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Inequalities Involving Hyperbolic Functions and Trigonometric Functions II

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Abstract. Several inequalities involving trigonometric and hyperbolic functions are derived. Main results are obtained with the aid of certain inequalities for the Schwab-Borchardt mean and the new bivariate mean introduced recently by this author in [**25**].

1. Introduction

In recent years a significant progress has been made in the area of inequalities involving circular functions, hyperbolic functions and their inverse functions as well. A list of published research papers which deal with this topic is too long to be included here. The interested reader is referred to [**1, 5, 10, 12, 15, 16, 32, 33, 34**] and the references therein.

In particular, the following results

$$1 < \frac{1}{3}\big(2\frac{\sin x}{x} + \frac{\tan x}{x}\big) \tag{1.1}$$

and

$$1 < \frac{1}{2}\Big[\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x}\Big] \tag{1.2}$$

($0 < |x| < \pi/2$) have attracted attention of several researchers. Inequalities (1) and (2) have been obtained, respectively, by C. Huygens [**7**] and J.B. Wilker [**37**]. Several proofs of these results can be found in mathematical literature (see, e.g., [**4, 6, 15, 19, 30, 38, 39, 40, 41, 42**] and the references therein). In [**30**] the authors called inequalities (1) and (2) the first Huygens and the first Wilker inequalities, respectively, for the trigonometric functions.

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The second Huygens and the second Wilker inequalities for the trigonometric functions also appear in mathematical literature. They read, respectively, as follows

$$1 < \frac{1}{3}\big(2\frac{x}{\sin x} + \frac{x}{\tan x}\big) \tag{1.3}$$

and

$$1 < \frac{1}{2}\big[\Big(\frac{x}{\sin x}\Big)^2 + \frac{x}{\tan x}\big] \tag{1.4}$$

($0 < |x| < \pi/2$). For the proofs of the last two results the interested reader is referred to [**30**] and [**38**], respectively. Counterparts of inequalities (1)-(4) for hyperbolic functions also appear in mathematical literature. Recently the Huygens and Wilker type inequalities have been proven for the lemniscate functions (see [**20**]), generalized trigonometric functions (see [**8**]) Jacobian elliptic functions (see [**11, 14, 21**]) and for the Jacobian theta functions (see [**21**]).

Inequalities (1) and (2) follow easily from the left inequality in

$$(\cos x)^{1/3} < \frac{\sin x}{x} < \frac{\cos x + 2}{3} \tag{1.5}$$

($0 < |x| < \pi/2$) which has been established by D.D. Adamović and D.S. Mitrinović (see, e.g., [**9**]). The second inequality in (5) is due to N. Cusa and C.Huygens (see [**7**] for more details regarding this result).

This paper is the second part of our paper [**15**] and is organized as follows. In Section 2 we give definitions of some bivariate means. Among means included there the Schwab-Borchardt mean and a new mean introduced recently by this author play a crucial role in proofs of the main results of this paper. The latter are presented in Section 3. Altogether about a dozen of new inequalities for the trigonometric and hyperbolic functions are established.

2. Bivariate means used in this paper

In what follows the letters a and b will always stand for positive and unequal numbers.

The important mean utilized in this paper is called the Schwab-Borchardt mean and is defined as follows:

$$SB(a,b) \equiv SB = \begin{cases} \dfrac{\sqrt{b^2 - a^2}}{\cos^{-1}(a/b)} & \text{if } \ a < b, \\ \dfrac{\sqrt{a^2 - b^2}}{\cosh^{-1}(a/b)} & \text{if } \ b < a \end{cases} \tag{2.1}$$

(see, e.g., [**2**], [**3**]). It is well known that the mean SB is strict, nonsymmetric and homogeneous of degree one in its variables.

Another bivariate mean used in this paper is defined as follows:

$$N(a,b) \equiv N = \frac{1}{2}\big(a + \frac{b^2}{SB(a,b)}\big) \tag{2.2}$$

(see [**25**]). It's easy to see that mean N is also strict, nonsymmetric and homogeneous of degree one in its variables.

