BULLETIN OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN 2303-4874 (p), ISSN (o) 2303-4955
www.imvibl.org / JOURNALS / BULLETIN
Vol. 6(2016), 189-198

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

COMMON COUPLED FIXED POINT THEOREMS SATISFYING $(CLRg)$ PROPERTY IN COMPLEX VALUED b - METRIC SPACES

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Abstract. In this paper we obtain a common coupled fixed point theorems for the two pair of mappings satisfying CLRg property in complex valued b-metric spaces. In this theorem we have not used continuity of any mapping, completeness of the whole space or range space of any mapping and one range set contained in the other range set of mappings. In the second theorem we obtain a common coupled fixed point theorem for the two pair of mappings in which one pair of maps satisfying CLRg property one range set contained in the other range set of one pair of maps. In this theorem also continuity of any mapping, completeness of the whole space or range space of any mapping are not necessary .

1. Introduction and Preliminaries

Banach contraction principle in [**4**] was the starting point for many researchers during last decades in the field of non linear analysis.In 1989, Bakthin ([**5**]) introduced the concept of b - metric space as a generalization of metric spaces. The concept of complex valued b - metric space was introduced in 2013 by Rao et al. ([**7**]), which was more general than the well-known complex valued metric spaces that were introduced in 2011 by Azam et.al ([**3**]).

An ordinary metric d is a real -valued function from a set $X \times X \to R$, where X is a non empty set. That is $d : X \times X \to R$. A Complex number $z \in C$ is an ordered pair of real numbers, whose first co-ordinate is called $Re(z)$ and second co-ordinate is called $Im(z)$. Thus a complex-valued metric d is a function from a set $X \times X$ into C,where X is non empty set and C is the set of complex number.That

2010 *Mathematics Subject Classification.* 47H10; 54E50; 54H25.

Key words and phrases. complex valued metric space, (S, f) and (T, g) are w-compatible, $(CLRg)$ property, common fixed points.

is $d : X \times X \to C$. Let $z_1, z_2 \in C$. Define a partial order $\preceq$ on C follows:

$$z_1 \preceq z_2 \text{ if and only if } Re(z_1) \leqslant Re(z_2), Im(z_1) \leqslant Im(z_2).$$

Thus $z_1 \preceq z_2$if one of the following holds:

(1) $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$,
(2) $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$,
(3) $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$,
(4) $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.

We will write $z_1 \precsim z_2$ if $z_1 \neq z_2$ and one of (2), (3) and (4) is satisfed; also we will write $z_1 \prec z_2$ if only (4) is satisfed.

We use the following definition.

Definition 1.1. If $z_1 = a + ib$, $z_2 = \alpha + i\beta$ then
$max\{z_1, z_2\} = max\{a,\ \alpha\} + i\ max\{b,\ \beta\}$

Definition 1.2. ([**7**]) Let X be a non empty set. A function $d : X \times X \to \mathbf{C}$ is called a complex valued metric on X if for all $x, y, z \in X$ the following conditions are satisfied:

(i) $0 \preceq d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, y) \preceq d(x, z) + d(z, y)$.
The pair $(X, d)$is called a complex valued metric space.

Example 1.1. (See [**6**]) Let $X = \mathbf{C}$. Define the mapping $d : X \times X \to \mathbf{C}$ by $d(x, y) = i\,|x - y|\,, \forall x, y \in X$. Then (X, d) is a complex valued metric space.

Definition 1.3. ([**7**]) Let X be a non empty set and let $s \geqslant 1$ be a given real number. A function $d : X \times X \to \mathbf{C}$ is called a complex valued b - metric on X if for all $x, y, z \in X$ the following conditions are satisfied:

(i) $0 \preceq d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, y) \preceq s[d(x, z) + d(z, y)]$.
The pair (X, d) is called a complex valued b - metric space.

Example 1.2. ([**7**]) Let $X = [0, 1]$. Define the mapping $d : X \times X \to \mathbf{C}$ by $d(x, y) =|x - y|^2 + i|x - y|^2, \forall x, y \in X$. Then (X, d) is called a complex valued b - metric space with $s = 2$.

Definition 1.4. ([**7**]) Let (X, d) be a complex valued b - metric space. Consider the following.

