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STRONG CONVERGENCE RESULTS FOR NEARLY WEAK UNIFORMLY L-LIPSCHITZIAN MAPPINGS

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Dedicated to my dearest great teacher Professor J. A. Adepoju on his 70th Birthday

Abstract. In this paper, we prove some strong convergence theorems of the modified Mann iteration with errors for a new class of nonlinear mappings in real Banach spaces. Our results not only employ a simple proof technique, but also extend some well known results in this area of research.

1. Introduction and Preliminaries

We denote by J the normalized duality mapping from X into 2^{X^*} by

$$J(x) = \{f \in \mathrm{X}^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\},$$

for all $x \in X$, where X^* denotes the dual space of real Banach space X and $\langle ., .\rangle$ denotes the generalized duality pairing between elements of X and X^*. The normalized duality mapping J has the following properties:
(i) J is an odd mapping, i.e., $J(-x) = -J(x)$.
(ii) J is positive homogenous, i.e., for any $\lambda > 0$, $J(\lambda x) = \lambda J(x)$.
(iii) J is bounded, i.e., for any bounded subset A of X, $J(A)$ is a bounded ubset of X^*.
(iv) If X is smooth (or X^* is strictly convex), then J is single-valued.

In the sequel, we denote the single-valued normalized duality mapping by j. In the Hilbert space H, j is the identity mapping.

Goebel- Kirk [5] and Schu [17] introduced the asymptotically nonexpansive and asymptotically pseudocontractive mappings respectively.

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Let K be a nonempty subset of real Banach space X.

Definition 1.1. A mapping T is called asymptotically nonexpansive if for each $x, y \in K$

$$\|T^n x - T^n y\| \leqslant k\|x-y\| \leqslant k_n\|x-y\|^2, \forall n \geqslant 1,$$

where $(k_n) \subset [1,\infty)$ with $lim_{n\to\infty}k_n = 1$.

Definition 1.2. A mapping T is called asymptotically pseudocontractive with the sequence$(k_n) \subset [1,\infty)$ if and only if $lim_{n\to\infty}k_n = 1$, and for all $n \in N$ and all $x, y \in K$, there exists $j(x-y) \in J(x-y)$ such that

$$< T^n x - T^n y, j(x-y) > \leqslant k_n\|x-y\|^2, \forall n \geqslant 1.$$

Definition 1.3. A mapping T is called uniformly $L-$ Lipschitzian if, for any $x, y \in K$, there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leqslant L\|x-y\|, \forall n \geqslant 1.$$

It is easy to see that every asymptotically nonexpansive mapping is asymptotically pseudocontractive. However, the converse is not true in general. Therefore, it is of interest to study these mappings in the theory of fixed point and its applications.

In recent years, some authors have given much attention to iterative methods for approximating fixed points of Lipschitz asymptotically type of some nonlinear mappings (see [1-5, 7, 8, 10-19]).

In [1], Chang extended the results of Schu [17] to a real uniformly smooth Banach space and proved the following theorem:

Theorem 1.1. ([1]). *Let E be a real uniformly smooth Banach space, K be a nonempty bounded closed convex subset of E,* $T : K \to K$ *be an asymptotically pseudocontractive mapping with a sequence* $k_n \subset [1,\infty)$ *with* $k_n \to 1$ *and* $F(T) \neq \emptyset$, *where* $F(T)$ *is the set of fixed points of* T *in K. Let* $\{\alpha_n\}_{n=0}^{\infty}$ *be a sequence in* $[0,1]$ *satisfying the following conditions:*
(i) $\lim_{n\to\infty} \alpha_n = 0$
(ii)$\sum_{n=0}^{\infty} \alpha_n = \infty$.
For any $x_0 \in K$, *let* $\{x_n\}_{n=0}^{\infty}$ *be the iterative sequence defined by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n, \quad n \geqslant 0.$$

If there exists a strictly increasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ such that

$$< T^n x_n - \rho, j(x_n - \rho) > \leqslant k_n\|x_n - \rho\|^2 - \Phi(\|x_n - \rho\|), \quad n \geqslant 0 \tag{1.1}$$

where $\rho \in F(T)$ is some fixed point of T in K, then $x_n \to \rho$ as $n \to \infty$.

