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BI-IDEALS IN Γ-SEMIRINGS

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Abstract. The concept of a bi-ideal in a Γ-semiring is introduced. Properties of bi-ideals in a regular Γ-semiring are studied. Bi-ideals are used to characterize special types of Γ-semirings viz intra-regular Γ-semiring,duo Γ-semiring, simple Γ-semiring , bi-simple Γ-semiring , division Γ-semiring.

1. Introduction

The class of Γ-rings were introduced by Nobusawa ([**8**]) contains not only rings but also ternary rings ([**11**]). As a generalization of a ring , the concept of a semiring was introduced and studied in detail by Vandiver ([**14**]). A Γ-semiring was introduced by Rao ([**9**]) as a generalization of the concepts of ring, Γ-ring and semiring.

Ideals play an important role in algebraic structures. Steinfeld coined the concept of a quasi-ideal in a semigroup ([**12**]) and in a ring ([**13**]). Semiring being a generalization of a ring, Iseki ([**4**]) introduced the notion of a quasi-ideal in a semiring without zero. Shabir, Ali, Batool in [**10**] characterized semiring by using the concept of a quasi-ideal. As Γ-semiring is an extension of a semiring, Chinram ([**1**]) successfully extended the concept of a quasi-ideal to a Γ-semiring. Authors in [**5**] discussed some properties of quasi-ideals and minimal quasi-ideals of a Γ-semiring.

As a generalization of the concept of quasi-ideal in different algebraic systems, bi-ideal is introduced. As per the development of quasi-ideals, the concept of bi-ideals is extended from semigroup to semiring by many authors. The notion of a bi-ideal in a semigroup was first introduced by Good and Hughes ([**3**]) and for rings by Lajos in [**7**]. Bi-ideal in a semigroup is a special case of (m, n) ideal in a semigroup defined by Lajos ([**6**]).Shabir,Ali and Batool gave some properties and characterizations of bi-ideals in a semiring in [**10**].

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Hence it is natural to study the concept of bi-ideals in a Γ-semiring . In this paper the concept of a bi-ideal is defined. Characterizations of bi-ideals in a regular Γ-semiring is the prime part of this paper. Efforts are also taken to characterize special types of Γ-semirings viz. intra-regular Γ-semiring,duo Γ-semiring, simple Γ-semiring, bi-simple Γ-semiring, division Γ-semiring using the concept of a bi-ideal .

2. Preliminaries

In this article we recall some definitions which we need in sequel. For this we follow Dutta and Sardar ([**2**]).

Definition 2.1. Let S and Γ be two additive commutative semigroups. S is a Γ-semiring if there exists a mapping $S \times \Gamma \times S \longrightarrow S$ (images to be denoted by $a\alpha b$; for all $a, b \in S$ and for all $\alpha \in \Gamma$) satisfying the following conditions:
(i) $a\alpha\,(b + c) = (a\alpha b) + (a\alpha c)$
(ii) $(b + c)\,\alpha a = (b\alpha a) + (c\alpha a)$
(iii) $a(\alpha + \beta)c = (a\alpha c) + (a\beta c)$
(iv) $a\alpha\,(b\beta c) = (a\alpha b)\,\beta c$; for all $a, b, c \in S$ and for all $\alpha, \beta \in \Gamma$.

Obviously, every semiring S is a Γ-semiring.

Let S be a semiring and Γ be a commutative semigroup. Define a mapping $S \times \Gamma \times S \longrightarrow S$ by, $a\alpha b = ab$ for all $a, b \in S$ and $\alpha \in \Gamma$. Then S is a Γ-semiring.

Definition 2.2. An element $0 \in S$ is an absorbing zero if

$$0\alpha a = 0 = a\alpha 0, a + 0 = 0 + a = a, \text{ for all } a \in S \text{ and } \alpha \in \Gamma.$$

Throughout the paper S denotes any Γ-semiring unless otherwise stated.

Definition 2.3. A non-empty subset T of S is a sub-Γ-semiring of S if $(T, +)$ is a subsemigroup of $(S, +)$ and $a\alpha b \in T$ for all $a, b \in T$ and $\alpha \in \Gamma$.

Definition 2.4. A non-empty subset T of S is a left (respectively right) ideal of S if T is a subsemigroup of $(S, +)$ and $x\alpha a \in T$ (respectively $a\alpha x \in T$) for all $a \in T$, $x \in S$ and $\alpha \in \Gamma$.

Definition 2.5. If T is both left and right ideal of S, then T is an ideal of S.

