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PRIME BI-IDEALS IN Γ - SEMIRINGS

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Abstract. In this paper we introduce the concepts of prime, semiprime, strongly prime, irreducible and strongly irreducible bi-ideals in a Γ-semiring. Characterizations of a Γ-semiring using these concepts are furnished. A topology on the set of strongly prime bi-ideals is defined and a property of the space of strongly prime bi-ideals of a Γ-semiring is furnished.

1. Introduction

The notion of Γ-rings was introduced by Nobusawa in [10]. The class of Γ-rings contains not only all rings but also ternary rings. As a generalization of rings, semirings were introduced by Vandiver [14] and he obtained many results about it. Further as a generalization of Γ-rings and semirings, the notion of a Γ-semiring was introduced by Rao [11]. It is well known that ideals play an important role in any abstract algebraic structures. Characterizations of ideals in a semigroup were given by Lajos [8], while ideals in semirings were characterized by Iseki [4, 5]. Prime and semiprime ideals in Γ-semirings were discussed by Dutta and Sardar [2]. Authors were studied quasi-ideals and bi-deals in Γ-semirings [6, 7]. The notion of a bi-ideal was first introduced for semigroups by Good and Hughes [3].The concept of a bi-ideal for ring was given by Lajos [9] and for semirings by Shabir , Ali and Batool [12]. The concept of a bi-ideal in a semigroup (ring and semiring) is a generalization of one sided ideal and two sided ideal in a semigroup (ring and semiring). Prime bi-ideals in a Γ-ring was introduced by Booth and Groenewald [1] and in a semigroup by Shabir and Kanwal [13].

In this paper efforts are made to extend the notion of prime ideals and semiprime ideal in Γ-semirings to prime bi-ideal and semiprime bi-ideal respectively in Γ-semirings. Also we define strongly prime bi-ideal in Γ-semirings and discuss some

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properties of it. Finally we prove a topological property of the space of strongly prime bi-ideals of a Γ-semiring.

2. Preliminaries

First we recall some definitions of the basic concepts of Γ-semirings that we need in sequel. For this we refer Dutta and Sardar [2].

DEFINITION 2.1. *Let S and Γ be two additive commutative semigroups. S is called a Γ - semiring if there exists a mapping $S \times \Gamma \times S \longrightarrow S$ denoted by $a\alpha b$; for all $a, b \in S$ and $\alpha \in \Gamma$ satisfying the following conditions:*

(i) $a\alpha\,(b+c) = (a\alpha b) + (a\alpha c)$
(ii) $(b+c)\,\alpha a = (b\alpha a) + (c\alpha a)$
(iii) $a(\alpha+\beta)c = (a\alpha c) + (a\beta c)$
(iv) $a\alpha\,(b\beta c) = (a\alpha b)\,\beta c$ *; for all* $a, b, c \in S$ *and for all* $\alpha, \beta \in \Gamma$.

Obviously, every semiring S is a Γ-semiring.

Let S be a semiring and Γ be a commutative semigroup. Define a mapping $S \times \Gamma \times S \longrightarrow S$ by, $a\alpha b = ab$; *for all* $a, b \in S$ *and* $\alpha \in \Gamma$. Then S is a Γ-semiring.

DEFINITION 2.2. *An element $0 \in S$ is said to be an absorbing zero if $0\alpha a = 0 = a\alpha 0$, $a + 0 = 0 + a = a$; for all $a \in S$ and $\alpha \in \Gamma$.*

Now onwards S denotes a Γ-semiring with absorbing zero unless otherwise stated.

DEFINITION 2.3. *A non-empty subset T of S is said to be a sub-Γ-semiring of S if (T,+) is a subsemigroup of (S,+) and $a\alpha b \in T$; for all $a, b \in T$ and $\alpha \in \Gamma$.*

DEFINITION 2.4. *A non-empty subset T of S is called a left (respectively right) ideal of S if T is a subsemigroup of $(S,+)$ and $x\alpha a \in T$ (respectively $a\alpha x \in T$) for all $a \in T$, $x \in S$ and $\alpha \in \Gamma$.*

DEFINITION 2.5. *If T is both left and right ideal of S, then T is known as an ideal of S.*

If M, N are non-empty subsets of S, then

$$M\Gamma N = \{\sum_{i=1}^{n} x_i\alpha_i y_i | x_i \in M, \alpha_i \in \Gamma, y_i \in N\} .$$

Principle left ideal, right ideal and two sided ideal generated by $a \in S$ denoted by $(a)_l$,$(a)_r$ and (a) respectively.

