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A NEW ANALYSIS OF REGULAR COEQUALITY RELATION

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Abstract. This investigation is in the Bishop's constructive mathematics. We discuss about co-order, co-quasiorder and coequality relations on set X with appartness. A connection between the family of all co-quasiorder relations and the family of all coequality relations on set X is given. In addition, a connection between the family of all co-quasiorder relations included in the co-order α and the family of all regular coequality relation on X with respect to α is also given.

1. Introduction and Preliminaries

This investigation is in the Constructive mathematics ([**1, 2, 3, 4, 5, 8, 9, 17**]) with the Constructive (Intuitionistic) logic ([**8, 17**]). It is the improvement of our previously published articles [**12, 14**] and a continuation of our forthcoming articles [**15, 16**]. So, in this text use notions and notations as in our recently published articles [**6, 7**] and in our mentioned forthcoming articles [**15, 16**].

In this investigation we continue our intention to research relations on set with apartness. In order to gain insight into the characters of our previous studies, the reader can look at texts [**10, 11, 12**] and earlier mentioned articles [**6, 7, 15, 16**].

A relation α on set X is a *co-order relation* (In our earlier articles [**11, 12, 13**] we used term *anti-order.*) if

$$\alpha \subseteq \neq, \ \alpha \subseteq \alpha * \alpha, \ \neq \subseteq \alpha \cup \alpha^{-1} \text{ (linearity)}$$

where "$*$" is the operation of relations $\alpha \subseteq X \times Y$ and $\beta \subseteq Y \times Z$, called *filled product* of relations α and β, are relation on $X \times Z$ defined by

$$(a, c) \in \beta * \alpha \Longleftrightarrow (\forall b \in X)((a, b) \in \alpha \vee (b, c) \in \beta).$$

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In that case we say that (X, α) is *co-ordered* set or X is a *ordered set under co-order* α.

A relation τ on X is a *coquasiorder* ([**7, 8**]) on X if

$$\tau \subseteq (\alpha \subseteq) \neq,\ \tau \subseteq \tau * \tau.$$

(In our earlier articles [**10, 11, 12, 13**] we used term *quasi-antiorder relation.*) In that case, we say that set (X, τ) is a *coquasiordered set* or set X is *ordered set under coquasiorder* τ. A relation q on X is a *coequality relation* on X if and only if it is consistent, symmetric and cotransitive:

$$q \subseteq \neq, \quad q = q^{-1}, \quad q \subseteq q * q.$$

It is clear that each coequality relation q on set X is a coquasiorder relation on X and the apartness is a trivial co-order relation on X. For an equivalence e and a coequalence q on set X we say that they are *associated* if $e \circ q \subseteq q$ holds, where the notation $'\circ'$ is the standard product of relations.

Ler q be a coequaliuty relation on a set X. Then we can ([**10, 11, 12**]) construct the family $X/q = \{aq : a \in X\}$ of strongly extensional subsets $aq = \{x \in X : (a, b) \in q\}$ generated by the element a of X. This family has ([**15**], Theorem 2.1) the following properties:

$$a \triangleright aq, \quad aq = qa, \quad aq \neq bq \Longrightarrow aq \cup qb = X.$$

The family X/q is a set with an equality and a coequality relations defined by

$$aq = bq \Longleftrightarrow (a, b) \triangleright q,\ aq \neq bq \Longleftrightarrow (a, b) \in q.$$

Let g be a strongly extensional mapping of relational system from (X, α) into relational system (Y, β). For g we say that it is:

(1) *isotone* if the following holds $(\forall a, b \in X)((a, b) \in \alpha \Longrightarrow (g(a), g(b)) \in \beta)$;

(2) *reverse isotone* if $(\forall a, b \in X)((g(a), g(b)) \in \beta \Longrightarrow (a, b) \in \alpha)$.

What is special about the work presented in this paper? In the first place, it is the application of intuitionistic logic instead of classical logic. In mathematics based on intuitionistic logic, there is a need to determine so-called negative concepts through a positive approach. In intuitionistic logic formula 'The principle of the third excluded' is neither an axiom nor a valid formula. Therefore, in this case, a set looks like as the relational system $(X, =, \neq)$ where the '$\neq$' is apartness relation (extensive to equality on the set in the following sense: $= \circ \neq \subseteq \neq$).

