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CONVEXITY OF ONE MEAN WITH RESPECT TO ANOTHER MEAN

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ABSTRACT. Convexity/Concavity nature of different means are generally discussed. But in this paper relative study of Convexity/Concavity between different means are found and these results are interpreted in Van der Monde determinants.

1. Introduction

The well-known means are presented by Pappus of Alexandria in his books in the fourth century A.D., which is the main contribution of the ancient Greeks. In Pythagorean School, on the basis of proportion, ten Greek means are defined out of which six means are named and four means are un-named of which Arithmetic mean, Geometric mean, Harmonic mean and Contra harmonic mean respectively given below:

$$A(a,b) = \frac{a+b}{2} \tag{1.1}$$

$$G(a,b) = \sqrt{ab} \tag{1.2}$$

$$H(a,b) = \frac{2ab}{a+b} \tag{1.3}$$

and

$$C(a,b) = \frac{a^2+b^2}{a+b} \tag{1.4}$$

Have their own importance in the literature (See, [**1**–**5**, **7**, **8**, **10**, **13**]).

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Results on convexity of one function with respect to another function were in detail discussed by Bullen [**1**] and also some convexity results on various important means and their applications to mean inequalities were found in [**9**, **11**, **12**, **14**].

Zhen-Gang Xiao et al. [**15**] and various other authors have obtained some interesting and valuable results on generalization of Heron mean $H_e(a,b) = \frac{a+\sqrt{ab}+b}{3}$, using the generalized Van der Monde's determinants. These type of generalizations and applications have generated an impressive amount of work in this field.

Let φ be a continuous function on an interval I $\subseteq$ R, $a = (a_0, a_1, a_2, ..., a_n)$ and $a_i \in$ I, $a_i \neq a_j$ for $i \neq j$ (see [**6**]). Setting

$$V(a;\varphi) = \begin{vmatrix} 1 & a_0 & a_0^2 & \cdots & a_0^{n-1} & \varphi(a_0) \\ 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} & \varphi(a_1) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} & \varphi(a_n) \end{vmatrix}. \tag{1.5}$$

Let $\varphi(x) = x^{n+r} \ln^k x$ in 1.5, we have

$$V(a;r,k) = \begin{vmatrix} 1 & a_0 & a_0^2 & \cdots & a_0^{n-1} & a_0^{n+r} \ln^k a_0 \\ 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} & a_1^{n+r} \ln^k a_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} & a_n^{n+r} \ln^k a_n \end{vmatrix}. \tag{1.6}$$

2. Definitions and Lemmas

In this section, we presented some definitions and lemmas, which are necessary to develop this paper.

DEFINITION 2.1. A mean is defined as a function

$$M : R^2 \to R^+,$$

which has the property

$$a \wedge b \leqslant M(a,b) \leqslant a \vee b, \forall a, b > 0,$$

where

$$a \wedge b = min(a,b) \quad and \quad a \vee b = max(a,b).$$

LEMMA 2.1. *For $\varphi(x) = x^2$ and $a = (a_0, a_1, a_2)$, Eq. (1.5) is the Van der Monde's determinant of order three in the form:*

$$V(a;r=0,k=0) = \begin{vmatrix} 1 & a_0 & a_0^2 \\ 1 & a_1 & a_1^2 \\ 1 & a_2 & a_2^2 \end{vmatrix} \tag{2.1}$$

which is equivalently

$$V(a; r = 0, k = 0) = (a_1 - a_0)(a_2 - a_0)(a_2 - a_1) \tag{2.2}$$

LEMMA 2.2. *For $\varphi(x) = x^{1/2} = \sqrt{x}$ and $a = (a_0, a_1, a_2)$, Eq. (1.5) is the Van der Monde's determinant of order three in the form:*

$$V(a; r = -3/2, k = 0) = \begin{vmatrix} 1 & a_0 & \sqrt{a_0} \\ 1 & a_1 & \sqrt{a_1} \\ 1 & a_2 & \sqrt{a_2} \end{vmatrix} \tag{2.3}$$

which is equivalently

$$V(a; r = -3/2, k = 0) = (\sqrt{a_1} - \sqrt{a_0})(\sqrt{a_2} - \sqrt{a_0})(\sqrt{a_1} - \sqrt{a_2}) \tag{2.4}$$

