre-image set of X are not dominated. Hence, S is not a dominating set.

Case 3. $S\cap V(C_p^{'})\neq\phi$ and $S\cap V(C_p^{''})\neq\phi$.
Let $S\cap V(C_p^{'})=A$, $S\cap V(C_p^{''})=B$ and $A'\subset V(C_p^{''})$ be the mirror image of A. If $A'\cap B=\emptyset$, then $\langle A\cup B\rangle$ is not two regular. If $A'\cap B\neq\emptyset$, then also $\langle A\cup B\rangle$ is not two regular. Hence, there exists no BCD-set S, with $|S|\leqslant p-1$. Hence, $V(C_p^{'})$ is a minimum BCD-set satisfying property $\mathcal{P}_5$. Hence, $\gamma(G^*:\mathcal{P}_5)=p$. $\square$

THEOREM 6.3. *Let G be a $\mathcal{P}_5$-graph. Then*

$$3\leqslant\gamma(G:\mathcal{P}_5)\leqslant p-3.$$

PROOF. Let G be a $\mathcal{P}_5$- graph. Clearly, $\gamma(G:\mathcal{P}_5)\geqslant 3$ because the subgraph induced by a BCD-set D of G is cyclic. Further, we construct cycle C_3 and C_p; $p\geqslant 3$ and make all the vertices of C_p adjacent to a single vertex of C_3. The set of vertices of C_3 in the obtained graph is the minimum dominating set D such that both $\langle D\rangle$ and $\langle V-D\rangle$ are cyclic. Therefore, D is the BCD-set of $\mathcal{P}_5$-graph with $\gamma(G:\mathcal{P}_5)\geqslant 3$. Hence, $\gamma(G:\mathcal{P}_5)=3$. Thus the lower bound follows.

Now we prove $\gamma(G:\mathcal{P}_5) \leqslant p-3$. Suppose on the contrary $\gamma(G:\mathcal{P}_5) > p-3$. Then the subgraph induced by complement of D contains at most two vertices and hence $\langle V-D\rangle$ can not be cyclic, which is a contradiction. Also, consider the cycle C_3 and cycle C_p; $p \geqslant 3$ and make all the vertices of C_3 adjacent to a single vertex of C_p. The set of vertices of C_p in the obtained graph is the minimum dominating set D such that both $\langle D\rangle$ and $\langle V-D\rangle$ are cyclic. Therefore, D is the minimum BCD-set. Hence, $\gamma(G:\mathcal{P}_5) = p-3$. Thus the upper bound follows. □

To prove our next result we make use of the following definition.

An $(n-p)$-*cycle net* is the graph obtained by taking n copies of a cycle C_p one inside the other and joining the corresponding copies of the vertices in every two consecutive cycles.

THEOREM 6.4. *Let G be a $\mathcal{P}_5$-graph with $(n-p)$-cycle net. Then*

$$\gamma(G:\mathcal{P}_5) = \begin{cases} \dfrac{np}{2}, & \text{if } n \text{ is even} \\ \dfrac{p(n-1)}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

PROOF. In a $(n-p)$-cycle net, take the vertices of first cycle and the vertices of every alternate cycles into the set $V-D$ and the vertices of all other cycles into the set D, we get D as minimum BCD-set of G. Therefore, every $(n-p)$- cycle net is a $\mathcal{P}_5$-graph. Further, if n is even, then both $\langle D\rangle$ and $\langle V-D\rangle$ contain equal number of cycles and hence $\gamma(G:\mathcal{P}_5) = |D| = \dfrac{np}{2}$.

If n is odd, then $\langle D\rangle$ contains $\dfrac{n-1}{2}$ cycles and hence $\gamma(G:\mathcal{P}_5) = |D| = \dfrac{p(n-1)}{2}$. □

7. Bi-Acyclic Domination

A set $D \subseteq V$ is a bi-acyclic dominating (BAD) set of G, if it satisfies the property $\mathcal{P}_6$. The minimum cardinality taken over all BAD-sets is called the bi-acyclic domination number and is denoted by $\gamma(G:\mathcal{P}_6)$. For more details, we refer to [**4**], [**12**] and [**13**].

OBSERVATION 7.1. *Not all graphs have a BAD-set.*
For example, complete graph K_p with $p \geqslant 5$ vertices, has no BAD-set. We can reduce a graph G which has no BAD-set to a graph H having BAD- set, by deleting edges.