We will give now another formula for mean SB (see [**25**]):

$$SB(a,b) \equiv SB = \begin{cases} b\dfrac{\sin r}{r} = a\dfrac{\tan r}{r} & \text{if } \ a < b, \\ b\dfrac{\sinh s}{s} = a\dfrac{\tanh s}{s} & \text{if } \ b < a, \end{cases} \tag{2.3}$$

where

$$\cos r = a/b \qquad \text{if} \qquad a < b \qquad \text{and} \qquad \cosh s = a/b \qquad \text{if} \qquad a > b. \tag{2.4}$$

For the later use let us also record similar formulas for the mean N. We have ([**25**]

$$N(a,b) \equiv N = \frac{1}{2}b\big(\cos r + \frac{r}{\sin r}\big) = \frac{1}{2}a\big(1 + \frac{r}{\sin r \cos r}\big) \tag{2.5}$$

provided $a < b$. Similarly, if $a > b$, then

$$N(a,b) \equiv N = \frac{1}{2}b\big(\cosh s + \frac{s}{\sinh s}\big) = \frac{1}{2}a\big(1 + \frac{s}{\sinh s \cosh s}\big). \tag{2.6}$$

In what follows he symbols G, A, and Q will be used to denote, respectively, the geometric, arithmetic, and the root-square means of a and b. Recall that

$$G = \sqrt{ab}, \qquad A = \frac{a+b}{2}, \qquad Q = \sqrt{\frac{a^2+b^2}{2}}.$$

For the sake of presentation let us recall definitions of certain bivariate means of a and b.

Two Seiffert means P and T are defined as follows:

$$P = A\frac{v}{\sin^{-1} v}, \qquad T = A\frac{v}{\tan^{-1} v} \tag{2.7}$$

(see [**35**] and [**36**]), where

$$v = \frac{a-b}{a+b}.$$

Clearly $0 < |v| < 1$. We shall also use the logarithmic mean L and the Neuman-Sándor mean M, introduced in [**28**] and studied in [**13, 17, 18, 22, 23, 29, 31**]. The last two means are defined as follows

$$L = \frac{a-b}{\log a - \log b} = A\frac{v}{\tanh^{-1} v}, \qquad M = A\frac{v}{\sinh^{-1} v}. \tag{2.8}$$

It is known (see [**28**]) that

$$G < L < P < A < M < T < Q. \tag{2.9}$$

Thus the means listed in the last chain are comparable. Moreover, four means which appear in (8) and (9) are generated by the Schwab-Borchardt mean. The following result

$$\begin{aligned} L &= SB(A,G), \qquad P = SB(G,A), \\ M &= SB(Q,A), \qquad T = SB(A,Q) \end{aligned} \tag{2.10}$$

has been established in [**28**].

In what follows, let $k \geqslant 1$. The following means

$$\Lambda_k \equiv \Lambda_k(a,b) = A(1 - \frac{1}{k}v^2) \tag{2.11}$$

and

$$\Omega_k \equiv \Omega_k(a,b) = A(1 + \frac{1}{k}v^2). \tag{2.12}$$

have been introduced in [**24**].

3. Main results

In this section we shall establish several inequalities for functions under discussion. Those results are obtained with the aid of inequalities satisfied by bivariate means included in the previous section.

Our first result reads as follows

THEOREM 3.1. *The following inequality*

$$\frac{1 + 2\cosh t}{2 + \cosh t} < \frac{\sinh t}{t} \tag{3.1}$$

$(0 < |t| < \operatorname{arcsinh}(1))$ *is valid.*

PROOF. In order to prove the desired result we shall utilize the following inequality

$$\frac{2Q + A}{Q + 2A} < \frac{M}{A}$$

which is established in [**26**]. This is equivalent to the inequality

$$\frac{2\sqrt{1+v^2} + 1}{\sqrt{1+v^2} + 2} < \frac{v}{\sinh^{-1} v}.$$

Here we have used a formula $Q = A\sqrt{1+v^2}$ and the last part of (13). We let now $v = \sinh t$. Then the inequality $0 < |v| < 1$ implies that $0 < |t| < \sinh^{-1}(1)$. The proof is complete. □

In the proof of the next result we shall employ the following result established in [**27**]

$$SB(b,a) < A < N(a,b), \qquad \text{if} \qquad a < b \tag{3.2}$$

and

$$N(a,b) < A < SB(b,a), \qquad \text{if} \qquad a > b. \tag{3.3}$$

We are in a position to prove the following result

THEOREM 3.2. *If* $0 < |t| < \pi/2$, *then*

$$\frac{\sin t}{\tanh^{-1}(\sin t)} < \frac{1 + \cos t}{2} < \frac{1}{2}\Big(\cos t + \frac{t}{\sin t}\Big). \tag{3.4}$$