Setting $a = x$ and $b = 1$ in eqs 1.1 to 1.4, Arithmetic mean, Geometric mean, Harmonic mean and Contra harmonic mean takes the following functions form;

$$A(x, 1) = \frac{x + 1}{2} \tag{2.5}$$

$$G(x, 1) = \sqrt{x} \tag{2.6}$$

$$H(x, 1) = \frac{2x}{x + 1} \tag{2.7}$$

and

$$C(x, 1) = \frac{x^2 + 1}{x + 1} \tag{2.8}$$

LEMMA 2.3. *Let $f(x)$ and $g(x)$ are two functions, then $f(x)$ is said to be convex with respect to $g(x)$ for $a \leqslant b \leqslant c$ if and only if*

$$\begin{vmatrix} 1 & f(a) & g(a) \\ 1 & f(b) & g(b) \\ 1 & f(c) & g(c) \end{vmatrix} \geqslant 0 \tag{2.9}$$

which is equivalently

$$\begin{vmatrix} 1 & f(a) & g(a) \\ 0 & f(b) - f(a) & g(b) - g(a) \\ 0 & f(c) - f(a) & g(c) - g(a) \end{vmatrix} \geqslant 0 \tag{2.10}$$

or

$$([f(b) - f(a)][g(c) - g(a)] - [f(c) - f(a)][g(b) - g(a)]) \geqslant 0 \tag{2.11}$$

3. Main Results

In this section, the necessary and sufficient conditions for convexity of one mean with respect to another mean among the Arithmetic mean, Geometric mean, Harmonic mean and Contra harmonic mean are discussed and the conditions are expressed in terms of Vander Monde's determinants.

THEOREM 3.1. *The arithmetic mean is convex (concave) with respect to geometric mean if and only if* $V(a; r = -3/2, k = 0) \geqslant (\leqslant) 0$.

PROOF. Consider the Arithmetic mean and Geometric mean in the form;

$$A(x,1) = \frac{x+1}{2} \quad \text{and} \quad G(x,1) = \sqrt{x}$$

Let

$$f(x) = \frac{x+1}{2} \quad \text{and} \quad g(x) = \sqrt{x}$$

then by Lemma 2.3 we have

$$\begin{vmatrix} 1 & f(a) & g(a) \\ 0 & f(b)-f(a) & g(b)-g(a) \\ 0 & f(c)-f(a) & g(c)-g(a) \end{vmatrix} = \begin{vmatrix} 1 & \frac{a+1}{2} & \sqrt{a} \\ 0 & \frac{b-a}{2} & \sqrt{b}-\sqrt{a} \\ 0 & \frac{c-a}{2} & \sqrt{c}-\sqrt{a} \end{vmatrix}$$

on simplifying the determinant leads to

$$\frac{1}{2}(\sqrt{c}-\sqrt{a})(\sqrt{b}-\sqrt{a})(\sqrt{b}-\sqrt{c}) \leqslant 0$$

then by Lemma 2.2,

$$(\sqrt{c}-\sqrt{a})(\sqrt{b}-\sqrt{a})(\sqrt{b}-\sqrt{c}) = V(a; r=-3/2, k=0) \leqslant 0. \tag{3.1}$$

similarly by considering

$$f(x) = \sqrt{x} \quad \text{and} \quad g(x) = \frac{x+1}{2}$$

then by Lemma 2.3 we have

$$(\sqrt{c}-\sqrt{a})(\sqrt{b}-\sqrt{a})(\sqrt{c}-\sqrt{b}) = V(a; r=-3/2, k=0) \geqslant 0. \tag{3.2}$$

By combining eqs 3.1 and 3.2, the proof of Theorem 3.1 completes. □

THEOREM 3.2. *The arithmetic mean is convex (concave) with respect to harmonic mean if and only if* $V(a; r = 0, k = 0) \geqslant (\leqslant) 0$.