Secondly, the duality of the relationship, which appears with this aspect of observation on concepts and processes in mathematics based on Intuitionistic logic, opens possibilities for us to analyze the specific relationships that do not appear in classical mathematics. So, we are interested to study some specific relations that appear on sets with the apartness. In addition, we are also interested to analyze structures based on those specific relations.

2. On Regular Coequality Relations

For a given ordered set $(X, =, \neq, \alpha)$ under co-order α, it is essential to know if there exists a coequality relation q on X such that X/q be a co-ordered set. This plays an important role for studying the structure of co-ordered sets. The following question is natural: If $(X,=,\neq,\alpha)$ is a co-ordered set and q a coequality relation on X, is the family X/q a co-ordered set? Naturally, co-order on X/q should be the relation Θ on X/q defined by means of the co-order α on X, such that $\Theta = \{(xq, yq) \in X/q \times X/q : (x,y) \in \alpha\}$, but it is not held in general case. The following question appears: Is there a coequality relation q on X for which the family X/q is a co-ordered set such that the natural mapping $\vartheta : X \longrightarrow X/q$, defined by $\vartheta(a) = aq \ (a \in X)$, is reverse isotone? According to Lemma 2.4 in article [**16**], if $(X,=,\neq)$ is a set and σ a coquasiorder on X, then the relation q on X, defined by $q = \sigma \cup \sigma^{-1}$, is a coequality relation on X and the set X/q is an ordered set under co-order θ defined by $(xq, yq) \in \theta \Longleftrightarrow (x,y) \in \sigma$. So, each coquasiorder σ on a set X induces a coequality relation $q = \sigma \cup \sigma^{-1}$ on X such that the family X/q is an ordered set under co-order θ.

THEOREM 2.1. *Let q be a coequality relation on a co-ordered set $(X,=,\neq,\alpha)$. Then the relation $\Theta = \vartheta \circ \alpha \circ \vartheta^{-1}$ is a co-order relation on the family X/q if and only if the relation $\tau = q^{\triangleright} \circ \alpha \circ q^{\triangleright}$ is a coquasiorder relation on X such that $\tau \cup \tau^{1} = q$.*

PROOF. (1) Let q be a coequality relation on X such that the relation $\Theta = \vartheta \circ \alpha \circ \vartheta^{-1}$ is a co-order relation on X/q. Hence, the relation $\vartheta^{-1}(\Theta) = \{(a,b) \in X \times X : (aq, bq) \in \Theta\}$ is a coquasiorder relation on X under Θ such that $q = \tau \cup \tau^{-1}$ and the mapping $\vartheta : X \longrightarrow X/q$ is a reverse isotone strongly extensional surjective function. At the other hand, we have:

$$
\begin{aligned}
&(a,b) \in \vartheta^{-1}(\Theta) \Longleftrightarrow \\
&(aq, bq) \in \Theta = \vartheta \circ \alpha \circ \vartheta^{-1} \Longleftrightarrow \\
&(\exists x, y \in X)((aq, x) \in \vartheta^{-1} \wedge (x,y) \in \alpha \wedge (y, bq) \in \vartheta) \Longrightarrow \\
&(\exists x, y \in X)((a, aq) \in \vartheta \wedge (aq, x) \in \vartheta^{-1} \wedge (x,y) \in \alpha \\
&\qquad\qquad\qquad\qquad \wedge (y, bq) \in \vartheta \wedge (bq, b) \in \vartheta^{-1}) \Longrightarrow \\
&(a,b) \in \vartheta^{-1} \circ \vartheta \circ \alpha \circ \vartheta^{-1} \circ \vartheta \Longleftrightarrow \\
&(a,b) \in q^{\triangleright} \circ \alpha \circ q^{\triangleright}.
\end{aligned}
$$

