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APPLICATION OF ELZAKI TRANSFORM TO FIRST ORDER CONSTANT COEFFICIENTS COMPLEX EQUATIONS

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Abstract. In this work, we present a reliable Elzaki transform method to solve first order constant coefficients complex equations. This method provides an effective and efficient way of solving a wide range of linear operator equations.

1. Introduction

In real, general solutions of some equations, especially type of elliptic, are not found. For example,

$$u_{xx} + u_{yy} = 0$$

Laplace equation hasn't got general solution in R^2, but it can be written

$$u_{z\overline{z}} = 0$$

and the solution of this equation is

$$u = f(z) + g(\overline{z})$$

where f is analytic, g is anti analytic arbitrary functions [6]. That is, an equation which has not general solution in real can has general solution in complex space. A partial differential equation system which has two real dependant and two real independant variables can be transformed to a complex equation. For example,

$$\begin{aligned} u_x - v_y &= 0 \\ u_y + v_x &= 0 \end{aligned}$$

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$$w_{\overline{z}} = 0$$

where $w = u + iv, z = x + iy$. All solutions of this complex equation are analytic functions [6].

Moreover any order complex differential equation can be transformed to real partial differential equation system which has two unknowns, two independent variables by seperating the real and imaginer parts. The solution of complex equation can be put forward helping solutions of this real system [6].

Elzaki transform method which is used several areas of mathematics is a integral transform. We can solve linear differential equations, integral equations, integro-differantial equations with elzaki transform [1, 2, 3]. This method can not suitable for solution of nonlinear differential equations because of nonlinear terms. But nonlinear differential equations can be solved by using elzaki transform aid with differential transform method and homotopy perturbation method [4, 5]. In this study, we investigate solutions of first order constant coeffients complex equations. These equations were solved by laplace transform in [6]. The above mentioned equations are solved by Elzaki transform method in this paper. We obtain a formulazition for general first order constant coeffients complex equations. This paper is organized as follows: In Section 2, basic definitions and theorems are given. In Section 3, we get a formulazition for solve the first order constant coeffients complex partial differential equations and some examples have been given.

2. Basic Definitions and Theorems

Definition 2.1. *Let $F(t)$ be a function for $t > 0$. Elzaki transform of $F(t)$*

$$E(F(t)) = s \cdot \int_0^\infty e^{-\frac{t}{s}} \cdot f(t)\, dt \tag{2.1}$$

is defined.

Theorem 2.1. *Elzaki transforms of some functions are given in following.*

$F(t)$	$E(f(t)) = T(s)$
1	s^2
t	s^3
t^n	$n!s^{n+2}$
$\cos at$	$\frac{s^2}{1+a^2s^2}$
$\sin at$	$\frac{as^3}{1+a^2s^2}$

Theorem 2.2. *Elzaki transforms of partial derivatives of $f(x,t)$ are following.*

i) $E\left[\frac{\partial f}{\partial t}\right] = \frac{1}{s}T(x,s) - sf(x,0)$

ii) $L\left[\frac{\partial f}{\partial x}\right]=\frac{\partial T(x,s)}{\partial x}$,

where $T(x,s)=E\left[f(x,t)\right].$

2.1. Complex Derivatives. Let $w=w(z,\overline{z})$ be a complex function. Here

$$z=x+iy,\ w(z,\overline{z})=u(x,y)+i\cdot v(x,y).$$

First order derivatives according to z and $\overline{z}$ of $w(z,\overline{z})$ are defined as following:

$$\frac{\partial w}{\partial z}=\frac{1}{2}(\frac{\partial w}{\partial x}-i\frac{\partial w}{\partial y}), \tag{2.2}$$

$$\frac{\partial w}{\partial \overline{z}}=\frac{1}{2}(\frac{\partial w}{\partial x}+i\frac{\partial w}{\partial y}). \tag{2.3}$$