(i) A point $x \in X$ is called interior point of a set $A \subseteq X$ whenever there exists $0 \prec r \in \mathbf{C}$ such that $B(x, r) = \{y \in X : d(x, y) \prec r\} \subseteq A$.
(ii) A point $x \in X$ is called limit point of a set A whenever,
for every $0 \prec r \in \mathbf{C}, B(x, r) \cap (A - X) \neq 0$.
(iii) A sub set $A \subseteq X$ is called open whenever each element of A is an interior point of A.

(iv) A sub set $A \subseteq X$ is called closed whenever each limit point of A is belongs to A.

(v) A subbasis for a Hausdroff topology τ on X is a family $F = \{B(x, r) : x \in X \text{ and } 0 \prec r\}$.

Definition 1.5. ([**7**]) Let (X, d) be a complex valued b - metric space and $\{x_n\}$ a sequence in $X$and $x \in X$. Consider the following.

(i) If for every $c \in \mathbf{C}$, with $0 \prec c$, there is $N \in \mathbf{N}$ such that, for all $n > \mathbf{N}, d(x_n, x) \prec c$, then $\{x_n\}$ is said to be convergent ,$\{x_n\}$ converge to x, and x is the limit point of $\{x_n\}$. We denote this by $\lim\limits_{n\to\infty} \{x_n\} = x$ or $\{x_n\} \to x$ as $n \to \infty$.

(ii) If for every $c \in \mathbf{C}$, with $0 \prec r$, there is $N \in \mathbf{N}$ such that, for all $n > \mathbf{N}, d(x_n, x_{n+m}) \prec c$, where $m \in \mathbf{N}$, there $\{x_n\}$ is said to be Cauchy sequence.

(iii) If every Cauchy sequence in X is convergent , then (X, d) is said to be a complete complex valued b-metric space.

One can easily prove the following Lemmas.

Lemma 1.1. ([**7**]) *Let (X, d) be a complex valued b - metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n, y)| \to s\,|d(x, y)|$ as $n \to \infty$.*

Lemma 1.2. ([**7**]) *Let (X, d) be a complex valued b - metric space and let $\{x_n\}$, $\{y_n\}$ be sequences in X.Then$\{x_n\}$,$\{y_n\}$ converges to x, y respectively if and only if $|d(x_n, y_n)| \to s^2\,|d(x, y)|$ as $n \to \infty$.*

Definition 1.6. ([**1**]) The mappings $S : X \times X \to X$ and $f : X \to X$ are called w-compatible if $f(S(x, y)) = S(fx, fy)$ whenever $f(x) = S(x, y)$ and $f(y) = S(y, x)$.

Definition 1.7. ([**2**]) Let (X, d) be a complex valued b - metric space .Four maps $S, T : X \times X \to X$ and $f, g : X \to X$ the pairs (S, f) and (T,g) are said to satisfy (CLRg) property, if there exist sequences $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{v_n\}$ in X such that

$$\lim_{n\to\infty} S(x_n, y_n) = \lim_{n\to\infty} fx_n = fa,\ \lim_{n\to\infty} S(y_n, x_n) = \lim_{n\to\infty} fy_n = fb,$$
$$\lim_{n\to\infty} T(u_n, v_n) = \lim_{n\to\infty} gu_n = ga,\ \lim_{n\to\infty} T(v_n, u_n) = \lim_{n\to\infty} gv_n = gb$$

for some $a, b \in X$.

2. Main Results

Theorem 2.1. *Let (X, d) be a complex valued b - metric space with the coefficient $s \geqslant 1$, $\lambda s^4 < 1$ and let*

$$S : X \times X \to X,\ T : X \times X \to X,\ f : X \to X,\ g : X \to X$$

be mappings satisfying

(2.1.1) *the pairs (S, f) and (T, g) satisfy (CLRg) property w.r.t f and g respectively,* (2.1.2) $d\left(S(x,y),T(u,v)\right) \preceq \lambda d_{u,v}^{x,y}$,

$$\text{where } d_{u,v}^{x,y} = max\left\{\begin{array}{l} d(fx,gu), d(fy,gv), d(fx,S(x,y)), \\ d(fy,S(y,x)), d(gu,T(u,v)), \\ d(gv,T(v,u)), d(fx,T(u,v)), d(fy,T(v,u)), \\ d(gu,S(x,y)), d(gv,S(y,x)) \end{array}\right\}$$