The iteration process of Theorem 1.1 is a modification of the well-known Mann iteration process (see, e.g., [9]).

Remark 1.1. Theorem 1.1, as stated is a modification of Theorem 2.1 of Chang [1] who actually included error terms in his iteration process.

Ofoedu [14] used the modified Mann iteration process (1.1) introduced by Schu [17] to obtain a strong convergence theorem for uniformly Lipschitzian asymptotically pseudocontractive mapping in real Banach space setting. He proved the following theorem:

THEOREM 1.2. ([14]). *Let E be a real Banach space, K be a nonempty closed convex subset of E, $T : K \to K$, be a uniformly L-Lipschitzian asymptotically mappings with a sequence $k_n \subset [1,\infty)$, $k_n \to 1$ such that $\rho \in F(T)$, where $F(T)$ is the set of fixed points of T in K. Let $\{\alpha_n\}_{n=0}^{\infty}$ be a sequence in $[0,1]$ satisfying the following conditions:*
(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$
(ii) $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$
(iii) $\sum_{n=0}^{\infty} \beta_n < \infty$
(iv) $\sum_{n=0}^{\infty} \alpha_n(k_n - 1) < \infty$.
For any $x_0 \in K$, let $\{x_n\}_{n=0}^{\infty}$ be the iterative sequence defined by (1.1).

If there exists a strictly increasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ such that

$$< T^n x_n - \rho, j(x_n - \rho) > \leqslant k_n\|x_n - \rho\|^2 - \Phi(\|x_n - \rho\|)$$

for all $x \in K$, then $\{x_n\}_{n=0}^{\infty}$ converges strongly to ρ.

Obviously, this result extends Theorem 1.2 of Chang [1] from a real uniformly smooth Banach space to an arbitrary real Banach space and removes the boundedness condition imposed on K.
It is important to note the following remark:

REMARK 1.2. (Remark 2, p. 567, of Rafiq [15]). One can see that, with $\sum_{n=0}^{\infty} \alpha_n = \infty$ the conditions $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$ and $\sum_{n=0}^{\infty} \alpha_n(k_n - 1) < \infty$ are not always true. Let us take $\alpha_n = \frac{1}{\sqrt{n}}$ and $k_n = 1 + \frac{1}{\sqrt{n}}$, then obviously $\sum_{n=0}^{\infty} \alpha_n = \infty$, but $\sum_{n=0}^{\infty} \alpha_n^2 = \infty = \sum_{n=0}^{\infty} \alpha_n(k_n - 1)$. Hence the results of Ofoedu [14] and Chang et al. [3] need to be improved.

Sahu [16] recently introduced the following new class of nonlinear map which is more general than the class of uniformly L- Lipschitzian mappings.

Let K be a subset of a normed space X and let $\{a'_n\}_{n\geqslant 0}$ be a sequence in $[0,\infty)$ such that $lim_{n\to\infty} a'_n = 0$.

A mapping $T : K \to K$ is called nearly Lipschitzian with respect to $\{a'_n\}$ if for each $n \in N$, there exists a constant $k_n \geqslant 0$ such that

$$\|T^n x - T^n y\| \leqslant k_n(\|x - y\| + a'_n), \quad \forall \quad x, y \in K. \tag{1.2}$$

Define

$$\mu(T^n) = sup\{\frac{||T^n x - T^n y||}{||x - y|| + a'_n} : x, y \in K, x \neq y\}.$$

Observe that for any sequence $\{k_n\}_n \geqslant 1$ satisfying (1.1) $\mu(T^n) \leqslant k_n$ $\forall n \in N$ and that

$$\|T^n x - T^n y\| \leqslant \mu(T^n)(\|x - y\| + a'_n), \quad \forall \quad x, y \in K$$