PROPOSITION 7.1. *For any graph G,*

$$\gamma(G) \leqslant \gamma_a(G) \leqslant \gamma(G:\mathcal{P}_6).$$

THEOREM 7.1. *Let G be a $\mathcal{P}_6$-graph. If s is the number of components in D and t is the number of components in $V-D$ with $\{s,t\} \geqslant 2$. Then*

$$2p-q-s-t \leqslant \gamma(G:\mathcal{P}_6).$$

Further more, the lower bound is attained if and only if there exists a BAD-set D of a graph G such that every vertex in $V-D$ is adjacent to exactly one vertex in D.

Proof. Let D be a BAD-set. If the number of edges in $\langle D\rangle$ and $\langle V-D\rangle$ are $|D|-s$ and $|V-D|-t$ respectively. Hence, the lower bound for the number of edges in a graph G is given by

$$\begin{aligned} q &\geqslant |D|-s+|V-D|-t+|V-D| \\ &\geqslant |D|+2|V-D|-s-t \\ 2|D|+q &\geqslant |D|+2|D|+2|V-D|-s-t \\ 2|D|+q &\geqslant |D|+2p-s-t \\ |D| &\geqslant 2p-q-s-t \end{aligned}$$

Hence the lower bound follows.

Now we prove the next part of the theorem. Suppose the lower bound is attained. On contrary, suppose there exists a vertex in $V-D$ adjacent to at least two vertices in D, then clearly $q>|D|-s+|V-D|-t+|V-D|$, which is a contradiction. Hence, every vertex in $V-D$ is adjacent to exactly one vertex in D. Hence the result follows.

Conversely, suppose every vertex in $V-D$ is adjacent to exactly one vertex in D, then the number of edges in the graph G is given by $|D|-s+2|V-D|-t$. Thus the result follows. □

Theorem 7.2. *If a graph G contains K_p with $p\geqslant 5$ vertices, as its induced subgraph, then G has no a BAD-set.*

Proof. For any complete graph K_p with $p\geqslant 5$, $\gamma(K_p:\mathcal{P}_6)$ does not exist because in K_p, every three vertices form a cycle ($i.e, C_3$), which is not a tree (acyclic). □

To prove our next result we make use of the following definition.

The minimum number of edges to be removed from a graph G, which has no a BAD-set, to get a graph H which has a BAD-set, is called the bi-acyclic number and is denoted by $\xi_a(G)$.

Theorem 7.3. *For any complete graph K_p,*

$$\xi_a(K_p)=\begin{cases} (\frac{p}{2}-1)(\frac{p}{2}-2), & \text{if } p\geqslant 4 \text{ is even} \\ \frac{1}{4}(p-3)^2, & \text{if } p\geqslant 5 \text{ is odd} \\ 1, & \text{if } p=3 \end{cases}$$

Proof. Let K_p be a complete graph. Then the following cases arise.

Case 1. Suppose p is even. Then split the vertex set of K_p into two disjoint subsets S_1 and S_2 with cardinalities $\frac{p}{2}$ and $\frac{p}{2}$, respectively, to get a BAD-set of K_p with $p\geqslant 4$ vertices, the induced subgraphs $\langle S_1\rangle$ and $\langle S_2\rangle$ must be acyclic. Since

$\langle S_1\rangle \cong K_{\frac{p}{2}}$ and $\langle S_2\rangle \cong K_{\frac{p}{2}}$, the number of edges to be removed from $\langle S_1\rangle$ is given by

$$\frac{1}{2}\frac{p}{2}\left(\frac{p}{2}-1\right)-\left(\frac{p}{2}-1\right)=\frac{1}{2}\left(\frac{p}{2}-1\right)\left(\frac{p}{2}-2\right).$$

Similarly the number of edges to be removed from $\langle S_2\rangle$ is given by

$$\frac{1}{2}\left(\frac{p}{2}-1\right)\left(\frac{p}{2}-2\right).$$

So, the total number of edges to be removed from K_p is given by

$$\left(\frac{p}{2}-1\right)\left(\frac{p}{2}-2\right).$$

Case 2. Suppose p is odd. Then split the vertex set of K_p into two disjoint subsets S_1 and S_2 with cardinalities $\frac{p-1}{2}$ and $\frac{p-1}{2}+1$, respectively, and to get a BAD-set of K_p with $p \geqslant 4$ vertices, the induced subgraphs $\langle S_1\rangle$ and $\langle S_2\rangle$ must be acyclic. Since $\langle S_1\rangle \cong K_{\frac{p-1}{2}}$ and $\langle S_2\rangle \cong K_{\frac{p-1}{2}+1}$, the number of edges to be removed from $\langle S_1\rangle$ is given by

$$\frac{1}{2}\left(\frac{p-1}{2}\right)\left(\frac{p-1}{2}-1\right)-\left(\frac{p-1}{2}-1\right).$$