for all $x,y,u,v \in X,\ \lambda \in (0,1)$,

(2.1.3) (S,f) *and* (T,g) *are w-compatible.*

Then there exists unique $x \in X$ *such that*

$$S(x,x) = T(x,x) = fx = gx = x.$$

Proof. Since the pairs (S,f) and (T,g) satisfy $(CLRg)$ property, there exist sequences $\{x_n\}$, $\{y_n\}$,$\{u_n\}$ and $\{v_n\}$ in X such that

$\lim\limits_{n\to\infty} S(x_n,y_n) = \lim\limits_{n\to\infty} fx_n = fa$, $\lim\limits_{n\to\infty} S(y_n,x_n) = \lim\limits_{n\to\infty} fy_n = fb$,

$\lim\limits_{n\to\infty} T(u_n,v_n) = \lim\limits_{n\to\infty} gu_n = ga$ *and* $\lim\limits_{n\to\infty} T(v_n,u_n) = \lim\limits_{n\to\infty} gv_n = gb$

for some $a,b \in X$.

Suppose $\max\{|d(fa,ga)|,|d(fb,gb)|\} > 0$.

From (2.1.2) we have

$$|d(S(x_n,y_n),T(u_n,v_n))| \leqslant \lambda |d_{u_n,v_n}^{x_n,y_n}| \tag{2.1}$$

where

$$\begin{aligned} |d_{u_n,v_n}^{x_n,y_n}| &= \max\left\{\begin{array}{l} |d(fx_n,gu_n)|, |d(fy_n,gv_n)|, |d(fx_n,S(x_n,y_n))|, \\ |d(fy_n,S(y_n,x_n))|, |d(gu_n,T(u_n,v_n))|, \\ |d(gv_n,T(v_n,u_n))|, |d(fx_n,T(u_n,v_n))|, \\ |d(fy_n,T(v_n,u_n))|, |d(gu_n,S(x_n,y_n))|, \\ |d(gv_n,S(y_n,x_n))| \end{array}\right\}. \\ \lim_{n\to\infty} |d_{u_n,v_n}^{x_n,y_n}| &= \max\left\{\begin{array}{l} s^2|d(fa,ga)|, s^2|d(fb,gb)|, s^2|d(fa,fa)|, \\ s^2|d(fb,fb)|, s^2|d(ga,ga)|, s^2|d(gb,gb)|, \\ s^2|d(fa,ga)|, s^2|d(fb,gb)|, s^2|d(ga,fa)|, \\ s^2|d(gb,fb)| \end{array}\right\} \\ &= \max\left\{\begin{array}{l} s^2|d(fa,ga)|, s^2|d(fb,gb)|, 0,0,0,0, s^2|d(fa,ga)|, \\ s^2|d(fb,gb)|, s^2|d(ga,fa)|, s^2|d(gb,fb)| \end{array}\right\} \\ &= s^2 \max\left\{\ |d(fa,ga)|, |d(fb,gb)|\ \right\}. \end{aligned}$$

Letting $n \to \infty$ in (2.1), we get

$\frac{1}{s^2}|d(fa,ga)| \leqslant \lambda s^2 max\{|d(fa,ga)|,|d(fb,gb)|\}$.

$|d(fa,ga)| \leqslant \lambda s^4 max\{|d(fa,ga)|,|d(fb,gb)|\}$.

Similarly $|d(fb,gb)| \leqslant \lambda s^4 max\{|d(fa,ga)|,|d(fb,gb)|\}$.

Thus we have

$max\{|d(fa,ga)|,|d(fb,gb)|\} \leqslant \lambda s^4 max\{|d(fa,ga)|,|d(fb,gb)|\}$.

Since $\lambda s^4 < 1$, we have $max\{|d(fa,ga)|,|d(fb,gb)|\} = 0$.