If M, N are non-empty subsets of S, then

$$M\Gamma N = \{\textstyle\sum_{i=1}^{n} x_i\alpha_i y_i \mid x_i \in M, \alpha_i \in \Gamma, \mathrm{y_i} \in \mathrm{N}\} .$$

Principle left ideal, right ideal and two sided ideal generated by $a \in S$ is denoted by $(a)_l$,$(a)_r$ and (a) respectively.

Definition 2.6. An element a of a Γ-semiring is a regular if $a \in a\Gamma S\Gamma a$.

If all elements of Γ-semiring S are regular, then S is a regular Γ-semiring.

A quasi-ideal Q of a Γ-semiring S is defined as follows.

Definition 2.7. A non-empty subset Q of S is a quasi-ideal if Q is a subsemigroup of $(S, +)$ and $(S\Gamma Q) \cap (Q\Gamma S) \subseteq Q$.

Obviously, every quasi ideal of S is a sub Γ-semiring of S.

Example 2.1. Let N be the set of natural numbers and $\Gamma = 2N$. Then N is a Γ-semiring with respect to usual addition and $a\alpha b$ = usual product of a, α, b for $a, b \in N, \alpha \in \Gamma$. Then $Q = 3N$ is a quasi-ideal of a Γ-semiring N. $\diamond$

3. Bi-ideals

In this article we define a bi-ideal, a minimal bi-ideal in a Γ-semiring and study some of their properties.

Definition 3.1. A non-empty subset B of Γ-semiring S is said to be a bi-ideal of S if B is a sub-Γ-semiring of S and $B\Gamma S\Gamma B \subseteq B$.

Clearly every bi-ideal is a sub-Γ-semiring but not conversely. For this consider the following example.

Example 3.1. Consider the semiring $S = M_{2\times 2}(N_0)$, where N denotes the set of all natural numbers and $N_0 = N \cup \{0\}$. If $\Gamma = S$, then S forms a Γ-semiring with $A\alpha B$ = usual matrix product of A, α, B; for all $A, \alpha, B \in S$. $B = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \mid x\ ,\ y \in N_0 \right\}$ is a sub-Γ-semiring of S. But B is not a bi-ideal of S. $\diamond$

In the following theorem we mention some properties of bi-ideals in S. The proof being straight forward so omitted.

Theorem 3.1. *In S the following statements hold*
(1) Any one sided (two sided) ideal of S is a bi-ideal of S.
(2) Intersection of a right ideal and a left ideal of S is a bi-ideal of S.
(3) Every quasi-ideal of S is a bi-ideal of S.
(4) Arbitrary intersection of bi-ideals of S is also a bi-ideal of S and hence the set of all bi-ideals of S forms a complete lattice.
(5) If B is a bi-ideal of S, then $B\Gamma s$ and $s\Gamma B$ are bi-ideals of S, for any $s \in S$.
(6) If B is a bi-ideal of S, then $b\Gamma B\Gamma c$ is a bi-ideal of S, for $b, c \in S$.
(7) If B is a bi-ideal of S and if T is a sub Γ-semiring, then $B \cap T$ is a bi-ideal of T.
(8) If A and B are bi-ideals of S, then $A\Gamma B$ and $B\Gamma A$ are bi-ideals of S.
(9) For any $a \in S$, $S\Gamma a$ is a left ideal and $a\Gamma$S is a right ideal of S.

Remark 3.1. (I) Converse of the statement (1) in Theorem 3.1 need not be true. For this consider the following examples.
(1) Consider the semiring $S = M_{2\times 2}(N_0)$, where N denotes the set of all natural numbers and $N_0 = N \cup \{0\}$. If $\Gamma = S$, then S forms a Γ-semiring with

$$A\alpha B = \text{usual matrix product of } A, \alpha, B \text{ ; for all } A, \alpha, B \in S.$$

(i) $B = \left\{ \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix} \mid x\ ,\ y \in N_0 \right\}$ is a bi-ideal of S. B is a left ideal but not a right ideal of S. Therefore B is not an ideal of S.

(ii) $B = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix} \mid x \in N_0 \right\}$ is a bi-ideal of S but B is neither a left ideal nor a right ideal of S. Hence B is not an ideal of S.

(II) Converse of the statement (3) in Theorem 3.2 need not be true. For this consider the following example.
(2) Let $S = \{0, 1, 2, 3\}$. Define two binary operations $+$ and $\cdot$ on S as follows.

+	0	1	2	3
0	0	0	0	0
1	0	1	0	0
2	0	0	2	0
3	0	0	0	3

.	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	0	1
3	0	0	1	2

Then S forms a semiring. If $\Gamma = S$ and $x\alpha y = xy$, for all x, y in S then S forms a Γ-semiring. $B = \{0, 2\}$ is a bi-ideal of S. Hence B is not a quasi-ideal of S because $3\alpha 2 = 2\alpha 3 = 1 \notin B$, for all $\alpha \in \Gamma$. This implies $(S\Gamma B) \cap (B\Gamma S) \nsubseteq B$. Also B is a two sided ideal of an ideal $I = \{0, 1, 2\}$ of S. But B is not an ideal of S since $3\alpha 2 = 1 \notin B$, for all $\alpha \in \Gamma$.