Moreover, if $t \neq 0$, *then*

$$\frac{1}{2}\Big(\cosh t + \frac{t}{\sinh t}\Big) < \frac{1 + \cosh t}{2} < \frac{\sinh t}{\sin^{-1}(\tanh t)}. \tag{3.5}$$

PROOF. Inequality (21) can be obtained using (19) with $a = \cos t$ and $b = 1$. Making use of (6) and (7) we easily obtain

$$SB(1, \cos t) = \frac{\sin t}{\tanh^{-1}(\sin t)} \tag{3.6}$$

and

$$N(\cos t, 1) = \frac{1}{2}\big(\cos t + \frac{t}{\sin t}\big). \tag{3.7}$$

The assertion now follows from (20). We shall now prove that the inequality (22) holds true using (20) with $a = \cosh t$ and $b = 1$ To this aim we employ formulas (6), (7), (10) and (11) to obtain

$$N(\cosh t, 1) = \frac{1}{2}\big(\cosh t + \frac{t}{\sinh t}\big) \tag{3.8}$$

and

$$SB(1, \cosh t) = \frac{\sinh t}{\cos^{-1}(1/\cosh t)} = \frac{\sinh t}{\sin^{-1}(\tanh t)}. \tag{3.9}$$

The desired result now follows using (20). □

We shall establish now the following

THEOREM 3.3. *Let* $0 < |t| < \pi/2$. *Then*

$$3 + \cos^2 t\frac{\tanh^{-1}(\sin t)}{\sin t} < 2\big(\cos t + \frac{t}{\sin t}\big). \tag{3.10}$$

Also, if $t \neq 0$*, then*

$$3 + \cosh^2 t\frac{\sin^{-1}(\tanh t)}{\sinh t} < 2\big(\cosh t + \frac{t}{\sinh t}\big). \tag{3.11}$$

PROOF. We shall prove both inequalities (27) and (28) using the following one (see [**26**])

$$b + N(b, a) < 2N(a, b) \tag{3.12}$$

which holds true for all $a \neq b$. Using (7) we write (29) as follows

$$3b + \frac{a^2}{SB(b, a)} < 2\big(a + \frac{b^2}{SB(a, b)}\big). \tag{3.13}$$

We let in (6) $a = \cos t$ and $b = 1$ to obtain

$$SB(\cos t, 1) = \frac{\sin t}{t} \tag{3.14}$$

and employ (23) to obtain, with the aid of (30), the desired result (27). Inequality (28) can be established in a similar manner. We use (30) with $a = \cosh t$ and $b = 1$ followed by application of (26) and an easy to verify formula

$$SB(\cosh t, 1) = \frac{\sinh t}{t} \tag{3.15}$$

to obtain the asserted result. The proof is complete. □

It is worth mentioning that the left hand sides of (27) and (28) are greater than the functions

$$4\frac{2+\cos t}{3}$$

and

$$4\frac{2+\cosh t}{3},$$

respectively. The last two inequalities follow easily from the following one

$$\frac{a+2b}{3} < \frac{b+N(b,a)}{2}$$

(see [**26**]). We leave the proofs to the interested readers.

Our next result reads as follows

Theorem 3.4. *The following inequality*

$$2\frac{t}{\sin t} < 1 + \frac{\sin t}{t}\Big(\frac{2}{1+\cos t}\Big)^2. \tag{3.16}$$

is valid provided $0 < |t| < \pi/2$. *A similar inequality*

$$2\frac{t}{\sinh t} < 1 + \frac{\sinh t}{t}\Big(\frac{2}{1+\cosh t}\Big)^2. \tag{3.17}$$

also holds true for all $t \neq 0$.