DEFINITION 2.6. *An element a of a Γ-semiring S is said to be regular if $a \in a\Gamma S\Gamma a$.*

If all elements of a Γ-semiring S are regular, then S is known as a regular Γ-semiring.

DEFINITION 2.7. *S is said to be an intra-regular Γ-semiring if for any $x \in S$, $x \in S\Gamma x\Gamma x\Gamma S$.*

LEMMA 2.1. *S is regular if and only if $R\Gamma L = R \cap L$, for a right ideal R and left ideal L of S.*

LEMMA 2.2. *Let $(a)_b$ denote the bi-ideal generated by $a \in S$.If S is a regular Γ-semiring, then $(a)_b = a\Gamma S\Gamma a$.*

3. Prime Bi-ideals

Here we recall the definition of a bi-ideal in Γ-semiring from [7].

DEFINITION 3.1. *A non-empty subset B of S is said to be a bi-ideal of S if B is a sub-Γ-semiring of S and $B\Gamma S\Gamma B \subseteq B$.*

EXAMPLE 3.1. Consider the semiring $S = M_{2\times 2}(N_0)$, where N denotes the set of all natural numbers and $N_0 = N \cup \{0\}$. If $\Gamma = S$, then S forms a Γ semiring with $A\alpha B =$ usual matrix product of A, α, B; for all $A, \alpha, B \in S$.

(1) $C = \left\{ \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix} \mid x\ ,\ y \in N_0 \right\}$ is a bi-ideal of S.

(2) $D = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix} \mid x \in N_0 \right\}$ is a bi-ideal of S.

Statements given in the following theorem are easy to verify.

THEOREM 3.1. *In Γ-semiring S following statements hold.*

1. *Any one sided (two sided) ideal of S is a bi-ideal of S.*
2. *Intersection of a right ideal and a left ideal of S is a bi-ideal of S.*
3. *Arbitrary intersection of bi-ideals of S is also a bi-ideal of S and hence the set of all bi-ideals of S forms a complete lattice.*
4. *If B is a bi-ideal of S, then $B\Gamma s$ and $s\Gamma B$ are bi-ideals of S, for any $s \in S$.*
5. *If B is a bi-ideal of S, then $b\Gamma B\Gamma c$ is a bi-ideal of S, for $b, c \in S$.*
6. *If B is a bi-ideal of S and if T is a sub-Γ-semiring of S, then $B \cap T$ is a bi-ideal of T.*
7. *If A, B are bi-ideals of S, then $A\Gamma B$ and $B\Gamma A$ are bi-ideals of S.*
8. *For any $a \in S$, $S\Gamma a$ is a left ideal and $a\Gamma S$ is a right ideal of S.*

DEFINITION 3.2. *A bi-ideal B of S is called a prime bi-ideal if $B_1\Gamma B_2 \subseteq B$ implies $B_1 \subseteq B$ or $B_2 \subseteq B$, for any bi-ideals B_1 and B_2 of S.*

DEFINITION 3.3. *A bi-ideal B of S is called a strongly prime bi-ideal if $(B_1\Gamma B_2) \cap (B_2\Gamma B_1) \subseteq B$ implies $B_1 \subseteq B$ or $B_2 \subseteq B$, for any bi-ideals B_1 and B_2 of S.*

DEFINITION 3.4. *A bi-ideal B of S is called a semiprime bi-ideal if for any bi-ideal B_1 of S, ${B_1}^2 = B_1\Gamma B_1 \subseteq B$ implies $B_1 \subseteq B$.*

Obviously every strongly prime bi-ideal in S is a prime bi-ideal and every prime bi-ideal in S is a semiprime bi-ideal.

Definition 3.5. *A bi-ideal B of S is called an irreducible bi-ideal if $B_1 \cap B_2 = B$ implies $B_1 = B$ or $B_2 = B$, for any bi-ideals B_1 and B_2 of S.*

Definition 3.6. *A bi-ideal B of S is called a strongly irreducible bi-ideal if for any bi-ideals B_1 and B_2 of S, $B_1 \cap B_2 \subseteq B$ implies $B_1 \subseteq B$ or $B_2 \subseteq B$.*

Obviously every strongly irreducible bi-ideal is an irreducible bi-ideal.