(III) Let S be a non regular Γ-semiring. Also let R and L be a minimal right ideal and a minimal left ideal of S. Then $R \cap L$ is a minimal quasi-ideal of S (see Theorem 4.2 in [**5**]). Hence $R \cap L$ is a bi-ideal of S by Theorem 3.1 (2). Then by Theorem 3.1 (8) $R\Gamma L$ is a bi-ideal of S. Always $R\Gamma L \subseteq R \cap L$. If $R\Gamma L$ is a quasi-ideal of S, then $R\Gamma L = R \cap L$ as $R \cap L$ is a minimal quasi-ideal. Therefore S is a regular Γ-semiring, which is a contradiction. Hence $R\Gamma L$ is not a quasi-ideal of S. Thus in a non regular Γ-semiring S, $R\Gamma L$ is a bi-ideal but not a quasi-ideal.

In the following theorems we show that every bi-ideal B in S induces bi-ideals in S.

Theorem 3.2. *If B is a bi-ideal and T is a sub-Γ-semiring of S, then $B\Gamma T$ and $T\Gamma B$ are bi-ideals of S.*

Proof. Let a , $b \in B\Gamma T$. Hence

$$a = \sum_{i=1}^{n} a_i\alpha_i s_i \ , \quad b = \sum_{j=1}^{m} b_j\beta_j t_j;$$

$a_i, s_i, b_j, t_j \in B$; $\alpha_i, \beta_j \in \Gamma$. Then $a + b = \sum_{i=1}^{n} a_i\alpha_i s_i + \sum_{j=1}^{m} b_j\beta_j t_j$ being a finite sum, $a + b \in B\Gamma T$. Hence$B\Gamma T$ is a subsemigroup of$(S, +)$. As

$$(B\Gamma T)\,\Gamma\,(B\Gamma T) = (B\Gamma T\Gamma B)\,\Gamma T \subseteq (B\Gamma S\Gamma B)\,\Gamma T \subseteq B\Gamma T,$$

we get $B\Gamma T$ is a sub-Γ-semiring of S. Now

$$(B\Gamma T)\,\Gamma S\Gamma\,(B\Gamma T) = B\Gamma\,(T\Gamma S)\,\Gamma\,(B\Gamma T) \subseteq (B\Gamma S\Gamma B)\,\Gamma T \subseteq B\Gamma T.$$

This shows that $B\Gamma T$ is a bi-ideal of$S$.
Similarly, we can prove that $T\Gamma B$ is a bi-ideal of bS. □

Theorem 3.3. *If B is a bi-ideal of S and C is a bi-ideal of B such that $C^2 = C\Gamma C = C$, then C is a bi-ideal of S.*

PROOF. B being a bi-ideal of S we get $B\Gamma S\Gamma B \subseteq B$. Let C be a bi-ideal of B such that $C^2 = C\Gamma C = C$. Hence $C\Gamma B\Gamma C \subseteq C$. As C is a sub-Γ-semiring of B, it is a sub-Γ-semiring of S. Further

$$C\Gamma B\Gamma C = (C\Gamma C)\,\Gamma S\Gamma\,(C\Gamma C) = C\Gamma\,(C\Gamma S\Gamma C)\,\Gamma C$$
$$\subseteq C\Gamma\,(B\Gamma S\Gamma B)\,\Gamma C \subseteq C\Gamma B\Gamma C \subseteq C.$$

Thus $C\Gamma S\Gamma C \subseteq C$. This shows that C is a bi-ideal of S. □

THEOREM 3.4. *Let A and C be two sub-Γ-semirings of S and $B = A\Gamma C$. If A is a left ideal or a right ideal of S, then B is a bi-ideal of S*

PROOF. Suppose that A is a left ideal of S. Hence A is a bi-ideal of S. Further

$$B\Gamma S\Gamma B = (A\Gamma C)\,\Gamma S\Gamma\,(A\Gamma C) = (A\Gamma C)\,\Gamma\,(S\Gamma A)\,\Gamma C$$
$$\subseteq (A\Gamma C)\,\Gamma\,(A\Gamma C) \subseteq (A\Gamma C) = B.$$

This shows that B is a bi-ideal of S. Similarly we can show that $B = A\Gamma C$ is a bi-ideal of S if A is a right ideal of S. □

REMARK 3.2. Similarly we can prove that $B = A\Gamma C$ is a bi-ideal of S if A and C be two sub-Γ-semirings of S and if C is a left ideal or a right ideal of S.