$\mu(T^n)$ is called the nearly Lipschitz constant of the mapping T. A nearly Lipschitzian mapping T is said to be
(i) nearly contraction if $\mu(T^n) < 1$ for all $n \in N$;
(ii) nearly nonexpansive if $\mu(T^n) = 1$ for all $n \in N$;
(iii) nearly asymptotically nonexpansive if $\mu(T^n) \geqslant 1$ for all $n \in N$ and $\lim_{n\to\infty} \mu(T^n) = 1$;
(iv) nearly uniformly $L-$ Lipschitzian if $\mu(T^n) \leqslant L$ for all $n \in N$;
(v) nearly uniformly $k-$ contraction if $\mu(T^n) \leqslant k < 1$ for all $n \in N$.
A nearly Lipschitzian mapping T with sequence $\{a'_n\}$ is said to be nearly uniformly $L-$ Lipschitzian if $k_n = L$ for all $n \in N$.

Observe that the class of nearly uniformly $L-$ Lipschitzian mapping is more general than the class of uniformly $L-$ Lipschitzian mappings.

EXAMPLE 1.1. (see Sahu[16]) Let $X = R,\ K = [0, 1]$. Define $T : K \to K$ by

$$Tx = \{ \begin{array}{r} 1/2,\ x \in [0, 1/2), \\ 0,\ x \in (1/2, 1]. \end{array}$$

It is obvious that T is not continuous, and thus not Lipschitz. However, T is nearly nonexpansive. In fact, for a real sequence $\{a'_n\}_n \geqslant 1$ with $a'_1 = \frac{1}{2}$ and $a'_n \to 0$ as $n \to \infty$, we have

$$||Tx - Ty|| \leqslant ||x - y|| + a'_1,\ \forall x, y \in K$$

and

$$||T^n x - T^n y|| \leqslant ||x - y|| + a'_n,\ \forall x, y \in K,\ n \geqslant 2.$$

This is because $T^n x = \frac{1}{2}, \forall x \in [0, 1],\ n \geqslant 2$.

REMARK 1.3. The class of nearly uniformly $L-$ Lipschitzian is not necessarily continuous. In recent times, some authors have given much attention to this new class of mappings in Banach spaces (see [10] and references there in).

The aim of this paper is, by using an easy quite different analytical method, to prove some strong convergence theorems for a new class of nonlinear map. Our results include some well known recent results in [1-3, 5, 7, 10-16].

For our main purpose, we recall the following.

DEFINITION 1.4. ([18]) For arbitrary $x_1 \in K$, the sequence $\{x_n\}_{n=1}^{\infty}$ in K defined by

$$x_{n+1} = (1 - a_n - c_n)x_n + a_n T^n x_n + c_n u_n,\ n \geqslant 1, \tag{1.3}$$

where $\{a_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ are sequences in $[0, 1]$ with $a_n + c_n \leqslant 1$ and $\{u_n\}_{n=1}^{\infty}$ is a bounded sequence of K.

We observe that the iteration process (1.3) is well defined and is a generalization of the modified Mann iteration (1.1). This is evident by specialising some of the parameters. Indeeed, when $c_n = 0$ and $\alpha_n = a_n$, then (1.3) reduces to (1.1) which is used by several authors working in this area of research.

LEMMA 1.1. ([1, 4]) *Let X be real Banach Space and $J : X \to 2^{X^*}$ be the normalized duality mapping. Then, for any $x, y \in X$*

$$\|x + y\|^2 \leqslant \|x\|^2 + 2 < y, j(x + y) >, \forall j(x + y) \in J(x + y).$$

LEMMA 1.2. ([12]) *Let $\Phi : [0,\infty) \to [0,\infty)$ be an increasing function with $\Phi(x) = 0 \Leftrightarrow x = 0$ and let $\{b_n\}_{n=0}^{\infty}$ be a positive real sequence satisfying*

$$\sum_{n=0}^{\infty} b_n = +\infty \quad \text{and} \quad \lim_{n\to\infty} b_n = 0.$$

Suppose that $\{a_n\}_{n=0}^{\infty}$ is a nonnegative real sequence. If there exists an integer $N_0 > 0$ satisfying

$$a_{n+1}^2 < a_n^2 + o(b_n) - b_n \Phi(a_{n+1}), \quad \forall n \geqslant N_0$$

where $\lim_{n\to\infty} \frac{o(b_n)}{b_n} = 0$, then $\lim_{n\to\infty} a_n = 0$.