Opposite, let (a,b) be an arbitrary element of $q^{\triangleright} \circ \alpha \circ q^{\triangleright}$. Then there exists x, y of X such that $(a,x) \in q^{\triangleright}$, $(x,y) \in \alpha$ and $(y,b) \in q^{\triangleright}$. Thus,
$aq = xq = \vartheta(x)$, $\vartheta(y) = bq = yq$ and $(x,y) \in \alpha$. Since $(aq, x) \in \vartheta^{1}$, $(x,y) \in \alpha$ and $(y, bq) \in \vartheta$ we have the following $(aq, bq) \in \vartheta \circ \alpha \circ \vartheta^{-1} = \Theta$. Hence, $(a,b) \in \vartheta^{1}(\Theta)$. Therefore, the relation $q^{\triangleright} \circ \alpha \circ q^{\triangleright} = \tau$ is a coquasiorder relation on X such that $\tau \cup \tau^{-1} = q$ and the mapping ϑ is a reverse isotone strongly extensional function from X onto X/q.

(2) Let $= q^{\triangleright} \circ \alpha \circ q^{\triangleright}$ be a coquasiorder relation on X such that $\tau \cup \tau^{-1} = q$. Then, the relation $\Theta = \{(aq, bq) \in X/q \times X/q : (a,b) \in \tau\}$ is a co-order relation on X/q. Checking that the equality $\Theta = \vartheta \circ \alpha \circ \vartheta^{-1}$ is valid is analogously to the checking in the first part of this proof. So, the relation Θ is a co-order relation on X/q such that the mapping ϑ is a strongly extensional reverse isotone function. Therefore, for the coequality q, the relation $\Theta = \vartheta \circ \alpha \circ \vartheta^{-1}$ is a co-order relation on the family X/q. □

Each coequality relation q on a set $(X,=,\neq,\alpha)$, such that X/q is co-ordered set, induces a coquasiorder on X. Such coequality relation we call *regular with*

respect to α. A coequality relation q on a set X is *regular* if there is a co-order Θ on X/q satisfying the following conditions:
(1) $(X/q, =, \neq, \Theta)$ is a co-ordered set;
(2) The mapping $\vartheta : X \longrightarrow X/q$ is a reverse isotone and surjective function.
In this case, we call the co-order Θ on X/q a *regular co-order* with respect to a regular coequalitty q on X and the co-order α ([**12**]).

Remark 2.1. Let us note that any coequality relation q on a set $(X, =, \neq)$ is a trivial regular coequality on X with respect to the apartness $\neq$ on X. In fact: if q is a coequality relation on X, then $\neq$ on X/q is a co-order relation on X/q and, besides, out of $(xq \neq yq \Longrightarrow (x, y) \in q \subseteq \neq)$ we get that the natural mapping $\vartheta : X \longrightarrow X/q$ is reverse isotone.

The following lemma gives a result important for the second main result in this article. The result is important by itself:

Lemma 2.1. *Let q be a coequality relation on $(X, =, \neq, \alpha)$ and suppose that there exists a co-order Θ on X/q such that $(X/q, =, \neq, \Theta)$ is co-ordered set and the natural mapping $\vartheta : X \longrightarrow X/q$ is a reverse isotone function. Then there exists a coquasiorder $\sigma(\subseteq \alpha)$ on X such that $\sigma \cup \sigma^{-1} = q$ and $\Theta = \theta$.*

Proof. Let q be a coequality relation on set $(X, =, \neq)$ and let Θ be a co-order relation on X/q such that $(X/q, =, \neq, \Theta)$ is an ordered set under co-order Θ. Let σ be an relation on X defined by $(x, y) \in \sigma \Longleftrightarrow (xq, yq) \in \Theta$. Then:
(1) The relation σ is a coquasiorder relation on X. Indeed:

Let (x, y) be an arbitrary element of σ. Then, $(xq, yq) \in \Theta$. Thus, $xq \neq yq$. The last means $(x, y) \in q$. Since q is a consistent relation, then we have $x \neq y$.