3. Solution of complex differential equations from first order which is constant coeffients

THEOREM 3.1. *Let A,B,C are real constants, $F(z,\overline{z})$ is a polynomial of $z,\overline{z}$ and $w=u+iv$ is a complex function. Then the real and imaginal parts of solution of*

$$A\frac{\partial w}{\partial z}+B\frac{\partial w}{\partial \overline{z}}+Cw=F\left(z,\overline{z}\right)$$

$$w\left(x,0\right)=f\left(x\right)$$

are

$$\begin{aligned} u &= Rew=E^{-1}\left[\frac{(A+B)\frac{\partial}{\partial x}(2T_3+(A-B).s.v(x,0))}{\left[(A+B)D+2C\right]^2+(\frac{A-B}{s})^2}\right. \\ &\left.+\frac{2C\left(2T_3+(A-B).s.v(x,0)\right)-\left(\frac{A-B}{s}\right)\left(2T_4+(B-A).s.u(x,0)\right)}{\left[(A+B)D+2C\right]^2+(\frac{A-B}{s})^2}\right], \end{aligned}$$

$$\begin{aligned} v &= Imw=E^{-1}\left[\frac{(A+B)\frac{\partial}{\partial x}(2F_2^*+(B-A)u(x,0))+2C}{\left[(A+B)D\right]^2+s^2(A-B)^2}\right. \\ &\left.+\frac{2C\left(2F_2^*+(B-A)u(x,0)\right)-s(B-A)(2F_1^*+(A-B)v(x,0))}{\left[(A+B)D+2C\right]^2+s^2(A-B)^2}\right]. \end{aligned}$$

PROOF.

$$A\frac{\partial w}{\partial z}+B\frac{\partial w}{\partial \overline{z}}+Cw=F\left(z,\overline{z}\right). \tag{3.1}$$

If it is used equalities (2.2), (2.3) in equality (3.1), following equality is obtained .

$$A\cdot\frac{1}{2}(\frac{\partial w}{\partial x}-i\frac{\partial w}{\partial y})+B\cdot\frac{1}{2}(\frac{\partial w}{\partial x}+i\frac{\partial w}{\partial y})+Cw=F_1(x,y)+iF_2(x,y). \tag{3.2}$$

If $w=u+iv$ is written in (3.2) , then following equality is obtained .

$$A(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}-i\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y})+B(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}+i\frac{\partial u}{\partial y}-\frac{\partial v}{\partial y})+2C\left(u+iv\right) \tag{3.3}$$
$$= 2F_1\left(x,y\right)+2iF_2\left(x,y\right).$$

If (3.3) is separated to real and imaginer parts, then following equation system is obtained

$$(A+B)\frac{\partial u}{\partial x}+(A-B)\frac{\partial v}{\partial y}+2Cu=2F_1(x,y), \tag{3.4}$$

$$(A+B)\frac{\partial v}{\partial x}+(B-A)\frac{\partial u}{\partial y}+2Cv=2F_2(x,y). \tag{3.5}$$

If we use Elzaki transform for above equalities (3.4), (3.5), then we get following equalities

$$(A+B)\frac{\partial T_1}{\partial x}+(A-B)(\frac{T_2}{s}-sv(x,0))+2CT_1=2T_3 \tag{3.6}$$

$$(A+B)\frac{\partial T_2}{\partial x}+(B-A)(\frac{T_1}{s}-su(x,0))+2CT_2=2T_4, \tag{3.7}$$

where T_1,T_2,T_3,T_4 are Elzaki transform transforms of u,v,F_1,F_2 respectively. If $(3.6),(3.7)$ is rerugulate and is used Cramer rule, then equalities $(3.8),(3.9)$ are obtained.

$$(A+B)\frac{\partial T_1}{\partial x}+2CT_1+\frac{A-B}{s}T_2=2T_3+(A-B)\cdot s\cdot v(x,0)$$

$$\frac{B-A}{s}T_1+(A+B)\frac{\partial T_2}{\partial x}+2CT_2=2T_4+(B-A)\cdot s\cdot u(x,0)$$

$$\begin{vmatrix} (A+B)D+2C & \frac{A-B}{s} \\ \frac{B-A}{s} & (A+B)D+2C \end{vmatrix}=[(A+B)D+2C]^2+(\frac{A-B}{s})^2$$

$$T_1 = \frac{\begin{vmatrix} 2T_3+(A-B).s.v(x,0) & \frac{A-B}{s} \\ 2T_4+(B-A).s.u(x,0) & (A+B)D+2C \end{vmatrix}}{[(A+B)D+2C]^2+(\frac{A-B}{s})^2}$$