2. Main results

First of all, we give a new concept.

Let K be a subset of a normed space X and let $\{a'_n\}_{n\geqslant 1}$ be a sequence in $[0,\infty)$ such that $lim_{n\to\infty} a'_n = 0$.

DEFINITION 2.1. A mapping $T : K \to K$ is called nearly weak uniformly Lipschitzian with respect to $\{a'_n\}$ if for each $n \in N$, there exists a constant $k_n \geqslant 0$ such that

$$\|T^n x - T^n y\| \leqslant L(\|x - y\| + a'_n), \quad \forall \;\; x \in K, \;\; y \in F(T). \tag{1.2}$$

It is easy to see that if T has a bounded range, then it is nearly weak uniformly Lipschitzian. In fact, since $R(T^n) \subset R(T)$, then

$$sup_{x\in K}\|T^n x\| \leqslant sup_{x\in K}\|T^{n-1}x\| \leqslant \cdots \leqslant sup_{x\in K}\|Tx\| \leqslant x,$$

thus

$$\|T^n x - T^n y\| \leqslant \|Tx - Ty\| \leqslant (\|x - y\|) \leqslant L(\|x - y\| + a'_n),$$

where $x \in K, \;\; y \in F(T)$. On the contrary, it may not be true in general.

In the following, we prove the main result of this paper.

THEOREM 2.1. *Let X be a real Banach space, K be a nonempty closed convex subset of X, $T : K \to K$ be a nearly weak uniformly L-Lipschitzian mapping with sequence $\{a'_n\}$. Let $k_n \subset [1,\infty)$ and ϵ_n be sequences with $\lim_{n\to\infty} k_n = 1$, $\lim_{n\to\infty} \epsilon_n = 0$ and $F(T) = \{\rho \in K : T\rho = \rho\} \neq \emptyset$. Let $\{a_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ be two real sequences in $[0,1]$ satisfying the following conditions:*
(i) $a_n + c_n \leqslant 1$;
(ii) $a_n, \;\; c_n \to 0$ as $n \to \infty$ and $c_n = o(a_n)$;
(iii)$\sum_{n\geqslant 1} a_n = \infty$.
For arbitrary $x_1 \in K$, let $\{x_n\}_{n\geqslant 1}$ be iteratively defined by (1.3). If there exists a strictly increasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ such that

$$< T^n x - T^n \rho, j(x - \rho) > \leqslant k_n \|x - \rho\|^2 - \Phi(\|x - \rho\|) + \epsilon_n$$

for all $n \geqslant 1$, then, $\{x_n\}_{n\geqslant 1}$ converges strongly to ρ of T.

Proof. Since there exists a strictly increasing continuous function

$$\Phi : [0, \infty) \to [0, \infty)$$

with $\Phi(0) = 0$ such that

$$\langle T^n x - T^n \rho, j(x - \rho)\rangle \leqslant k_n \|x - \rho\|^2 - \Phi(\|x - \rho\|) + \epsilon_n, \tag{2.1}$$

for $x \in K,\ \rho \in F(T)$, that is

$$\epsilon_n + \langle k_n(x - \rho) - (T^n x - \rho), j(x - \rho)\rangle \geqslant \Phi(\|x - \rho\|). \tag{2.2}$$