Let (x, z) be an arbitrary element of σ , i.e let $(xq, zq) \in \Theta$ hold. Hence, by cotransitivity, we have $(\forall yq \in X/q)((xq, yq) \in \Theta \vee (yq, zq) \in \Theta)$. Therefore, we have $(\forall y \in X)((x, y) \in \sigma \vee (y, z) \in \sigma)$.
(2) $q = \sigma \cup \sigma^{-1}$. Indeed:

Let (a, b) be an arbitrary element of q, i.e. let $aq \neq bq$. Thus, by linearity of Θ we have $(aq, bq) \in \Theta \vee (bq, aq) \in \Theta$. This means $(a, b) \in \sigma \vee (b, a) \in \sigma$. So, we have that $(a, b) \in \sigma \cup \sigma^{-1}$;

Opposite, let (x, y) be an element in $\sigma \cup \sigma^{-1}$, i.e. let $(x, y) \in \sigma \vee (y, x) \in \sigma$. Thus, we have $(xq, yq) \in \Theta \vee (yq, xq) \in \Theta$. Therefore, by definition of co-order relation, we have $xq \neq yq$. Finally, we have $(x, y) \in q$.
(3) $\Theta = \theta$. In fact:

$$\begin{aligned}(aq, bq) \in \Theta &\Longleftrightarrow (a, b) \in \sigma \\ &\Longleftrightarrow (a(\sigma \cup \sigma^{-1}), b(\sigma \cup \sigma^{-1})) \in \theta \\ &\Longleftrightarrow (aq, bq) \in \theta.\end{aligned}$$

(4) $\sigma \subseteq \alpha$. In fact: if $(x, y) \in \sigma$, then $(xq, yq) \in \theta$ and $(x, y) \in \alpha$ because the natural mapping ϑ is a reverse isotone function. □

Remark 2.2. Recall that any class aq of coequality relation q, generated by the element $a \in X$, is a strongly extensional subset of X. Besides, we have the following assertion, which is crucial in characterization of regular coequality relation on an co-ordered set $(X, =, \neq, \alpha)$. If q is a regular coequality relation on a co-ordered set X, then for every q–class aq in X we have

$$((x, y) \triangleright \alpha \ \wedge \ (y, z) \triangleright \alpha \ \wedge \ x \triangleright aq \ \wedge z \triangleright aq) \Longrightarrow y \triangleright aq$$

for any $x, y, z, a \in X$. If q is a regular coequality relation on a set X with respect to a co-order α, then there exists a co-order relation θ on the family X/q such that the natural function $\vartheta : X \longrightarrow X/q$ is a strongly extensive reverse isotone mapping. Further on, there exists a coquasiorder σ under α, defined by $(x, y) \in \sigma \Longleftrightarrow (xq, yq) \in \theta$ such that $\sigma \cup \sigma^{-1} = q$. Let t be an arbitrary element of aq. Then $(a, t) \in q = \sigma \cup \sigma^{-1}$. Thus $(a, t) \in \sigma \ \vee \ (t, a) \in \sigma$. Hence, we have

$(a, t) \in \sigma \Longrightarrow (a, x) \in \sigma \ \subseteq q \ \vee \ (x, y) \in \sigma \ \subseteq \alpha \ \vee \ (y, t) \in \sigma \ \subseteq q \Longrightarrow t \neq y$;

$(t, a) \in \sigma \Longrightarrow (t, y) \in \sigma \ \vee \ (y, z) \in \sigma \ \subseteq \ \alpha \ \vee \ (z, a) \in \sigma \ \subseteq \ q \Longrightarrow t \neq y.$

So, in both cases, we have that $t \in aq \Longrightarrow t \neq y$. Therefore, $y \triangleright aq$.

On the other side, we have

$$((x, y) \triangleright \alpha \ \wedge \ (y, z) \triangleright \alpha \ \wedge \ y \in aq) \Longrightarrow x \in aq \ \vee \ z \in aq.$$

for any $x, y, z, a \in X$. Indeed, if $x, y, z, a \in X$ such that $(x, y) \triangleright \alpha$ and $(y, z) \triangleright \alpha$ and $y \in aq$, then

$$(a, y) \in q = \sigma \cup \sigma^{-1} \Longrightarrow ((a, y) \in \sigma \ \vee \ (y, a) \in \sigma).$$