Hence

$$fa = ga \text{ and } fb = gb \tag{2.2}$$

Now suppose that $|d(fa,fb)|>0$.
From (2.1.2) we have

$$|d(S(y_n,x_n),T(u_n,v_n))| \leqslant \lambda |d^{y_n,x_n}_{u_n,v_n}| \qquad (2.3)$$

where

$$\begin{aligned}
|d^{y_n,x_n}_{u_n,v_n}| &= max\left\{\begin{array}{l} |d(fy_n,gu_n)|,|d(fx_n,gv_n)|,|d(fy_n,S(y_n,x_n))|,\\ |d(fx_n,S(x_n,y_n))|,|d(gu_n,T(u_n,v_n))|,\\ |d(gv_n,T(v_n,u_n))|,|d(fx_n,T(v_n,u_n))|,\\ |d(fy_n,T(u_n,v_n))|,|d(gu_n,S(y_n,x_n))|,\\ |d(gv_n,S(x_n,y_n))| \end{array}\right\}.\\
\lim_{n\to\infty}|d^{y_n,x_n}_{u_n,v_n}| &= max\left\{\begin{array}{l} s^2|d(fb,ga)|,s^2|d(fa,gb)|,s^2|d(fb,fb)|,\\ s^2|d(fa,fa)|,s^2|d(ga,ga)|,s^2|d(gb,gb)|,\\ s^2|d(fa,gb)|,s^2|d(fb,ga)|,s^2|d(ga,fb)|,\\ s^2|d(gb,fa)| \end{array}\right\}\\
&= max\left\{\begin{array}{l} s^2|d(fb,ga)|,s^2|d(fa,gb)|,0,0,0,0,\\ s^2|d(fa,gb)|,s^2|d(fb,ga)|,s^2|d(ga,fb)|,\\ s^2|d(gb,fa)| \end{array}\right\}\\
&= s^2 max\left\{\ |d(fb,ga)|,|d(fa,gb)|\ \right\}\\
&= s^2|d(fa,fb)|, from (2.2)
\end{aligned}$$

Letting $n\to\infty$ in (2.3), we get

$\frac{1}{s^2}|d(fb,fa)| \leqslant \lambda s^2|d(fa,fb)|$

$|d(fb,fa)| \leqslant \lambda s^4|d(fa,fb)|$.

Since $\lambda s^4<1$, we have $|d(fa,fb)|=0$ so that $fa=fb$.
Thus

$$ga=fa=fb=gb \qquad (2.4)$$

Now we will show that $fa=S(a,b)$ and $fb=S(b,a)$.
Suppose $max\{|d(fa,S(a,b))|,|d(fb,S(b,a))|\}>0$.
From (2.1.2), we have

$$|d(S(a,b),T(u_n,v_n))| \leqslant \lambda |d^{a,b}_{u_n,v_n}| \qquad (2.5)$$

where

$$\begin{aligned}
|d^{a,b}_{u_n,v_n}| &= \max\left\{\begin{array}{c} |d(fa,gu_n)|,|d(fb,gv_n)|,|d(fa,S(a,b))|,\\ |d(fb,S(b,a))|,|d(gu_n,T(u_n,v_n))|,\\ |d(gv_n,T(v_n,u_n))|,|d(fa,T(u_n,v_n))|,\\ |d(fb,T(v_n,u_n))|,|d(gu_n,S(a,b))|,|d(gv_n,S(b,a))| \end{array}\right\}\\
\lim_{n\to\infty}|d^{a,b}_{u_n,v_n}| &= max\left\{\begin{array}{l} s|d(fa,fa)|,s|d(fb,fb)|,|d(fa,S(a,b))|,\\ |d(fb,S(b,a))|,s^2|d(fa,fa)|,s^2|d(fb,fb)|,\\ s|d(fa,fa)|,s|d(fa,S(a,b))|,\\ s|d(fb,S(b,a))|,s|d(fb,fb)| \end{array}\right\}\\
&\leqslant s^2\ max\{|d(fa,S(a,b))|,|d(fb,S(b,a))|\}.
\end{aligned}$$