PROOF. Consider the Arithmetic mean and Harmonic mean in the form;

$$A(x,1) = \frac{x+1}{2} \quad \text{and} \quad H(x,1) = \frac{2x}{x+1}$$

Let

$$f(x) = \frac{x+1}{2} \quad \text{and} \quad g(x) = \frac{2x}{x+1}$$

then by Lemma 2.3 we have

$$\begin{vmatrix} 1 & f(a) & g(a) \\ 0 & f(b)-f(a) & g(b)-g(a) \\ 0 & f(c)-f(a) & g(c)-g(a) \end{vmatrix} = \begin{vmatrix} 1 & \frac{a+1}{2} & \frac{2a}{a+1} \\ 0 & \frac{b-a}{2} & \frac{2b}{b+1}-\frac{2a}{a+1} \\ 0 & \frac{c-a}{2} & \frac{2c}{c+1}-\frac{2a}{a+1} \end{vmatrix}$$

on simplifying the determinant leads to

$$2\frac{(b-a)(c-a)(b-c)}{(a+1)(b+1)(c+1)} \leqslant 0$$

then by Lemma 2.2,

$$2\frac{(b-a)(c-a)(c-b)}{(a+1)(b+1)(c+1)} = 2\frac{V(a; r=0, k=0)}{(a+1)(b+1)(c+1)} \tag{3.3}$$

assume that $0 \leqslant a \leqslant b \leqslant c$ implies that $[(a+1)(b+1)(c+1)] \geqslant 0$ similarly by considering

$$f(x) = \frac{2x}{x+1} \quad \text{and} \quad g(x) = \frac{x+1}{2}$$

then by Lemma 2.3 we have

$$2\frac{(b-a)(c-a)(c-b)}{(a+1)(b+1)(c+1)} = 2\frac{V(a; r=0, k=0)}{(a+1)(b+1)(c+1)}. \tag{3.4}$$

By combining eqs 3.3 and 3.4, the proof of Theorem 3.2 completes. □

THEOREM 3.3. *The geometric mean is convex (concave) with respect to harmonic mean if and only if* $V(a; r=-3/2, k=0)(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}-1) \geqslant (\leqslant) 0$.

PROOF. Consider the Geometric mean and Harmonic mean in the form;

$$G(x,1) = \sqrt{x} \quad \text{and} \quad H(x,1) = \frac{2x}{x+1}$$

Let

$$f(x) = \sqrt{x} \quad \text{and} \quad g(x) = \frac{2x}{x+1}$$

then by Lemma 2.3 we have

$$\begin{vmatrix} 1 & f(a) & g(a) \\ 0 & f(b)-f(a) & g(b)-g(a) \\ 0 & f(c)-f(a) & g(c)-g(a) \end{vmatrix} = \begin{vmatrix} 1 & \sqrt{a} & \frac{2a}{a+1} \\ 0 & \sqrt{b}-\sqrt{a} & \frac{2b}{a+1}-\frac{2a}{a+1} \\ 0 & \sqrt{c}-\sqrt{a} & \frac{2c}{a+1}-\frac{2a}{a+1} \end{vmatrix}$$

on simplifying the determinant leads to

$$2\frac{(\sqrt{c}-\sqrt{a})(\sqrt{b}-\sqrt{a})(\sqrt{c}-\sqrt{b})(\sqrt{a}\sqrt{b}+\sqrt{b}\sqrt{c}+\sqrt{c}\sqrt{a}-1)}{(a+1)(b+1)(c+1)} \leqslant 0$$

then by Lemma 2.2,

$$(\sqrt{c}-\sqrt{a})(\sqrt{b}-\sqrt{a})(\sqrt{b}-\sqrt{c}) = V(a; r=-3/2, k=0) \tag{3.5}$$

assume that $0 \leqslant a \leqslant b \leqslant c$ implies that $(a+1)(b+1)(c+1) \geqslant 0$ similarly by considering

$$f(x) = \frac{2x}{x+1} \quad \text{and} \quad g(x) = \sqrt{x}$$

then by Lemma 2.3 we have

$$2\frac{(\sqrt{c}-\sqrt{a})(\sqrt{b}-\sqrt{a})(\sqrt{b}-\sqrt{c})(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}-1)}{(a+1)(b+1)(c+1)} \leqslant 0. \tag{3.6}$$