Theorem 3.2. *The intersection of any family of prime bi-ideals of S is a semiprime bi-ideal.*

Proof. Let $\{P_i | i \in \Lambda\}$ be the family of prime bi-ideals of S. For any bi-ideal B of S, $B^2 \subseteq \bigcap_i P_i$ implies $B^2 \subseteq P_i$, for all $i \in \Lambda$. As P_i are prime bi-ideals, P_i are semiprime bi-ideals. Therefore $B \subseteq P_i$, for all $i \in \Lambda$.
Hence $B \subseteq \bigcap_i P_i$. □

Theorem 3.3. *Every strongly irreducible, semiprime bi-ideal of S is a strongly prime bi-ideal.*

Proof. Let B be a strongly irreducible and semiprime bi-ideal of S. For any bi-ideals B_1 and B_2 of S, $(B_1 \Gamma B_2) \cap (B_2 \Gamma B_1) \subseteq B$. Hence by Theorem 3.1(3), $B_1 \cap B_2$ is a bi-ideal of S. Since

$$(B_1 \cap B_2)^2 = (B_1 \cap B_2)\,\Gamma\,(B_1 \cap B_2) \subseteq B_1 \Gamma B_2.$$

Similarly we get $(B_1 \cap B_2)^2 \subseteq B_2 \Gamma B_1$. Therefore

$$(B_1 \cap B_2)^2 \subseteq (B_1 \Gamma B_2) \cap (B_2 \Gamma B_1) \subseteq B.$$

As B is a semiprime bi-ideal of S, $B_1 \cap B_2 \subseteq B$. But B is a strongly irreducible bi-ideal. Therefore $B_1 \subseteq B$ or $B_2 \subseteq B$. Hence B is a strongly prime bi-ideal of S. □

Theorem 3.4. *If B is a bi-ideal of S and $a \in S$ such that $a \notin B$, then there exists an irreducible bi-ideal I of S such that $B \subseteq I$ and $a \notin I$.*

Proof. Let $\mathcal{B}$ be the family of all bi-ideals of S which contain B but do not contain an element a. Then $\mathcal{B}$ is a non-empty as $B \in \mathcal{B}$. This family of all bi-ideals of S forms a partially ordered set under the inclusion of sets. Hence by Zorn's lemma , there exists a maximal bi-ideal say I in $\mathcal{B}$. Therefore $B \subseteq I$ and $a \notin I$. Now to show that I is an irreducible bi-ideal of S. Let C and D be any two bi-ideals of S such that $C \cap D = I$. Suppose that C and D both contain I properly. But I is a maximal bi-ideal in $\mathcal{B}$. Hence we get $a \in C$ and $a \in D$. Therefore $a \in C \cap D = I$ which is absurd . Thus either $C = I$ or $D = I$. Therefore I is an irreducible bi-ideal of S. □

Theorem 3.5. *Any proper bi-ideal B of S is the intersection of all irreducible bi-ideals of S containing B.*

Proof. Let B be a bi-ideal of S and $\{B_i | i \in \Lambda\}$ be the collection of irreducible bi-ideals of S containing B, where Λ denotes an indexing set. Then $B \subseteq \bigcap_{i\in\Lambda} B_i$. Suppose that $a \notin B$. Then by Theorem 3.4, there exists an irreducible bi-ideal A of S containing B but not a. Therefore $a \notin \bigcap_{i\in\Lambda} B_i$. Thus $\bigcap_{i\in\Lambda} B_i \subseteq B$. Hence $\bigcap_{i\in\Lambda} B_i = B$. □

Theorem 3.6. *In S following statements are equivalent.*

(1) *S is regular and intra-regular.*
(2) *$B^2 = B$, for any bi-ideal B of S.*
(3) *$B_1 \cap B_2 = (B_1 \Gamma B_2) \cap (B_2 \Gamma B_1)$, for any bi-ideals B_1 and B_2 of S.*
(4) *Each bi-ideal of S is semiprime.*
(5) *Each proper bi-ideal of S is the intersection of irreducible semiprime bi-ideals of S which contain it.*