Thus, we have

$((a, y) \in \sigma \ \vee \ (y, a) \in \sigma) \Longrightarrow$

$((a, x) \in \sigma \ \subseteq q \ \vee \ (x, y) \in \sigma \ \subseteq \ \alpha)) \ \vee \ ((y, z) \in \sigma \ \subseteq \alpha \ \vee \ (z, a) \in \sigma \ \subseteq \ q) \Longrightarrow$

$x \in aq \ \vee \ z \in aq.$

In this section we analyze a special case of regular coequality relation on co-ordered set X. For a regular coequality q we say that it is a *strongly regular coequality relation* on X if

$$\alpha \circ q^{\triangleright} \ \subseteq \ q^{\triangleright} \circ \alpha$$

holds. For a strongly regular coequality q we have the following assertion: If coequality relation q on a set $(X, =, \neq)$ is a strongly regular, then the relation $\alpha \circ q^{\triangleright}$ is a coquasiorder relation on X. In this part of the section we start with the following result important for our main result of this paper and interesting by itself:

LEMMA 2.2. *For any three relations $\alpha \subseteq X_1 \times X_2$, $\beta \subseteq X_2 \times X_3$ and $\gamma \subseteq X_3 \times X_4$ the following inclusion*

$$\gamma * (\beta \circ \alpha) \supseteq (\gamma * \beta) \circ \alpha \ \textit{and} \ (\gamma \circ \beta) * \alpha \supseteq \gamma \circ (\beta * \alpha)$$

are valid.

For a strongly regular coequality q on ordered set X under co-order, we have:

THEOREM 2.2. *If the coequality relation q is a strongly regular, then the relation $\alpha \circ q^{\triangleright}$ is a coquasiorder relation on X and the relation $\Theta = \vartheta \circ \alpha \circ \vartheta^{-1}$ is the maximal co-order relation on X/q.*

Proof. (I) Let q be a regular coequality relation on a ordered set $(X,=,\neq,\alpha)$ under co-order α. Then there exists a co-order Θ on X/q such that the natural mapping $\vartheta : X \longrightarrow X/q$ is a reverse isotone function. So, it holds $(\forall aq, bq \in X/q)((aq,bq) \in \Theta) \Longrightarrow (a,b) \in \alpha)$. Hence, there exists a coquasiorder $\vartheta^{-1}(\Theta)$ on X, defined by $(aq,bq) \in \Theta \Longleftrightarrow (a,b) \in \vartheta^{-1}(\Theta)$ such that
$q = \{(a,b) \in X \times X : \vartheta(a) \neq \vartheta(b)\} = \{(a,b) \in X \times X : (aq,bq) \in \Theta \cup \Theta^{-1}\}$
$= \{(a,b) \in X \times X : (aq,bq) \in \Theta\} \cup \{(a,b) \in X \times X : (aq,bq) \in \Theta\}$
$= \vartheta^{-1}(\Theta) \cup (\vartheta^{-1}(\Theta))^{-1}$.
On the other hand, we have $\vartheta^{-1} \circ \alpha \circ \vartheta = \vartheta^{-1}(\Theta) \subseteq \alpha$. Therefore, we have the inclusion $\Theta \subseteq \vartheta \circ \alpha \circ \vartheta^{-1}$. Besides, we have $\vartheta^{-1}(\Theta) \subseteq q^{\triangleright} \circ \alpha \circ q^{\triangleright}$. Indeed: Let (a,b) be an arbitrary element of $\vartheta^{-1}(\Theta)$. Then $(aq,bq) \in \Theta \subseteq \vartheta \circ \alpha \circ \vartheta^{-1}$. Thus we conclude that there exist elements $x,y \in X$ such that $(aq,x) \in \vartheta^{-1}$, $(x,y) \in \alpha$ and $(y,bq) \in \vartheta$. Since $(a,aq) \in \vartheta$ and $(bq,b) \in \vartheta^{-1}$, we have $(a,b) \in \vartheta^{-1} \circ \vartheta \circ \alpha \circ \vartheta^{-1} \circ \vartheta$ $= q^{\triangleright} \circ \alpha \circ q^{\triangleright}$. Expect that, we have:

(1) $\alpha \circ q^{\triangleright} \subseteq q^{\triangleright} \circ q^{\triangleright} \circ q^{\triangleright} \subseteq q^{\triangleright} \circ \alpha \circ q^{\triangleright} \subseteq q^{\triangleright} \circ (\Theta * \Theta) \circ q^{\triangleright} \subseteq (q^{\triangleright}) \circ \alpha) * (\alpha \circ q^{\triangleright}) \subseteq (\alpha \circ q^{\triangleright}) * (\alpha \circ q^{\triangleright})$.