To ensure that $\Phi^{-1}(r_0)$ is well defined, choose some $x_1 \in K$ and $x_1 \neq Tx_1$ such that $r_0 = \epsilon_n + (k_n + L)\|x_1 - \rho\|^2 + L\|x_1 - \rho\|^2$, where $R(\Phi)$ is the range of Φ. Indeed, if $\Phi(r) \to +\infty$ as $r \to \infty$, then $r_0 \in R(\Phi)$; if $\sup\{\Phi(r) : r \in [0, \infty]\} = r_1 < +\infty$ with $r_1 < r_0$, then $\rho \in K$, there exists a sequence $\{\eta_n\}$ in K such that $\eta_n \to \rho$ as $n \to \infty$ with $\eta_n \neq \rho$. Clearly, $T\eta_n \to T\rho$ as $n \to \infty$ thus $\{\eta_n - T\eta_n\}$ is a bounded sequence. Therefore, there exists a natural number n_0 such that $\epsilon_n + (k_n + L)\|\eta_n - \rho\|^2 + L\|\eta_n - \rho\|^2 < \frac{r_1}{2}$ for $n \geqslant n_1$, and then we redefine $x_1 = \eta_{n_0}$ and $\epsilon_n + (k_n + L)\|x_1 - \rho\|^2 + L\|x_1 - \rho\|^2 \in R(\Phi)$.

Step 1. We first show that $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence.
Set $R = \Phi^{-1}(r_0)$, then from above (2.2), we obtain that $\|x_1 - \rho\| \leqslant R$. Denote

$$B_1 = \{x \in K : \|x - \rho\| \leqslant R\}, \quad B_2 = \{x \in K : \|x - \rho\| \leqslant 2R\},$$

$$M^* = sup_n\{\|u_n - \rho\|\}. \tag{2.3}$$

Now, we want to prove that $x_n \in B_1$. If $n = 1$, then $x_1 \in B_1$. Now assume that it holds for some n, that is, $x_n \in B_1$. Suppose that, it is not the case, then $\|x_{n+1} - \rho\| > R$.
Since $\{a'_n\} \in [0, \infty]$ with $a'_n \to 0$ as $n \to \infty$, set $M = sup_n\{a'_n : n \in N\}$. Denote

$$\tau_0 \quad = \quad min\{1, \tfrac{R}{(L(R+M)+M^*)}, \tfrac{\Phi(R)}{12R^2}, \tfrac{\Phi(R)}{16R(R+M^*)}, \tfrac{\Phi(R)}{16RL[(2+L)R+(2M+M^*)]}, \tfrac{\Phi(R)}{8}\}. \tag{2.4}$$

Since $\lim_{n\to\infty} a_n, c_n = 0$ and $\lim_{n\to\infty} k_n = 1$. Without loss of generality, let $0 \leqslant a_n, c_n, k_n - 1, \epsilon_n \leqslant \tau_0,\ c_n < a_n\tau_0$ for any $n \geqslant 1$. Thus, we get

$$\begin{aligned}
\|x_{n+1} - \rho\| &\leqslant (1 - a_n - c_n)\|x_n - \rho\| + a_n\|T^n x_n - T^n \rho\| + c_n\|u_n - \rho\| \\
&\leqslant R + a_n L(R + M) + cM^* \\
&\leqslant R + \tau_0(L(R + M) + M^*) \\
&\leqslant 2R,
\end{aligned}$$

and

$$\begin{array}{rcl}
\|T^n x_{n+1} - T^n x_n\| & \leqslant & L(\|x_{n+1} - x_n\| + a'_n) \\
& \leqslant & a_n L\|T^n x_n - x_n\| + c_n L\|u_n - x_n\| + a'_n L) \\
& \leqslant & a_n L(\|x_n - \rho\| + \|T^n x_n - T^n \rho\|) \\
& & +c_n L(\|u_n - \rho\| + \|x_n - \rho\|) + a'_n L) \\
& \leqslant & a_n L(R + L(R+M)) + c_n L(M^* + R) + a'_n L) \\
& \leqslant & \tau_0 L[((1+L)R + M) + (M^* + R) + M] \\
& = & \tau_0 L[(2+L)R + (2M + M^*)] \\
& \leqslant & \frac{\Phi(R)}{16R}.
\end{array} \tag{2.4}$$