This implies

$$\frac{1}{2}\left(\frac{p-1}{2}-1\right)\left(\frac{p-1}{2}-2\right).$$

Similarly the number of edges to be removed from $\langle S_2\rangle$ is given by

$$\frac{1}{2}\left(\frac{p-1}{2}+1\right)\left(\frac{p-1}{2}+1-1\right)-\left(\frac{p-1}{2}+1-1\right).$$

This implies

$$\frac{1}{2}\left(\frac{p-1}{2}\right)\left(\frac{p+1}{2}-2\right).$$

So, the total number of edges to be removed from K_p is given by

$$\frac{1}{2}\left[\left(\frac{p-1}{2}-1\right)\left(\frac{p-1}{2}-2\right)+\left(\frac{p-1}{2}\right)\left(\frac{p+1}{2}-2\right)\right]=\frac{1}{4}(p-3)^2.$$

□

To prove our next result we make use of the following definition.

Definition 7.1. *The* corona $G_1 \circ G_2$ *is the graph G obtained by taking one copy of G_1 of order p_1 and p_1 copies of G_2, and then joining the i^{th} vertex of G_1 to every vertex in the i^{th} copy of G_2.*

Theorem 7.4. *Let G_1 and G_2 be two k-regular graphs. Then the corona graph $G = G_1 \circ G_2$ is*

(i) $\mathcal{P}_1$ *- graph if* $k = 0$
(ii) $\mathcal{P}_3$ *- graph if* $k = 1$
(iii) $\mathcal{P}_5$ *- graph if* $k = 2$
(iv) $\gamma(G:\mathcal{P}_1)=\gamma(G:\mathcal{P}_3)=\gamma(G:\mathcal{P}_5)=m$, *where m is the order of G_1.*

Proof. Let G_1 and G_2 be any two k-regular graphs of order m and n, respectively. In $G_1 \circ G_2$, $D = V(G_1)$ is the minimum dominating set and $\langle V(G_1 \circ G_2) - V(G_1)\rangle$ is the subgraph consisting of disjoint copies of G_2. Hence, both $\langle D\rangle$ and $\langle V(G_1 \circ G_2) - D\rangle$ are k-regular. Hence, $G_1 \circ G_2$ is a $\mathcal{P}_1$-graph, $\mathcal{P}_3$-graph and $\mathcal{P}_5$-graph for $k = 0, 1, 2$ respectively. Thus $(i) - (iii)$ follow.

Since $D = V(G_1)$ is the minimum dominating set of a corona graph $G_1 \circ G_2$, $\gamma(G : \mathcal{P}_1) = \gamma(G : \mathcal{P}_3) = \gamma(G : \mathcal{P}_5) = m$. Hence (iv) follows. □

Theorem 7.5. *Let G be a nontrivial graph. Then prism of $\mathcal{P}_i$ - graph is a $\mathcal{P}_{i+2}$ - graph, $i = 1, 3$.*

Proof. For $i = 1$, let G be a $\mathcal{P}_1$ - graph and H be the prism of G. In G, there exists a dominating set D such that both D and $V(G) - D$ are independent sets. Hence in H, $D^{'} = D \cup f(D)$, where $f(D)$ is the mirror image of D in the prism, is the dominating set such that both $\langle D^{'}\rangle$ and $\langle V(H) - D^{'}\rangle$ have perfect matching. Hence, H is a $\mathcal{P}_3$ - graph.

For $i = 3$, let G be a $\mathcal{P}_3$-graph and H be the prism of G. In G, there exists a dominating set D such that both $\langle D\rangle$ and $\langle V(G) - D\rangle$ have perfect matching. Hence in H, $D^{'} = D \cup f(D)$, is dominating set such that both $\langle D^{'}\rangle$ and $\langle V(H) - D^{'}\rangle$ contain only cycles of length four. Hence H is a $\mathcal{P}_5$-graph. □

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