(2) Let us prove that the implication $\alpha \circ q^{\triangleright} \subseteq q^{\triangleright} \circ \alpha \Longrightarrow \alpha \circ q^{\triangleright} = q^{\triangleright} \circ \alpha \circ q^{\triangleright}$ is valid. In fact:

(i)$\alpha \circ q^{\triangleright} = \triangle_X \circ \alpha \circ q^{\triangleright} \subseteq q^{\triangleright} \circ \alpha \circ q^{\triangleright}$;

(ii) $q^{\triangleright} \circ \alpha \circ q^{\triangleright} \subseteq q^{\triangleright} \circ q^{\triangleright} \circ \alpha \subseteq q^{\triangleright} \circ \alpha$.

Therefore, if the relation q is a strongly regular coequality relation on co-ordered set $(X,=,\neq,\alpha)$, then holds $\alpha \circ q^{\triangleright} = q^{\triangleright} \circ \alpha \circ q^{\triangleright}$.

(II) Let Ξ be a co-order relation on the family X/q such that the mapping $\vartheta : X \longrightarrow X/q$ is a reverse isotone surjection. Then there exists a coquasiorder $\sigma = q^{\triangleright} \circ \alpha \circ q^{\triangleright}$ on X such that $\Xi = \vartheta \circ \alpha \circ \vartheta^{-1} = \Theta$. □

3. On Connections

The family $Coquas(X)$ of all coquasiorder relations in set X is a lattice ([**15**], Corollary 2.1) and the family $Coeq(X)$ all coequality relations in X is also a lattice ([**16**], Corollary 2.2)

Remark 3.1. Some of the colleagues who are engaged in constructive mathematics deny the existence of the concept of the cotransitive fulfillment of the relation that we used in our works [**12, 13, 14**]. However, one can not deny the existence of the maximal consistent and cotransitive relation on the set of X under given relation since the following statements is true.

Let $\{\sigma_k\}_{k \in J}$ be a family of coquasiorders on a set $(X,=,\neq)$ all included in a relation R. Then $\bigcup_{k \in J} \sigma_k$ is a coquasiorder in X included in α.

Indeed. It is easy to check that $\bigcup_{k \in J} \sigma_k$ is a consistent relation in X. Let (x,z) be an arbitrary elements of $X \times X$ such that $(x,z) \in \bigcup_{k \in J} \sigma_k$. then there exists k in J such that $(x,z) \in \sigma_k$. Hence, for every $y \in X$ we have $(x,y) \in \sigma_k \vee (y,z) \in \sigma_k$. So, $(x,y) \in \bigcup_{k \in J} \sigma_k$ or $(y,z) \in \bigcup_{k \in J} \sigma_k$. So, the relation $\bigcup_{k \in J} \sigma_k$ is cotransitive. At the other side, for every k in J holds $\sigma_k \subseteq R$. From this we conclude $\bigcup_{k \in J} \sigma_k \subseteq R$. Therefore, the relation $\bigcup_{k \in J} \sigma_k$ is the maximal coquasiorder relation in set X included in the relation R.

For a set $(X, =, \neq, \alpha)$ by $Coeq(X, \alpha)$ we denote the family of all regular coequality relations q on X with respect to α and by $Coquas(X, \alpha)$ denotes the family of all coquasiorder relations on X included in α.

As a direct consequence of thinking in above remark we have the following statements:

THEOREM 3.1. *Let X be an ordered set under co-order α. Then the family $Coquas(X, \alpha)$ is a completely lattice. The maximal element of this lattice is the relation α and the last element is $\emptyset$.*

PROOF. Let A be a subfamily of the family $Coquas(X, \alpha)$. Then the relation $\bigcup_{a \in A} \sigma_a$ is a coquasiorder relation included in the relation α. Also, there exist the maximal coquasiorder relation included in $\bigcap_{a \in A} \sigma_a$. □

As a consequence of the Theorem 3.1, we have

THEOREM 3.2. *The family $Coquas(X)$ of all coquasiorders in the set X is a completely lattice. The maximal element of this lattice is the relation $\neq$ and the last element is $\emptyset$.*

In addition, a connection between families $Coquas(X)$ and $Coeq(X)$ is given.