Proof. (1) ⇒ (2) Suppose S is regular and intra-regular. Let B be any bi-ideal of S. Always $B^2 = B\Gamma B \subseteq B$. Let $a \in B$. As S is regular and intra-regular, $a \in a\Gamma S\Gamma a$ and $a \in S\Gamma a\Gamma a\Gamma S$. Hence

$$a \in a\Gamma S\Gamma a \subseteq (a\Gamma S)\,\Gamma\,(a\Gamma S\Gamma a) \subseteq (a\Gamma S)\,\Gamma\,(S\Gamma a\Gamma a\Gamma S)\,\Gamma\,(S\Gamma a).$$

Then

$$\begin{aligned} a \in (B\Gamma S)\,\Gamma\,(S\Gamma B\Gamma B\Gamma S)\,\Gamma\,(S\Gamma B) &= B\Gamma\,(S\Gamma S)\,\Gamma B\Gamma B\Gamma\,(S\Gamma S)\,\Gamma B \\ &\subseteq (B\Gamma S\Gamma B)\,\Gamma\,(B\Gamma S\Gamma B) \subseteq B\Gamma B. \end{aligned}$$

Therefore

$$a \in B^2 = B\Gamma B. \text{ Hence } B \subseteq B\Gamma B. \text{ Thus } B^2 = B\Gamma B = B.$$

(2) ⇒ (1) Suppose $B^2 = B$, for any bi-ideal B of S. Let R be a right ideal and L be a left ideal of S. Then $R \cap L$ is a bi-ideal of S. Therefore by assumption

$$(R \cap L)^2 = R \cap L.\; R \cap L = (R \cap L)^2 = (R \cap L)\,\Gamma\,(R \cap L) \subseteq R\Gamma L.$$

But always $R\Gamma L \subseteq R \cap L$. Hence $R\Gamma L = R \cap L$. Hence by the Lemma 2.1, S is regular.

Let $a \in S$. As S is regular , by Lemma 2.2 $(a)_b = a\Gamma S\Gamma a$. By assumption

$$(a)_b = (a)_b\Gamma(a)_b = (a\Gamma S\Gamma a)\Gamma(a\Gamma S\Gamma a) \subseteq S\Gamma a\Gamma a\Gamma S.$$

But $a \in (a)_b \subseteq S\Gamma a\Gamma a\Gamma S$. This shows that S is an intra-regular Γ-semiring.

(2) ⇒ (3) Suppose that $B^2 = B$, for any bi-ideal B of S. Let B_1 and B_2 be any two bi-ideals of S. Hence by Theorem 3.1(3), $B_1 \cap B_2$ is a bi-ideal of S. Therefore by (2),

$$(B_1 \cap B_2)^2 = B_1 \cap B_2.\;\; B_1 \cap B_2 = (B_1 \cap B_2)^2 = (B_1 \cap B_2)\,\Gamma\,(B_1 \cap B_2) \subseteq B_1\Gamma B_2.$$

Similarly we have $B_1 \cap B_2 \subseteq B_2\Gamma B_1$. Hence $B_1 \cap B_2 \subseteq (B_1\Gamma B_2) \cap (B_2\Gamma B_1)$. By Theorem 3.1(7), $B_1\Gamma B_2$ and $B_2\Gamma B_1$ are bi-ideals of S. Therefore by Theorem 3.1(3), $(B_1\Gamma B_2) \cap (B_2\Gamma B_1)$ is a bi-ideal of S. Hence by (2),

$$(B_1\Gamma B_2)\cap(B_2\Gamma B_1) = ((B_1\Gamma B_2)\cap(B_2\Gamma B_1))\,\Gamma\,(B_1\Gamma B_2)\cap(B_2\Gamma B_1)\,).$$
$$(B_1\Gamma B_2)\cap(B_2\Gamma B_1)\subseteq(B_1\Gamma B_2)\,\Gamma\,(B_2\Gamma B_1)\subseteq B_1\Gamma S\Gamma B_1\subseteq B_1.$$

Similarly we show that $(B_1\Gamma B_2)\cap(B_2\Gamma B_1)\subseteq B_2$. Thus $(B_1\Gamma B_2)\cap(B_2\Gamma B_1)\subseteq B_1\cap B_2$. Hence $B_1\cap B_2=(B_1\Gamma B_2)\cap(B_2\Gamma B_1)$.