$$T_1 = \frac{(A+B)\frac{\partial}{\partial x}(2T_3+(A-B).s.v(x,0))+2C\left(2T_3+(A-B).s.v(x,0)\right)}{[(A+B)D+2C]^2+(\frac{A-B}{s})^2}$$
$$-\frac{\left(\frac{A-B}{s}\right)\left(2T_4+(B-A).s.u(x,0)\right)}{[(A+B)D+2C]^2+(\frac{A-B}{s})^2} \tag{3.8}$$

$$T_2 = \frac{\begin{vmatrix} (A+B)D+2C & 2T_3+(A-B).s.v(x,0) \\ \frac{B-A}{s} & 2T_4+(B-A).s.u(x,0) \end{vmatrix}}{\left[(A+B)D+2C\right]^2+(\frac{A-B}{s})^2}$$

$$T_2 = \frac{(A+B)\frac{\partial}{\partial x}(2T_4+(B-A).s.u(x,0))+2C\,(2T_4+(B-A).s.u(x,0))}{\left[(A+B)D+2C\right]^2+(\frac{A-B}{s})^2}$$

$$-\frac{\frac{(B-A)}{s}(2T_3+(A-B).s.v(x,0))}{\left[(A+B)D+2C\right]^2+(\frac{A-B}{s})^2}. \tag{3.9}$$

Followings are obtained from inverse Elzaki of (3.8) , (3.9)

$$u(x,y) = E^{-1}\left[\frac{(A+B)\frac{\partial}{\partial x}(2T_3+(A-B).s.v(x,0))}{\left[(A+B)D+2C\right]^2+(\frac{A-B}{s})^2}\right.$$

$$\left.+\frac{2C\,(2T_3+(A-B).s.v(x,0))-\left(\frac{A-B}{s}\right)(2T_4+(B-A).s.u(x,0))}{\left[(A+B)D+2C\right]^2+(\frac{A-B}{s})^2}\right] \tag{3.10}$$

$$v(x,y) = E^{-1}\left[\frac{(A+B)\frac{\partial}{\partial x}(2T_4+(B-A).s.u(x,0))}{\left[(A+B)D+2C\right]^2+(\frac{A-B}{s})^2}\right.$$

$$\left.+\frac{2C\,(2T_4+(B-A).s.u(x,0))-\frac{(B-A)}{s}(2T_3+(A-B).s.v(x,0))}{\left[(A+B)D+2C\right]^2+(\frac{A-B}{s})^2}\right]. \tag{3.11}$$

□

Example 3.1. Solve the following equation

$$4w_z + w_{\overline{z}} = 0$$

with the condition

$$w\,(x,0) = -\frac{1}{3x}.$$

Solution 3.1. *Coefficients of equation are* $A=4, B=1, C=0$ *and* $F(z,\overline{z}) = 0$. *From theorem 3.1 we have*

$$\begin{aligned} u(x,y) &= E^{-1}[\frac{-\frac{9}{3x}}{25D^2+\frac{9}{s^2}}] \\ &= E^{-1}[\frac{-\frac{3}{x}}{\frac{9}{s^2}\left(1+\frac{25s^2D^2}{9}\right)}] \\ &= E^{-1}[-\frac{s^2}{3}(1-\frac{25s^2D^2}{9}+\left(\frac{5s}{3}\right)^4 D^4-\left(\frac{5s}{3}\right)^6 D^6+...)\frac{1}{x}] \\ &= E^{-1}[-\frac{s^2}{3}(\frac{1}{x}-\left(\frac{5s}{3}\right)^2.\frac{2}{x^3}+\left(\frac{5s}{3}\right)^4.\frac{4!}{x^5}-\left(\frac{5s}{3}\right)^6.\frac{6!}{x^7}+...)] \end{aligned}$$