THEOREM 3.3. *The mapping*

$$f : Coquas(X) \longrightarrow Coeq(X),$$

defined by $f(\tau) = \tau \cup \tau^{-1}$, is a strongly extensional surjective mapping. Relations

$$Kerf = \{(\tau, \sigma) \in Coquas(X) \times Coquas(X) : \tau \cup \tau^{-1} = \sigma \cup \sigma^{-1}\}$$

$$Cokerf = \{(\tau, \sigma) \in Coquas(X) \times Coquas(X) : \tau \cup \tau^{-1} \neq \sigma \cup \sigma^{-1}\}$$

are associate equality and diversity relation on $Coquas(X)$ and there the isomorphism $Coquas(X)/(Kerf, Cokerf) \cong Coeq(X)$.

PROOF. The mapping f is well-defined strongly extensional function:

If τ is a coquasiorder relation on X, then $f(\tau) = \tau \cup \tau^{-1}$ is a coequality relation on X.

Let τ and σ be elements of $Coquas(X)$ such that $\tau = \sigma$. Then $f(\tau) = \tau \cup \tau^{-1} = \sigma \cup \sigma^{-1} = f(\sigma)$. So, the correspondence f is a function.

Suppose that $f(\tau) = \tau \cup \tau^{-1} \neq \sigma \cup \sigma^{-1} = f(\sigma)$ for some $\tau, \sigma \in Coquas(X)$. Then there exists an element $(x, y) \in X \times X$ such that

$$((x, y) \in \tau \cup \tau^{-1} \wedge (x, y) \rhd \sigma \cup \sigma^{-1}) \vee ((x, y) \in \sigma \cup \sigma^{-1} \wedge (x, y) \rhd \tau \cup \tau^{-1}).$$

In the first case, we have:

$$\begin{aligned}
&((x, y) \in \tau \vee (x, y) \in \tau^{-1}) \wedge (x, y) \rhd \sigma \wedge (x, y) \rhd \sigma^{-1} \Longrightarrow \\
&\quad ((x, y) \in \tau \wedge (x, y) \rhd \sigma) \vee ((x, y) \in \tau^{-1}) \wedge (x, y) \rhd \sigma^{-1}) \Longleftrightarrow \\
&\quad ((x, y) \in \tau \wedge (x, y) \rhd \sigma) \vee ((y, x) \in \tau \wedge (y, x) \rhd \sigma) \Longrightarrow \\
&\quad \tau \neq \sigma\ .
\end{aligned}$$

In the second case we derive similar implication analogously. So, the mapping f is strongly extensional.

f is a surjective mapping: Let q be a coequality relation on X. Then q is a coquasiorder relation on X and, by symmetric, $q^{-1} = q$. Thus we have $q \in Coquas(X)$ and $f(q) = q \cup q^{-1} = q$.

Finally, the correspondence $Coquas(X)/(Kerf, Cokerf) \longrightarrow Coeq(X)$ is a strongly extensional, embedding, injective and surjective mapping. $\square$

In the following assertion we give another main result of this paper:

THEOREM 3.4. *The mapping*

$$g : Coquas(X, \alpha)/(Kerf, Cokerf) \longrightarrow Coeq(X, \alpha),$$

defined by $g(\tau Kerf) = \tau \cup \tau^{-1}$, *is a strongly extensional embedding, injective and surjective function.*

PROOF. (1) The mapping g is well-defined strongly extensional function:

If τ is a coquasiorder relation on X, then $q = \tau \cup \tau^{-1}$ is a coequality relation on X. Then there exists a co-order relation θ on X/q defined by $(aq, bq) \in \theta \Longleftrightarrow (a, b) \in \tau$ and the natural mapping $\vartheta : X \longrightarrow X/q$ is reverse isotone. This means that $g(\tau Kerf) = \tau \cup \tau^{-1} = q \in Coeq(X, \alpha)$.