By combining eqs 3.5 and 3.6, the proof of Theorem 3.3 completes. □

THEOREM 3.4. *The contra harmonic mean is convex (concave) with respect to harmonic mean if and only if* $V(a; r = 0, k = 0) \geqslant (\leqslant) 0$.

PROOF. Consider the Arithmetic mean and Harmonic mean in the form;

$$C(x,1) = \frac{x^2+1}{x+1} \quad \text{and} \quad H(x,1) = \frac{2x}{x+1}$$

Let

$$f(x) = \frac{x^2+1}{x+1} \quad \text{and} \quad g(x) = \frac{2x}{x+1}$$

then by Lemma 2.3 we have

$$\begin{vmatrix} 1 & f(a) & g(a) \\ 0 & f(b)-f(a) & g(b)-g(a) \\ 0 & f(c)-f(a) & g(c)-g(a) \end{vmatrix} = \begin{vmatrix} 1 & \frac{a^2+1}{a+1} & \frac{2a}{a+1} \\ 0 & \frac{b^2+1}{b+1}-\frac{a^2+1}{a+1} & \frac{2b}{a+1}-\frac{2a}{a+1} \\ 0 & \frac{c^2+1}{c+1}-\frac{a^2+1}{a+1} & \frac{2c}{a+1}-\frac{2a}{a+1} \end{vmatrix}$$

on simplifying the determinant leads to

$$2\frac{(\sqrt{c}-\sqrt{a})(\sqrt{b}-\sqrt{a})(\sqrt{c}-\sqrt{a})}{(a+1)(b+1)(c+1)} \leqslant 0$$

then by Lemma 2.2,

$$\frac{(b-a)(c-a)(c-b)}{(a+1)(b+1)(c+1)} = \frac{V(a; r=0, k=0)}{(a+1)(b+1)(c+1)} \leqslant 0. \tag{3.7}$$

assume that $0 \leqslant a \leqslant b \leqslant c$ implies that $(a+1)(b+1)(c+1) \geqslant 0$ similarly by considering

$$f(x) = \frac{2x}{x+1} \quad \text{and} \quad g(x) = \frac{x^2+1}{x+1}$$

then by Lemma 2.3 we have

$$2\frac{(\sqrt{c}-\sqrt{a})(\sqrt{b}-\sqrt{a})(\sqrt{c}-\sqrt{a})}{(a+1)(b+1)(c+1)} \geqslant 0. \tag{3.8}$$

By combining eqs 3.7 and 3.8, the proof of Theorem 3.4 completes. □

THEOREM 3.5. *The contra harmonic mean is convex (concave) with respect to arithmetic mean if and only if* $V(a; r = 0, k = 0) \geqslant (\leqslant) 0$.

PROOF. Consider the Contra Harmonic mean and Arithmetic mean in the form;

$$C(x,1) = \frac{x^2+1}{x+1} \quad \text{and} \quad A(x,1) = \frac{x+1}{2}$$

Let

$$f(x) = \frac{x^2+1}{x+1} \quad \text{and} \quad g(x) = \frac{x+1}{2}$$

then by Lemma 2.3 we have

$$\begin{vmatrix} 1 & f(a) & g(a) \\ 0 & f(b)-f(a) & g(b)-g(a) \\ 0 & f(c)-f(a) & g(c)-g(a) \end{vmatrix} = \begin{vmatrix} 1 & \frac{a^2+1}{a+1} & \frac{a+1}{2} \\ 0 & \frac{b^2+1}{b+1}-\frac{a^2+1}{a+1} & \frac{b+1}{2}-\frac{a+1}{2} \\ 0 & \frac{c^2+1}{c+1}-\frac{a^2+1}{a+1} & \frac{c+1}{2}-\frac{a+1}{2} \end{vmatrix}$$