Using Lemma 1.1 and the above estimates, we have

$$\begin{array}{rcl}
\|x_{n+1} - \rho\|^2 & \leqslant & (1-a_n)^2\|x_n - \rho\|^2 + 2a_n < T^n x_n - x_n, j(x_{n+1} - \rho) > \\
& & +2c_n < u_n - x_n, j(x_{n+1} - \rho) > \\
& = & (1-a_n)^2\|x_n - \rho\|^2 + 2a_n < T^n x_{n+1} - x_{n+1}, j(x_{n+1} - \rho) > \\
& & +2c_n < u_n - x_n, j(x_{n+1} - \rho) > \\
& & +2a_n < T^n x_n - T^n x_{n+1}, j(x_{n+1} - \rho) > \\
& \leqslant & (1-a_n)^2\|x_n - \rho\|^2 \\
& & +2a_n(k_n\|x_{n+1} - \rho\|^2 - \Phi(\|x_{n+1} - \rho\|) + \epsilon_n) \\
& & +2a_n(\|T^n x_{n+1} - T^n x_n\|)\|x_{n+1} - \rho\| \\
& & +2c_n\|u_n - x_n\|\|x_{n+1} - \rho\| \\
& \leqslant & (1-a_n)^2 R^2 + 2a_n(k_n\|x_{n+1} - \rho\|^2 - \Phi(R) + \epsilon_n) \\
& & +\frac{2a_n}{16R}\Phi(R)2R + 2c_n(R + M^*)2R \\
& \leqslant & (1-a_n)^2 R^2 + 2a_n(k_n\|x_{n+1} - \rho\|^2 - \Phi(R) + \epsilon_n) \\
& & +\frac{2a_n}{16R}\Phi(R)2R + 2a_n\tau_0(R + M^*)2R \\
& \leqslant & R^2 + 2a_n[(k_n - 1) + \frac{a_n}{2}]R^2 - 2a_n\Phi(R) + 2a_n\epsilon_n \\
& & +\frac{a_n}{4R}\Phi(R) + 2a_n\tau_0(R + M^*)2R \\
& \leqslant & R^2 + 2a_n[\tau_0 + \frac{\tau_0}{2}]R^2 - 2a_n\Phi(R) + 2a_n\tau_0 \\
& & +\frac{a_n}{4R}\Phi(R) + 2a_n\tau_0(R + M^*)2R \\
& = & R^2 + 3a_n\tau_0 R^2 - 2a_n\Phi(R) + 2a_n\tau_0 \\
& & +\frac{a_n}{4R}\Phi(R) + 2a_n\tau_0(R + M^*)2R \\
& \leqslant & R^2.
\end{array} \tag{2.5}$$

which is a contradiction. Hence $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence.

Step 2. We want to prove that $\|x_n - \rho\| \to 0$ as $n \to \infty$. Let

$$M_o = sup_n\{\|x_n - \rho\|\} + sup_n\{\|u_n - \rho\|\}.$$

Observe that

$$\begin{aligned} \|x_{n+1}-x_n\| &\leqslant a_nL\|T^nx_n-x_n\|+c_nL\|u_n-x_n\|+a_n'L) \\ &\leqslant a_nL(\|x_n-\rho\|+\|T^nx_n-T^n\rho\|) \\ &\quad +c_nL(\|u_n-\rho\|+\|x_n-\rho\|)) \\ &\leqslant a_nL(\|x_n-\rho\|+L(\|x_n-\rho\|+a_n') \\ &\quad +c_nL(\|u_n-\rho\|+\|x_n-\rho\|)) \\ &\leqslant a_nL((1+L)M_o+M))+2M_oc_nL. \end{aligned} \tag{2.6}$$