Proof. We shall establish first (33). To this aim let us introduce a mean

$$U = SB(1-u, 1+u),$$

where $0 < |u| < 1$. Letting

$$u = \frac{1-\cos t}{1+\cos t}$$

we have

$$U = SB\Big(\frac{2\cos t}{1+\cos t}, \frac{2}{1+\cos t}\Big) = \frac{2}{1+\cos t} SB(\cos t, 1). \tag{3.18}$$

Let us record an equation

$$U = \frac{2}{1+\cos t}\frac{\sin t}{t}. \tag{3.19}$$

In these computations we have utilized homogeneity of SB and formula (31), as well. We use now the invariance property of the Schwab-Borchardt mean

$$SB(a,b) = SB(A, \sqrt{Ab})$$

(see [**3**]) to obtain

$$U = SB(1-u, 1+u) = SB(1, \sqrt{1+u}) = SB(1, \sqrt{\lambda}),$$

where

$$\lambda = 1 + u = \frac{2}{1+\cos t}. \tag{3.20}$$

Next we shall utilize the inequality

$$(ab^2)^{1/3} < SB(a,b)$$

(see [**28**]) to obtain $\lambda^{1/3} < U$ or what is the same that

$$1 < \frac{U}{\lambda} U^2.$$

Application of the inequality of arithmetic and geometric means yields

$$1 < \frac{1}{2}\left(\frac{U}{\lambda} + U^2\right).$$

This in conjunction with (36) and (37) yields the assertion. Proof of the inequality (34) is very similar to the one presented above. First we define a mean

$$V = SB(1+u, 1-u),$$

where now we let

$$u = \frac{\cosh t - 1}{\cosh t + 1}.$$

From now it suffices to follow the lines used earlier. The proof is completed. □

We close this section with two inequalities involving two circular and two hyperbolic functions.

Theorem 3.5. *Let* $f(t)$ *stands either for* $\sin(t)$ $(0 < |t| < \pi/2)$ *or for* $\sinh(t)$ $(0 < |t| < \sinh^{-1}(1))$. *Then*

$$\frac{f(t)}{t} < \frac{6}{6 + f^2(t)} \tag{3.21}$$

Further let $g(t)$ *stands either for* $\tan(t)$ $(0 < |t| < \pi/4)$ *or for* $\tanh(t)$ $(t \neq 0)$. *Then*

$$\frac{g(t)}{t} < \frac{3}{3 - g^2(t)}. \tag{3.22}$$

Proof. To obtain the desired results we shall utilize some inequalities satisfied by the means Ω_k and Λ_k defined in (17) and (16), respectively. It has been proven in [**24**, (4.4)] that

$$P\Omega_6 < A^2$$

where the first Seiffert mean P is defined in (12) and $\Omega_6 = A\big(1 + \frac{1}{6}v^2\big)$ (see (17)). This implies the inequality

$$\frac{v}{\sin^{-1} v}\big(1 + \frac{1}{6}v^2\big) < 1.$$

Letting $v = \sin t$ $(0 < |t| < \pi/2)$ we obtain inequality (39) for $f(t) = \sin t$. In order to establish inequality (39) for $f(t) = \sinh(t)$ we utilize the following one

$$M\Lambda_6 < A^2,$$

(see [**24**, (4.4)]), where the formula for M is included in (13). Making use of (16) we get $\Lambda_6 = A\big(1 + \frac{1}{6}v^2\big)$. This yields

$$\frac{v}{\sinh^{-1} v}\big(1 + \frac{1}{6}v^2\big) < 1.$$

A substitution $v = \sinh t$, where

$$0 < |t| < \sinh^{-1}(1) \tag{3.23}$$

gives the asserted result. Let us note that the two-sided inequality (41) follows from the simultaneous inequality $0 < |v| < 1$. We shall prove now the second assertion. To this aim we employ the inequality

$$T\Lambda_3 < A^2,$$

(see [**24**, (4.2)]), where the formula for T is included in (12). Making use of (16) with $k = 3$ yields $\Lambda_3 = A\bigl(1 - \frac{1}{3}v^2\bigr)$. Hence

$$\frac{v}{\tan^{-1} v}\bigl(1 - \frac{1}{3}v^2\bigr) < 1.$$

Letting $v = \tan t$ and taking into account that $0 < |v| < 1$ we obtain the desired inequality when $g(t) = \tan t$. To complete the proof of the inequality (39) with $g(t) = \tanh t$ we use the inequality $L\Omega_3 < A^2$ (see [**24**, (4.1)]) and follow the lines already used above. We omit further details. □

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