Letting $n \to \infty$ in (2.5), we get
$\frac{1}{s}|d(S(a,b), fa)| \leqslant \lambda s^2 \ max\{|d(fa, S(a,b))|, |d(fb, S(b,a))|\}$
$|d(S(a,b), fa)| \leqslant \lambda s^3 \ max\{|d(fa, S(a,b))|, |d(fb, S(b,a))|\}.$

$$\leqslant \lambda s^4 \ max\{|d(fa, S(a,b))|, |d(fb, S(b,a))|\}.$$

Similarly $|d(S(b,a), fa)| \leqslant \lambda s^4 \ max\{|d(fa, S(a,b))|, |d(fb, S(b,a))|\}$.
Thus we have

$$max\{|d(fa, S(a,b))|, |d(fb, S(b,a))|\} \leqslant \lambda s^4 max\{|d(fa, S(a,b))|, |d(fb, S(b,a))|\}.$$

Since $\lambda s^4 < 1$, we have $max\{|d(fa, S(a,b))|, |d(fb, S(b,a))|\} = 0$.
so that

$$fa = S(a,b) \ and \ fb = S(b,a) \tag{2.6}$$

Now we will show that $ga = T(a,b)$ and $gb = T(b,a)$
Suppose $max\{|d(ga, T(a,b))|, |d(gb, T(b,a))|\} > 0$.
From (2.1.2) we have

$$|d(S(x_n, y_n), T(a,b))| \leqslant \lambda |d_{a,b}^{x_n,y_n}| \tag{2.7}$$

where

$$\begin{aligned} |d_{a,b}^{x_n,y_n}| &= max \left\{ \begin{array}{l} |d(fx_n, ga)|, |d(fy_n, gb)|, |d(fx_n, S(x_n, y_n))|, \\ |d(fy_n, S(y_n, x_n))|, |d(ga, T(a,b))|, \\ |d(gb, T(b,a))|, |d(fx_n, T(a,b))|, \\ |d(fy_n, T(b,a))|, |d(ga, S(x_n, y_n))|, \\ |d(gb, S(y_n, x_n))| \end{array} \right\}. \\ \lim_{n\to\infty} |d_{a,b}^{x_n,y_n}| &= max \left\{ \begin{array}{l} s|d(fa, fa)|, s|d(fb, fb)|, s^2|d(fa, fa)|, \\ s^2|d(fb, fb)|, |d(ga, T(a,b))|, |d(gb, T(b,a))|, \\ s|d(fa, T(a,b))|, s|d(fb, T(b,a))|, \\ s|d(ga, fa)|, s|d(gb, fb)| \end{array} \right\} \\ &\leqslant s^2 max \left\{ \ |d(ga, T(a,b))|, |d(gb, T(b,a))| \ \right\}. \end{aligned}$$

Letting $n \to \infty$ in (2.7), we get
$\frac{1}{s}|d(ga, T(a,b))| \leqslant \lambda s^2 max\left\{ \ |d(ga, T(a,b))|, |d(gb, T(b,a))| \ \right\}$.
$|d(ga, T(a,b))| \leqslant \lambda s^3 max\left\{ \ |d(ga, T(a,b))|, |d(gb, T(b,a))| \ \right\}$.

$$\leqslant \lambda s^4 max\left\{ \ |d(ga, T(a,b))|, |d(gb, T(b,a))| \ \right\}.$$

Similarly $|d(gb, T(b,a))| \leqslant \lambda s^4 max\left\{ \ |d(ga, T(a,b))|, |d(gb, T(b,a))| \ \right\}$.
Thus we have

$$max\{|d(ga, T(a,b))|, |d(gb, T(b,a))|\} \leqslant \lambda s^4 max\left\{ \ |d(ga, T(a,b))|, |d(gb, T(b,a))| \ \right\}.$$

Since $\lambda s^4 < 1$, we have $max\left\{ \ |d(ga, T(a,b))|, |d(gb, T(b,a))| \ \right\} = 0$.
so that $ga = T(a,b)$ and $gb = T(b,a)$.
Let $x = fa$.Then from (2.4),

$$x = fa = fb = ga = gb \tag{2.8}$$

Since (S, f) and (T, g) are w-compatible,we have

$$fx = ffa = f(S(a, b)) = S(fa, fb) = S(x, x),\ from(2.6), (2.8) \tag{2.9}$$

and

$$gx = gga = g(T(a, b)) = T(ga, gb) = T(x, x),\ from(2.6), (2.8) \tag{2.10}$$

Suppose $|d(fx, x)| > 0$.
From (2.1.2) we have

$$|d(fx, x)| = |d(S(x, x), ga)| = |d(S(x, x), T(x, x))| \leqslant \lambda |d^{x,x}_{a,b}| \tag{2.11}$$