Employing Lemma 1.2, (2.5) and (2.6), we have

$$\begin{aligned} \|x_{n+1}-\rho\|^2 &\leqslant (1-a_n)^2\|x_n-\rho\|^2+2a_n<T^nx_n-x_n,j(x_{n+1}-\rho)> \\ &\quad +2c_n<u_n-x_n,j(x_{n+1}-\rho)> \\ &= (1-a_n)^2\|x_n-\rho\|^2+2a_n<T^nx_{n+1}-x_{n+1},j(x_{n+1}-\rho)> \\ &\quad +2c_n<u_n-x_n,j(x_{n+1}-\rho)> \\ &\quad +2a_n<T^nx_n-T^nx_{n+1},j(x_{n+1}-\rho)> \\ &\leqslant (1-a_n)^2\|x_n-\rho\|^2 \\ &\quad +2a_n(k_n\|x_{n+1}-\rho\|^2-\Phi(\|x_{n+1}-\rho\|)+\epsilon_n) \\ &\quad +2a_n(\|T^nx_{n+1}-T^nx_n\|)\|x_{n+1}-\rho\| \\ &\quad +2c_n\|u_n-x_n\|\|x_{n+1}-\rho\| \\ &\leqslant \|x_n-\rho\|^2 \\ &\quad +2a_n(k_n-1)M_o^2+a_n^2M_o^2-a_n\Phi(\|x_{n+1}-\rho\|)+2a_n\epsilon_n) \\ &\quad +2a_nL(\|x_{n+1}-x_n\|+a_n')M_o \\ &\quad +2c_nM_o^2 \\ &\leqslant \|x_n-\rho\|^2-a_n\Phi(\|x_{n+1}-\rho\|)+Q_n \end{aligned} \tag{2.7}$$

where

$$Q_n=2a_n(k_n-1)M_o^2+a_n^2M_o^2+2a_n\epsilon_n)+2a_nL(\|x_{n+1}-x_n\|+a_n')M_o+2c_nM_o^2.$$

By Lemma 1.2, we obtain that

$$\lim_{n\to\infty}\|x_n-\rho\|=0,$$

i.e., $x_n\to\rho$ as $n\to\infty$. This completes the proof. □

We have the following corollary from Theorem 2.1.

Corollary 2.1. *Let X be a real Banach space, K be a nonempty closed convex subset of X, $T:K\to K$ be a nearly weak uniformly L-Lipschitzian mapping with sequence $\{a_n'\}$. Let $k_n\subset[1,\infty)$ and ϵ_n be sequences with $\lim_{n\to\infty}k_n=1$, $\lim_{n\to\infty}\epsilon_n=0$ and $F(T)=\{\rho\in K:T\rho=\rho\}\neq\emptyset$. Let $\{\alpha_n\}_{n\geqslant 1}$ be a real sequence in $[0,1]$ satisfying the following conditions: (i) $\lim_{n\to\infty}\alpha_n=0$; (ii)$\sum_{n\geqslant 1}\alpha_n=\infty$. For arbitrary $x_1\in K$, let $\{x_n\}_{n\geqslant 1}$ be iteratively defined by*

(1.1). If there exists a strictly increasing function $\Phi : [0, \infty) \to [0, \infty)$ *with* $\Phi(0) = 0$ *such that*

$$< T^n x - T^n \rho, j(x - \rho) >\leqslant k_n \|x - \rho\|^2 - \Phi(\|x - \rho\|) + \epsilon_n$$

for all $n \geqslant 1$*, then,* $\{x_n\}_{n \geqslant 1}$ *converges strongly to* ρ *of* T*.*

Now, we give an example to illustrate the validity of our Theorem 2.1.

Example 2.1. Let $X = R$ and $K = [0, \infty)$. Define a mapping $T : K \to K$ by

$$Tx = \frac{x^3}{1 + x^2}, \forall x \in K$$

Clearly, T is nearly weak uniformly Lipschitzian with $\rho = 0 \in K$ and $a'_n = 0$ for all n.
Define $\Phi : [0, \infty) \to [0, \infty)$ by

$$\Phi(t) = \frac{t^2}{1 + nt^2}$$

then, Φ is a strictly increasing function with $\Phi(0) = 0$. For all $x \in K, \rho \in F(T)$, we have that operator T in Theorem 2.1 satisfies

$$< T^n x - T^n \rho, j(x - \rho) >\leqslant k_n \|x - \rho\|^2 - \Phi(\|x - \rho\|) + \epsilon_n$$

with the sequences $k_n = 1$ and $\epsilon_n = \frac{x^2}{1+nx^2}$.