(3) $\Rightarrow$ (4) Let B be any bi-ideal of S. Suppose that ${B_1}^2=B_1\Gamma B_1\subseteq B$, for any bi-ideal B_1 of S. Therefore by (3), we have

$$B_1=B_1\cap B_1=(B_1\Gamma B_1)\cap(B_1\Gamma B_1)=B_1\Gamma B_1\subseteq B.$$

Hence every bi-ideal of S is semiprime.

(4) $\Rightarrow$ (5) Let B be a proper bi-ideal of S. Hence by the Theorem 3.5, B is the intersection of all proper irreducible bi-ideals of S which contains B. By assumption every bi-ideal of S is semiprime. Hence each proper bi-ideal of S is the intersection of irreducible semiprime bi-ideals of S which contain it.

(5) $\Rightarrow$ (2) Let B be a bi-ideal of S. If $B^2=S$, then clearly result holds. Suppose that $B^2\neq S$. Then B^2 is a proper bi-ideal of S. Hence by assumption, B^2 is the intersection of irreducible semiprime bi-ideals of S which contain it. $B^2=\cap\{B_i/B_i\ is\ an\ irreducible\ semiprime\ bi-ideal\}$. As each B_i is a semiprime bi-ideal, $B\subseteq B_i$, for all i. Therefore $B\subseteq\bigcap_i B_i=B^2$. $B^2\subseteq B$ always. Hence we have $B^2=B$.

Thus we get (1)$\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (2) $\Rightarrow$ (1). Hence all the statements are equivalent. □

THEOREM 3.7. *Let S be a regular and intra- regular Γ-semiring. Then for any bi-ideal B of S, B is strongly irreducible bi-ideal if and only if B is strongly prime bi-ideal.*

PROOF. Let S be a regular and intra-regular Γ-semiring. Suppose that B is a strongly irreducible bi-ideal of S. To show that B is a strongly prime bi-ideal of S. Let B_1 and B_2 be any two bi-ideals of S such that $(B_1\Gamma B_2)\cap(B_2\Gamma B_1)\subseteq B$. By Theorem 3.6, $(B_1\Gamma B_2)\cap(B_2\Gamma B_1)=B_1\cap B_2$. Hence $B_1\cap B_2\subseteq B$. But B is a strongly irreducible bi-ideal of S. Therefore $B_1\subseteq B$ or $B_2\subseteq B$. Thus B is a strongly prime bi-ideal of S.

Conversely, suppose that B is a strongly prime bi-ideal of S. Let $B_1$and B_2 be any two bi-ideals of S such that $B_1\cap B_2\subseteq B$ and $(B_1\Gamma B_2)\cap(B_2\Gamma B_1)=B_1\cap B_2\subseteq B$. As B is a strongly prime bi-ideal, $B_1\subseteq B$ or $B_2\subseteq B$. Therefore B is a strongly irreducible bi-ideal of S. □

THEOREM 3.8. *Every bi-ideal of S is a strongly prime bi-ideal if and only if S is both regular and intra-regular and the set of bi-ideals of S is a totally ordered set under the inclusion of sets.*

PROOF. Suppose that every bi-ideal of S is a strongly prime bi-ideal. Then every bi-ideal of S is a semiprime bi-ideal. Hence by the Theorem 3.6, S is regular and intra-regular. To show that the set of bi-ideals of S is a totally ordered set under inclusion of sets. Let B_1 and B_2 be any two bi-ideals of S from the set of bi-ideals of S. $B_1\cap B_2$ is also a bi-ideal of S (see Theorem 3.1(3)). Hence by assumption $B_1 \cap B_2$ is a strongly prime bi-ideal of S. Therefore by Theorem 3.6, $(B_1\Gamma B_2) \cap (B_2\Gamma B_1) = B_1 \cap B_2 \subseteq B_1 \cap B_2$. Then $B_1 \subseteq B_1 \cap B_2$ or $B_2 \subseteq B_1 \cap B_2$. Therefore $B_1 \cap B_2 = B_1$ or $B_1 \cap B_2 = B_2$. Thus either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. This shows that the set of bi-ideals of S is a totally ordered set under inclusion of sets.