where

$$\begin{aligned} |d^{x,x}_{a,b}| &= max \left\{ \begin{array}{l} |d(fx, ga)|, |d(fx, gb)|, |d(fx, S(x, x))|, \\ |d(fx, S(x, x))|, |d(ga, T(a, b))|, \\ |d(gb, T(b, a))|, |d(fx, T(a, b))|, \\ |d(fx, T(b, a))|, |d(ga, S(x, x))|, \\ |d(gb, S(x, x))| \end{array} \right\} \\ &= max \left\{ \begin{array}{l} |d(fx, x)|, |d(fx, x)|, |d(fx, x)|, \\ |d(fx, x)|, |d(ga, ga)|, |d(gb, gb)|, \\ |d(fx, x)|, |d(fx, x)|, |d(fx, x)|, |d(fx, x)| \end{array} \right\} \\ &= |d(fx, x)|. \end{aligned}$$

Thus $|d(fx, x)| \leqslant \lambda |d(fx, x)|$.

Hence

$$fx = x \tag{2.12}$$

Suppose $|d(gx, x)| > 0$.
From (2.1.2) we have

$$|d(x, gx)| = |d(fa, T(x, x))| = |d(S(a, b), T(x, x))| \leqslant \lambda |d^{a,b}_{x,x}| \tag{2.13}$$

where

$$\begin{aligned} |d^{a,b}_{x,x}| &= max \left\{ \begin{array}{l} |d(fa, gx)|, |d(fb, gx)|, |d(fa, S(a, b))|, \\ |d(fb, S(b, a))|, |d(gx, T(x, x))|, \\ |d(gx, T(x, x))|, |d(fa, T(x, x))|, \\ |d(b, T(x, x))|, |d(gx, S(a, b))|, \\ |d(gx, S(b, a))| \end{array} \right\}. \\ &= max \left\{ \begin{array}{l} |d(fx, x)|, |d(fx, x)|, |d(fx, x)|, \\ |d(fx, x)|, |d(ga, ga)|, |d(gb, gb)|, \\ |d(fx, x)|, |d(fx, x)|, |d(fx, x)|, |d(fx, x)| \end{array} \right\} \\ &= |d(x, gx)|. \end{aligned}$$

Thus $|d(x, gx)| \leqslant \lambda |d(x, gx)|$.

Hence

$$x = gx \tag{2.14}$$

From (2.9),(2.10),(2.12),(2.14), we have $S(x,x) = T(x,x) = fx = gx = x$.
Suppose there exists $y \in X$ such that $S(y,y) = T(y,y) = fy = gy = y$.
Then

$$\begin{aligned} d(x,y) = d(S(x,x),T(y,y)) &\leqslant \lambda d^{x,x}_{y,y} \\ &= \lambda \max \left\{ \begin{array}{l} d(x,y), d(x,y), d(x,x), d(x,x), \\ d(y,y), d(y,y), d(x,y), d(x,y), \\ d(y,x), d(y,x) \end{array} \right\} \\ &= \lambda d(x,y) \end{aligned}$$

which implies that $x = y$.
Thus there exists unique $x \in X$ such that $S(x,x) = T(x,x) = fx = gx = x$. □

THEOREM 2.2. *Let* (X,d) *be a complex valued metric space and let*

$$S : X \times X \to X,\ T : X \times X \to X,\ f : X \to X,\ g : X \to X$$

be mappings satisfying (2.1.2), (2.1.3)

(2.2.1)(a) $S(X \times X) \subseteq g(X)$ *and the pair* (S,f) *satisfies* $(CLRg)$ *property w.r.t* f.

(or)

(2.2.1)(b) $T(X \times X) \subseteq f(X)$ *the pair* (T,g) *satisfies* $(CLRg)$ *property w.r.t* g.

Then there exist $x \in X$ *such that* $S(x,x) = fx = x = gx = T(x,x)$.