Let τ and σ be elements of $Coquas(X, \alpha)$ such that $\tau Kerf = \sigma Kerf$. Then $(\tau, \sigma) \in Kerf$ and $g(\tau Kerf) = \tau \cup \tau^{-1} = \sigma \cup \sigma^{-1} = g(\sigma Kerf)$.

Suppose that $g(\tau Kerf) = \tau \cup \tau^{-1} \neq \sigma \cup \sigma^{-1} = g(\sigma Kerf)$ for some $\tau, \sigma \in Coquas(X)$. Then there exists an element $(x, y) \in X \times X$ such that

$$((x, y) \in \tau \cup \tau^{-1} \text{ and } (x, y) \triangleright \sigma \cup \sigma^{-1}) \text{ or } ((x, y) \in \sigma \cup \sigma^{-1} \text{ and } (x, y) \triangleright \tau \cup \tau^{-1}).$$

In the first case, we have:

$$\begin{aligned} &((x, y) \in \tau \vee (x, y) \in \tau^{-1}) \wedge (x, y) \triangleright \sigma \wedge (x, y) \triangleright \sigma^{-1} \Longrightarrow \\ &\qquad ((x, y) \in \tau \wedge (x, y) \triangleright \sigma) \vee ((x, y) \in \tau^{-1}) \wedge (x, y) \triangleright \sigma^{-1}) \Longleftrightarrow \\ &\qquad ((x, y) \in \tau \wedge (x, y) \triangleright \sigma) \vee ((y, x) \in \tau \wedge (y, x) \triangleright \sigma) \Longrightarrow \\ &\qquad \tau \neq \sigma. \end{aligned}$$

In the second case we derive similar implication analogously.

(2) g is an injective function. In fact: Let τ and σ be elements of $Coquas(X, \alpha)$ such that $g(\tau Kerf) = \tau \cup \tau^{-1} = \sigma \cup \sigma^{-1} = g(\sigma Kerf)$. Then, $(\tau, \sigma) \in Kerf$ and $\tau Kerf = \sigma Kerf$.

(3) g is an embedding. Indeed, let τ and σ be elements of $Coquas(X, \alpha)$ such that $\tau Ker \neq \sigma Kerf$, i.e such that $(\tau, \sigma) \in Cokerf$. It means $g(\tau Kerf) = \tau \cup \tau^{-1} \neq \sigma \cup \sigma^{-1} = g(\sigma Kerf)$.

(4) g is a surjective function: Let q be a regular coequality relation on X with respect to α , i.e. let q be a coequality relation on X such that there exists a co-order θ on X/q and the natural mapping $\vartheta : X \longrightarrow X/q$ is reverse isotone. Then there exists a coquasiorder σ $(\subseteq \alpha)$ on X such that $\sigma \cup \sigma^{-1} = q$ and $\Theta = \theta$. Thus, $\sigma \in Coquas(X, \alpha)$ and $g(\sigma Kerf) = \sigma \cup \sigma^{-1} = q$. $\square$

EXAMPLE 3.1. Let $X = \{a, b, c, d\}$ be a set,

$$\alpha = \{(a, b), (a, c), (a, d), (b, a), (c, a), (c, b), (d, a), (d, b), (d, c)\}$$

be a co-order relation on X and

$$q = \{(a,c),(a,d),(b,c),(b,d),(c,a),(c,b),(d,a),(d,b)\}$$
be a coequality relation on X. Then
$$X/q = \{aq = \{c,d\}, bq = \{c,d\}, cq = \{a,b\}, dq = \{a,b\}\}.$$
Now we can define co-oreder θ on X/q by
$$\theta = \{(aq,cq),(aq,dq),(bq,cq),(bq,dq)\}.$$
In addition, there exists the coquasiorder $\tau = \{(a,c),(a,d),(b,c),(b,d)\}$ on X such that $q = \tau \cup \tau^{-1}$. It is clear that the mapping $\vartheta : X \longrightarrow X/q$ is reverse isotone. So, the coequality relation q is regular.

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