Conversely, suppose that S is regular, intra-regular and the set of bi-ideals of S is a totally ordered set under inclusion of sets. Let B be any bi-ideal of S. B_1 and B_2 be any two bi-ideals of S such that $(B_1\Gamma B_2) \cap (B_2\Gamma B_1) \subseteq B$. By the Theorem 3.6, we have $(B_1\Gamma B_2) \cap (B_2\Gamma B_1) = B_1 \cap B_2$. Therefore $B_1 \cap B_2 \subseteq B$.But by assumption either $B_1 \subseteq B_2$or $B_2 \subseteq B_1$. Hence $B_1 \cap B_2 = B_1$ or $B_1 \cap B_2 = B_2$. Thus we get $B_1 \subseteq B$ or $B_2 \subseteq B$. Therefore B is a strongly prime bi-ideal of S. □

Theorem 3.9. *If the set of bi-ideals of S is a totally ordered set under inclusion of sets, then every bi-ideal of S is a strongly prime if and only if every bi-ideal of S is prime.*

Proof. Let the set of bi-ideals of S be a totally ordered set under inclusion of sets. As every strongly prime bi-ideal of S is prime , the proof of only if part is obvious.

Conversely, suppose that every bi-ideal of S is prime. Then every bi-ideal of S semiprime. Hence by the Theorem 3.6, S is both regular and intra-regular. Again by Theorem 3.8, every bi-ideal of S is a strongly prime bi-ideal. □

Theorem 3.10. *If the set of bi-ideals of S is a totally ordered set under inclusion of sets,then S is both regular and intra-regular if and only if each bi-ideal of S is prime.*

Proof. Let the set of all bi-ideals of S be a totally ordered set under inclusion of sets. Suppose S is both regular and intra-regular. Let B be any bi-ideal of S. For any bi-ideals B_1 and B_2 of S, $B_1\Gamma B_2 \subseteq B$. By the assumption we have either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. Assume $B_1 \subseteq B_2$. Then $B_1\Gamma B_1 \subseteq B_1\Gamma B_2 \subseteq B$. Hence by Theorem 3.6, B is a semiprime bi-ideal of S. Therefore $B_1 \subseteq B$. Hence B is a prime bi-ideal of S.

Conversely, suppose that every bi-ideal of S prime. Hence every bi-ideal of S semiprime. Therefore by Theorem 3.6, S is both regular and intra-regular. □

Theorem 3.11. *Following statements are equivalents in S.*
(1) The set of bi-ideals of S is totally ordered set under inclusion of sets.
(2)Each bi-ideal of S is strongly irreducible.
(3) Each bi-ideal of S is irreducible.

Proof. (1) ⇒(2). Suppose that the set of bi-ideals of S is a totally ordered set under inclusion of sets. Let B be any bi-ideal of S. To show that B is a strongly irreducible bi-ideal of S. Let B_1 and B_2 be any two bi-ideals of S such that $B_1 \cap B_2 \subseteq B$. But by the hypothesis we have either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. Therefore $B_1 \cap B_2 = B_1$ or $B_1 \cap B_2 = B_2$. Hence $B_1 \subseteq B$ or $B_2 \subseteq B$. Thus B is a strongly irreducible bi-ideal of S.

(2) ⇒ (3) Suppose that each bi-ideal of S is strongly irreducible. Let B be any bi-ideal of S such that $B = B_1 \cap B_2$, for any bi-ideals B_1 and B_2 of S. Hence by

(2) we have , $B_1 \subseteq B$ or $B_2 \subseteq B$. As $B \subseteq B_1$ and $B \subseteq B_2$, we have $B_1 = B$ or $B_2 = B$. Hence B is an irreducible bi-ideal of S.

(3) $\Rightarrow$ (1) Suppose that each bi-ideal of S is an irreducible bi-ideal. Let B_1 and B_2 be any two bi-ideals of S. Then $B_1 \cap B_2$ is also a bi-ideal of S (see Theorem 3.1(3)). Hence $B_1 \cap B_2 = B_1 \cap B_2$ implies $B_1 \cap B_2 = B_1$ or $B_1 \cap B_2 = B_2$ by assumption. Therefore either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$.This shows that the set of bi-ideals of S is a totally ordered set under inclusion of sets. □

THEOREM 3.12. *A prime bi-ideal B of S is a prime one sided ideal of S.*

PROOF. Let B be a prime bi-ideal of S. Suppose B is not a one sided ideal of S. Therefore $B\Gamma S \nsubseteq B$ and $S\Gamma B \nsubseteq B$. As B is a prime bi-ideal

$$(B\Gamma S)\,\Gamma(S\Gamma B) \nsubseteq B.$$

$$(B\Gamma S)\,\Gamma\,(S\Gamma B) = B\Gamma\,(S\Gamma S)\,\Gamma B \subseteq B\Gamma S\Gamma \mathrm{B} \subseteq B,$$

which is a contradiction. Therefore $B\Gamma S \subseteq B$ or $S\Gamma B \subseteq B$.Hence B is a prime one sided ideal of S. □