and

$$
\begin{aligned}
&= E^{-1}[-\frac{s^2}{3x}] + E^{-1}[\left(\frac{5}{3}\right)^2 \frac{2s^4}{3x^3}] - E^{-1}[\left(\frac{5}{3}\right)^4 \frac{4!s^6}{3x^5}] + E^{-1}[\left(\frac{5}{3}\right)^6 \frac{6!.s^8}{3x^7}] - ... \\
&= -\frac{1}{3x} + \frac{5^2}{3^3}\frac{y^2}{x^3} - \frac{5^4}{3^5}\frac{y^4}{x^5} + \frac{5^6}{3^7}\frac{y^6}{x^7} - ... \\
&= -\frac{1}{3x}\left(1 - \left(\frac{5y}{3x}\right)^2 + \left(\frac{5y}{3x}\right)^4 - \left(\frac{5y}{3x}\right)^6 + ...\right) = -\frac{\frac{1}{3x}}{1 + \frac{25y^2}{9x^2}} = -\frac{3x}{9x^2 + 25y^2}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
v(x,y) &= E^{-1}[\frac{5.\frac{\partial}{\partial x}\left(\frac{-3s}{-3x}\right)}{25D^2 + \frac{9}{s^2}}] \\
&= E^{-1}[\frac{-\frac{5s}{x^2}}{\frac{9}{s^2}\left(1 + \frac{25s^2D^2}{9}\right)}] \\
&= E^{-1}[\frac{-5s^3}{9}(1 - \frac{25s^2D^2}{9} + \left(\frac{5s}{3}\right)^4 D^4 - \left(\frac{5s}{3}\right)^6 D^6 + ...)\frac{1}{x^2}] \\
&= E^{-1}[\frac{-5s^3}{9}(\frac{1}{x^2} - \left(\frac{5}{3}\right)^2 \frac{s^2.3!}{x^4} + \left(\frac{5}{3}\right)^4 \frac{s^4.5!}{x^6} - \left(\frac{5}{3}\right)^6 \frac{s^6.7!}{x^8} + ...)] \\
&= E^{-1}[\frac{-5s^3}{9x^2} + \frac{5^3.s^5}{3^4}\frac{3!}{x^4} - \frac{5^5.s^7}{3^6}\frac{5!}{x^6} + ...] = \frac{-5y}{9x^2} + \frac{5^3.y^3}{3^4.x^4} - \frac{5^5.y^5}{3^6.x^6} + ... \\
&= \frac{-5y}{9x^2}\left(1 - \frac{5^2.y^2}{3^2.x^2} + \frac{5^4.y^4}{3^4.x^4} - \frac{5^6.y^6}{3^6.x^6} + ...\right) \\
&= \frac{-5y}{9x^2}\left(\frac{1}{1 + \frac{25y^2}{9x^2}}\right) = -\frac{5y}{9x^2 + 25y^2}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
w &= u + iv = -\frac{3x}{9x^2 + 25y^2} - \frac{5iy}{9x^2 + 25y^2} \\
&= -\frac{1}{3x - 5iy} = \frac{1}{z - 4\overline{z}}.
\end{aligned}
$$

Example 3.2. Solve the following problem

$$\frac{\partial w}{\partial z} - \frac{\partial w}{\partial \overline{z}} - w = 0$$

with the condition

$$w\,(x,0) = e^{3x}.$$

Solution 3.2. *Coefficients of equation are* $A = 1,\ B = -1,\ C = 1$ *and* $F(z, \overline{z}) = 0$. *From theorem 3.1 we have obtained that*

$$u(x,y) = E^{-1}\left(\frac{e^{3x}.s^2}{s^2+1}\right) = e^{3x}.E^{-1}\left(\frac{s^2}{s^2+1}\right) = e^{3x}.\cos y.$$

Similarly,

$$v(x,y) = E^{-1}\left(\frac{e^{3x}.s^3}{s^2+1}\right) = e^{3x}.E^{-1}\left(\frac{s^3}{s^2+1}\right) = e^{3x}.\sin y.$$

Hence

$$\begin{aligned} w &= u + iv = e^{3x}.\cos y + ie^{3x}.\sin y \\ &= e^{3x+iy} = e^{3\left(\frac{z+\overline{z}}{2}\right)+i\left(\frac{z-\overline{z}}{2i}\right)} \\ &= e^{2z+\overline{z}}. \end{aligned}$$

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