PROOF. Suppose (2.2.1)(a) holds.Then there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} S(x_n,y_n) = \lim_{n\to\infty} fx_n = fa. \tag{2.15}$$

and

$$\lim_{n\to\infty} S(y_n,x_n) = \lim_{n\to\infty} fy_n = fb \tag{2.16}$$

Since $S(x_n,y_n) \in S(X \times X) \subseteq g(X)$, there exists $\{u_n\}$ in X such that
$S(x_n,y_n) = gu_n, \forall n$
Since $S(y_n,x_n) \in S(X \times X) \subseteq g(X)$, there exists $\{v_n\}$ in X such that
$S(y_n,x_n) = gv_n, \forall n$.
From (2.15), (2.16), we have

$$\lim_{n\to\infty} gu_n = fa \tag{2.17}$$

and

$$\lim_{n\to\infty} gv_n = fb \tag{2.18}$$

Putting $x = x_n$, $y = y_n$, $u = u_n$, $v = v_n$ and $x = y_n$, $y = x_n$, $u = v_n$, $v = u_n$ in (2.1.2) and letting $n \to \infty$ and using (2.15), (2.16), (2.17), (2.18), we get

$$\lim_{n\to\infty} T(u_n,v_n) = fa \tag{2.19}$$

and

$$\lim_{n\to\infty} T(v_n, u_n) = fb \tag{2.20}$$

Putting $x = x_n$, $y = y_n$, $u = v_n$, $v = u_n$ in (2.1.2) and letting $n \to \infty$ and using (2.15), (2.16), (2.17), (2.18), (2.19) and (2, 20), we get

$$fa = fb \tag{2.21}$$

Putting $x = a$, $y = b$, $u = u_n$, $v = v_n$ and $x = b$, $y = a$, $u = v_n$, $v = u_n$ in (2.1.2) and letting $n \to \infty$ and using (2.17), (2.18), (2.19), (2.20), we get
$fa = S(a, b)$ and $fb = S(b, a)$.

Since $fa = S(a, b) \in S(X \times X) \subseteq g(X)$, there exists $a \in X$ such that

$$fa = ga \tag{2.22}$$

Since $fb = S(b, a) \in S(X \times X) \subseteq g(X)$, there exists $b \in X$ such that

$$fb = gb \tag{2.23}$$

Thus from (2.21), (2.22), (2.23), we have $Ta = Sa = Sb = Tb$.
The rest of the proof follows as in Theorem 2.1.
Similarly the proof follows (2.2.1)(b) holds. □

Now we give an example to illustrate our Theorem 2.1

Example 2.1. Let $X = [0, 1]$. Define the mapping $d : X \times X \to C$ by $d(x, y) = i|x - y|^2,\ x, y \in X$. Clearly (X, d) is a complete complex valued b-metric space with $s = 2$. Consider the mappings $S(x, y) = \frac{x^2+y^2}{24}$, $T(x, y) = \frac{x+y}{48}$, $fx = x^2$ and $gx = \frac{x}{2}$ for all $x, y \in X$. Clearly (S, f) and (T, g) satisfy $CLRg$ property with the sequences $\{x_n\} = \{\frac{1}{n}\}$, $\{y_n\} = \{\frac{1}{\sqrt{n}}\}$. It is also that clearly (S, f) and (T, g) are w-compatible.

Consider
$d(S(x, y), T(u, v)) = i|\frac{x^2+y^2}{24} - \frac{u+v}{48}|^2$
$= \frac{i}{576}|(x^2 + y^2) - \frac{u+v}{2}|^2$
$\leqslant \frac{i}{576}.2[|(x^2 - \frac{u}{2})|^2 + |(y^2 - \frac{v}{2})|^2], since\, |a + b|^2 \leqslant 2(|a|^2 + |b|^2)$
$= \frac{i}{576}.4[\frac{|(x^2-\frac{u}{2})|^2+|(y^2-\frac{v}{2})|^2}{2}]$
$\leqslant \frac{1}{144}[max\{d(fx, gu), d(fy, gv)\}]$
where $\lambda = \frac{1}{144}$ and $s = 2$ then $\lambda s^4 < 1$. Hence $(0, 0)$ is a common coupled fixed point.

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Received by editors 24.04.04.2016; Revised version 13.05.2016;
Available online 16.05.2016.

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