THEOREM 3.13. *A bi-ideal B of S is prime if and only if for a right ideal R and a left ideal L of S, $R\Gamma L \subseteq B$ implies $R \subseteq B$ or $L \subseteq B$.*

PROOF. Suppose that a bi-ideal of S is a prime bi-ideal of S. Let R be a right ideal and L be a left ideal of S such that $R\Gamma L \subseteq B$. Itself R and L are bi-ideals of S (seeTheorem 3.1(2)). Hence $R \subseteq B$ or $L \subseteq B$. Conversely, we have to show that a bi-ideal B of S is a prime bi-ideal of S. Let A and C be any two bi-ideal of S such that $A\Gamma C \subseteq B$. For any $a \in A$ and $c \in C$, $(a)_r \subseteq A$ and $(c)_l \subseteq C$, where $(a)_r$ and $(c)_l$ denotes the right ideal and left ideal generated by a and c respectively. Therefore $(a)_r\Gamma\,(c)_l \subseteq A\Gamma C \subseteq B$. Hence by the assumption, $(a)_r \subseteq B$ or $(c)_l \subseteq B$. Therefore $a \in B$ or $c \in B$. Thus $A \subseteq B$ or $C \subseteq B$. Hence B is a prime bi-ideal of S. □

THEOREM 3.14. *If B is a strongly irreducible bi-ideal of a regular and intra-regular Γ-semiring S, then B is a prime bi-ideal.*

PROOF. Let B be a strongly irreducible bi-ideal of a regular and intra-regular Γ-semiring S. Let B_1 and B_2 be any two bi-ideals of S such that $B_1\Gamma B_2 \subseteq B$. $B_1 \cap B_2$ is also a bi-ideal of S (see Theorem 3.1(3)). Therefore by Theorem 3.6, $(B_1 \cap B_2)^2 = (B_1 \cap B_2)$. Hence $B_1 \cap B_2 = (B_1 \cap B_2)^2 = (B_1 \cap B_2)\,\Gamma\,(B_1 \cap B_2) \subseteq B_1\Gamma B_2 \subseteq B$. As B is a strongly irreducible bi-ideal of S, we have $B_1 \subseteq B$ or $B_2 \subseteq B$. Hence B is a prime bi-ideal of S. □

4. Space of Strongly Prime Bi-ideals

Let $\mathcal{B}$ be the family of all bi-ideals of S. $\mathcal{B}$ is a partially ordered set under the inclusion of sets. Clearly $\mathcal{B}$ is a complete lattice under $\vee$ and $\wedge$ defined by

$$B_1 \vee B_2 = B_1 + B_2 = \langle B_1 \bigcup B_2\rangle_b \text{ and } B_1 \wedge B_2 = B_1 \cap B_2 \text{ , for all } B_1, B_2 \in \mathcal{B}.$$

Let S be a Γ-semiring and $\wp_S$ be the set of all strongly prime bi-ideals of S. For each bi-ideal B of S define

$$\Theta_B = \{J \in \wp_S / B \nsubseteq J\} \text{ and } \zeta(\wp_S) = \{\Theta_B \,/ B \; is\; a\; bi-ideal\; of\; S\}.$$

Theorem 4.1. *If S is both regular and intra-regular, then $\zeta(\wp_s)$ forms a topology on the set $\wp_S$. There is an isomorphism between lattice of bi-ideals $\mathcal{B}$ and $\zeta(\wp_S)$, the lattice of open subsets of $\wp_S$.*

Proof. Since $\{0\}$ is a bi-ideal of S and each bi-ideal of S contains $\{0\}$. Hence $\Theta_{\{0\}} = \{ J \in \wp_s / \{0\} \nsubseteq J\} = \Phi$. Therefore $\Theta_{\{0\}} = \Phi \in \zeta(\wp_S)$. As S itself bi-ideal, $\Theta_S = \{J \in \wp_S / S \nsubseteq J\} = \wp_S$ implies $\wp_S = \Theta_S \in \zeta(\wp_S)$. Now let $\Theta_{B_k} \in \zeta(\wp_S)$, for $k \in \Lambda$ (Λ is an indexing set) and B_k is a bi-ideal of S. Therefore $\Theta_{B_k} = \{J \in \wp_S / B_k \nsubseteq J\}$.