Remark 2.1. Theorem 2.2 remains true for the so-called modified Ishikawa-type iteration scheme. This is a modification of the scheme introduced by Ishikawa in [6]. There is no further generality obtained in using the cumbersome-Ishikawa iteration process, rather than the iteration process considered in this paper.

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References

[1] Chang, S. S. *Some results for asymptotically pseudocontractive mappings and asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc., **129**(3)(2001), 845-853.

[2] Chang, S. S., Cho, Y. J., Lee, B. S. and Kang, S. H. *Iterative approximation of fixed points and solutions for strongly accretive and strongly pseudo-contractive mappings in Banach spaces* , J. Math. Anal.Appl., **224**(1)(1998), 149-165.

[3] Chang, S. S., Cho, Y. J. and Kim, J. K. *Some results for uniformly L-Lipschitzian mappings in Banach spaces* , Appl.Math.Lett., **22**(1)(2009), 121-125.

[4] Chidume, C. E. and Chidume, C. O. *Convergence theorems for fixed points of uniformly continuous generalized* Φ *hemicontractive mappings*, J.Math.Anal.Appl., **303**(2)(2005), 545-554.

[5] Goebel, K. and Kirk, W. A. *A fixed point theorem for asymptotically nonexpansive mappings* , Proc.Amer.Math.Soc., **35**(1)(1972), 171-174.

[6] Ishikawa, S. *Fixed points by a new iteration method*, Proc.Amer.Math.Soc., **44**(1)(1974), 147-150.

[7] Kim, J. K., Sahu, D. R. and Nam, Y. M. *Convergence theorem for fixed points of nearly uniformly* $L-$ *Lipschitzian asymptotically generalized* $\Phi-$ *hemicontractive mappings*, Nonlinear Analysis: Theory, Methods and Applications, **71**(12)(2009), e2833-e2838.

[8] Liu, L. S. *Ishikawa and Mann iterative process with errors for nonlinear strongly accretive mapping in Banach spces*, J.Math.Anal.Appl., **194**(1)(1995), 114-125.

[9] Mann, W. R. *Mean value methods in iteration*, Proc.Amer.Math.Soc., **4**(1953), 506- 510.

[10] Mogbademu. A. A. *A convergence theorem for Multistep iterative scheme for nonlinear maps*, Publ.Inst.Math., Nouvelle serie, **98**(112)(2015), 281-285.

[11] Mogbademu. A. A. *Strong convergence results for nonlinear mappings in Banach spaces*, Creative Mathematics and Informatics, **25**(1)(2016), 85-92.

[12] Moore, C. and Nnoli, B. V. C. *Iterative solution of nonlinear equations involving set-valued uniformly accretive operators* , Comput. Math. Anal. Appl., **42**(1-2)(2001), 131-140 .

[13] Noor, M. A., Kassias, T. M. and Huang, Z. *Three-step iterations for nonlinear accretive operator equations* , J. Math. Anal. Appl., **274**(1)(2002), 59-68.

[14] Ofoedu, E. U. *Strong convergence theorem for uniformly L-Lipschitzian asymptotically pseudocontractive mapping in real Banach space* , J. Math. Anal. Appl., **321**(2)(2006), 722-728.

[15] Rafiq, A. *Fixed point iterations of three asymptotically pseudocontractive mappings*, Demonstratio Math., Vol. XLVI, **3**(2013), 563- 573.

[16] Sahu, D. R. *Fixed points of demicontinuous nearly Lipschitzian mappings in Banach spaces* , Comment. Math. Univ. Carolin, **46**(4)(2005), 653-666.

[17] Schu, J. *Iterative construction of fixed points of asymptotically nonexpansive mappings* , J. Math. Anal. Appl., **158**(2)(1991), 407-413.

[18] Xu, Y. G. *Ishikawa and Mann iterative process with errors for nonlinear strongly accretive operator equations*, J. Math. Anal. Appl., **224** (1)(1998), 91-101.

[19] Xue, Z., Rafiq, A. and Zhou, H. *On the convergence of multistep iteration for uniformly continuous* Φ*-hemicontractive mappings*, Abstr. Appl. Anal., **224**(2012), Article ID 386983, 9pp.

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