on simplifying the determinant leads to

$$\frac{1}{2}\frac{(c-a)(c-a)(b-c)}{(a+1)(b+1)(c+1)} \leqslant 0$$

then by Lemma 2.2,

$$\frac{(b-a)(c-a)(c-b)}{(a+1)(b+1)(c+1)} = \frac{V(a;r=0,k=0)}{(a+1)(b+1)(c+1)} \leqslant 0. \tag{3.9}$$

assume that $0 \leqslant a \leqslant b \leqslant c$ implies that $(a+1)(b+1)(c+1) \geqslant 0$ similarly by considering

$$f(x) = \frac{x+1}{2} \quad \text{and} \quad g(x) = \frac{x^2+1}{x+1}$$

then by Lemma 2.3 we have

$$2\frac{1}{2}\frac{(c-a)(c-a)(b-c)}{(a+1)(b+1)(c+1)} \geqslant 0. \tag{3.10}$$

By combining eqs 3.9 and 3.10, the proof of Theorem 3.5 completes. □

THEOREM 3.6. *The contra harmonic mean is convex (concave) with respect to geometric mean if and only if* $(abc+ab+ac+bc+a+b+c+2(\sqrt{ab}+\sqrt{bc}+\sqrt{ac})-1)V(a;r=-3/2,k=0) \geqslant (\leqslant) 0$.

PROOF. Consider the Contra harmonic mean mean and Geometric mean in the form;

$$C(x,1) = \frac{x^2+1}{x+1} \quad \text{and} \quad H(x,1) = \frac{2x}{x+1}$$

Let

$$f(x) = \frac{x^2+1}{x+1} \quad \text{and} \quad g(x) = \frac{2x}{x+1}$$

then by Lemma 2.3 we have

$$\begin{vmatrix} 1 & f(a) & g(a) \\ 0 & f(b)-f(a) & g(b)-g(a) \\ 0 & f(c)-f(a) & g(c)-g(a) \end{vmatrix} = \begin{vmatrix} 1 & \frac{a^2+1}{a+1} & \sqrt{a} \\ 0 & \frac{b^2+1}{b+1}-\frac{a^2+1}{a+1} & \sqrt{b}-\sqrt{a} \\ 0 & \frac{c^2+1}{c+1}-\frac{a^2+1}{a+1} & \sqrt{c}-\sqrt{a} \end{vmatrix}$$

on simplifying the determinant leads to

$$\frac{(\sqrt{c}-\sqrt{a})(\sqrt{b}-\sqrt{a})(\sqrt{c}-\sqrt{a})(abc+ab+ac+bc+a+b+c+2(\sqrt{ab}+\sqrt{bc}+\sqrt{ac})-1)}{(a+1)(b+1)(c+1)} \leqslant 0 \tag{3.11}$$

then by Lemma 2.2,

$$\frac{(abc+ab+ac+bc+a+b+c+2(\sqrt{ab}+\sqrt{bc}+\sqrt{ac})-1)V(a;r=-3/2,k=0)}{(a+1)(b+1)(c+1)} \leqslant 0.$$

assume that $0 \leqslant a \leqslant b \leqslant c$ implies that $(a+1)(b+1)(c+1) \geqslant 0$ similarly by considering

$$f(x) = \frac{2x}{x+1} \quad \text{and} \quad g(x) = \frac{x^2+1}{x+1}$$

then by lemma 2.3 we have

$$\frac{(\sqrt{c}-\sqrt{a})(\sqrt{b}-\sqrt{a})(\sqrt{c}-\sqrt{a})(abc+ab+ac+bc+a+b+c+2(\sqrt{ab}+\sqrt{bc}+\sqrt{ac})-1)}{(a+1)(b+1)(c+1)} \geqslant 0. \tag{3.12}$$

By combining eqs 3.11 and 3.12, the proof of Theorem 3.6 completes. □

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