At the other hand, we have

$$\bigcup_{k\in\Lambda} \Theta_{B_k} = \bigcup_{k\in\Lambda} \{J \in \wp_S / B_k \nsubseteq J\} = \{J \in \wp_S / B_k \nsubseteq J, for\; some\; k \in \Lambda \}.$$

Hence

$$\bigcup_{k\in\Lambda} \Theta_{B_k} = \left\{J \in \wp_S / \left\langle \bigcup_{k\in\Lambda} B_k \right\rangle_b \nsubseteq J\right\},$$

where $\left\langle \bigcup_{k\in\Lambda} B_k \right\rangle_b$ denotes the bi-ideal of S generated $\bigcup_{k\in\Lambda} B_k$. Therefore

$$\bigcup_{k\in\Lambda} \Theta_{B_k} = \Theta_{\left\langle \bigcup_{k\in\Lambda} B_k \right\rangle_b} \in \zeta(\wp_S).$$

Further let Θ_A, $\Theta_B \in \zeta(\wp_S)$. Let $J \in \Theta_A \bigcap \Theta_B$ imply $J \in \Theta_A$ and $J \in \Theta_B$. Then $A \nsubseteq J$ and B$\nsubseteq J$. Suppose that $A \cap B \subseteq J$. As S is both regular and intra-regular hence by the Theorem 3.6, $A \cap B = (A\Gamma B) \cap (B\Gamma A)$. Therefore $(A\Gamma B) \cap (B\Gamma A) \subseteq J$ and J is a strongly prime bi-ideal of S imply $A \subseteq J$ or $B \subseteq J$, which is a contradiction to $A \nsubseteq J$ and $B \nsubseteq J$. Hence $A \cap B \nsubseteq J$ implies $J \in \Theta_{A\cap B}$. Therefore $\Theta_A \bigcap \Theta_B \subseteq \Theta_{A\cap B}$. Now let $J \in \Theta_{A\cap B}$. Then $A \cap B \nsubseteq J$ implies $A \nsubseteq J$ and $B \nsubseteq J$. Therefore $J \in \Theta_A$ and $J \in \Theta_B$ imply $J \in \Theta_A \bigcap \Theta_B$ .Thus $\Theta_{A\cap B} \subseteq \Theta_A \bigcap \Theta_B$. Therefore we get $\Theta_A \bigcap \Theta_B = \Theta_{A\cap B} \in \zeta(\wp_S)$. Hence $\zeta(\wp_s)$ forms a topology on the set $\wp_S$.

Now we define a function $\phi : \mathcal{B} \longrightarrow \zeta(\wp_S)$ such that $\phi(B) = \Theta_B$. Let $A, B \in \mathcal{B}$. Then

$$\phi(A \cap B) = \Theta_{A\cap B} = \Theta_A \bigcap \Theta_B = \phi(A) \cap \phi(B)$$

and

$$\phi(A + B) = \phi(\langle A \cup B\rangle_b) = \Theta_{\langle A\cup B\rangle_b} = \Theta_A \cup \Theta_B = \phi(A) \cup \phi(B).$$

Therefore ϕ is a lattice homomorphism. Now let $\phi(A) = \phi(B)$. Hence we have $\Theta_A = \Theta_B$ Suppose that $A \neq B$. Then there exists $a \in A$ such that $a \notin B$. As B is a proper bi-ideal of S, by Theorem 3.4, there exists an irreducible bi-ideal J of S such that $B \subseteq J$ and $a \notin J$. By the Theorem 3.11, the set of all bi-ideals of S is totally ordered under inclusion of sets and also by the Theorem 3.8, J is a strongly prime bi-ideal of S. Hence $A \nsubseteq J$. $J \in \Theta_A = \Theta_B$ imply $B \nsubseteq J$. This contradicts to $B \subseteq J$. Therefore $A = B$. Hence ϕ is a lattice isomorphism. □

REMARK 4.1. *In the same way we can construct the space* $\wp_S$ *of strongly irreducible bi-ideals of* S.

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