Some characterizations of bi-ideals in a Γ-semiring are given in the following theorems.

THEOREM 3.5. *For any non-empty subset B of S, following statements are equivalent:*
(1) B is a bi-ideal of S.
(2) B is a left ideal of some right ideal of S.
(3) B is a right ideal of some left ideal of S.

PROOF. (1)⇔(2). Suppose that B is a bi-ideal of S. Therefore $B\Gamma S\Gamma B \subseteq B$. As $B\Gamma S$ is a right ideal of S (see Result 3.1 in [5]), $(B\Gamma S)\,\Gamma B \subseteq B$. This shows that B is a left ideal of a right ideal $B\Gamma S$ of S. Conversely, suppose that B is a left ideal of a right ideal R of S.Then $R\Gamma B \subseteq B$, $R\Gamma S \subseteq R$ and B is a sub-Γ-semiring of S. Hence $B\Gamma S\Gamma B \subseteq R\Gamma S\Gamma B \subseteq R\Gamma B \subseteq B$. This shows that B is a bi-ideal of S.

(1)⇔(3). Suppose that B is a bi-ideal of S. Then $(B\Gamma S)\,\Gamma B \subseteq B$. As $S\Gamma B$ is a left ideal of S (see Result 3.1 in [5]), $B\Gamma\,(S\Gamma B) \subseteq B$.This shows that B is a right ideal of a left ideal $S\Gamma B$ of S. Conversely, suppose B is a right ideal of a left ideal L of S. Then $B\Gamma L \subseteq B$, $S\Gamma L \subseteq L$ and B is a sub-Γ-semiring of S. Further $B\Gamma S\Gamma B \subseteq B\Gamma S\Gamma L \subseteq B\Gamma L \subseteq B$.

This shows that B is a bi-ideal of S. Thus we prove that **(**1) ⇔(2)⇔ (3). □

THEOREM 3.6. *A sub-Γ- semiring B of S is a bi-ideal of S if and only if there exist a left ideal L and a right ideal R of S such that $R\Gamma L \subseteq B \subseteq R \cap L$.*

PROOF. Suppose that B is a bi-ideal of S. Then $B\Gamma S\Gamma B \subseteq B$. Hence $R = B\Gamma S$ is a right ideal of S and $L = S\Gamma \mathrm{B}$ is a left ideal of S (see [**5**]).

$$R\Gamma L = (B\Gamma S)\,\Gamma\,(S\Gamma B) = B\Gamma\,(S\Gamma S)\,\Gamma B \subseteq B\Gamma S\Gamma B \subseteq B.$$

As B is a bi-ideal of S, B is a right ideal of a left ideal L and also a left ideal of a right ideal R (see Theorem 3.5). Therefore $B \subseteq R \cap L$. Thus $R\Gamma L \subseteq B \subseteq R \cap L$.

Conversely, suppose that $R\Gamma L \subseteq B \subseteq R \cap L$. Therefore

$$B\Gamma S\Gamma B \subseteq (R \cap L)\,\Gamma S\Gamma\,(R \cap L) \subseteq R\Gamma S\Gamma L \subseteq R\Gamma L \subseteq B.$$

This shows B is a bi-ideal of S. □

Definition 3.2. A bi-ideal B of S is minimal if there is no non zero proper bi-ideal of S contained in B.

A property of minimal bi-ideal in a Γ-semiring S is proved in the following theorem.

Theorem 3.7. *If B is a minimal bi-ideal of S, then any two non zero elements of B generate the same left (right) ideal of S.*

Proof. Let B be a minimal bi-ideal of S. Let x and y be non zero elements of B. Therefore $(x)_l \cap B$ is a bi-ideal of S . Hence $(x)_l \cap B \subseteq B$. B being a minimal bi-ideal of S, we have $(x)_l \cap B = B$. This implies $B \subseteq (x)_l$. $y \in B$ implies $y \in (x)_l$. Therefore $(y)_l \subseteq (x)_l$. Similarly we can show that $(x)_l \subseteq (y)_l$. Hence $(x)_l = (y)_l$. Thus any two non zero elements of B generate the same left ideal of S. In the same way we can show that any two non zero elements of a bi-ideal generate the same right ideal of S. □

4. Bi-ideals in a Regular Γ-semiring

In this article we discuss some characterizations of bi-ideals in a regular Γ-semiring.

Definition 4.1. Recall that a Γ-semiring S is regular if $a \in a\Gamma S\Gamma a$, for any $a \in S$.

Necessary and sufficient conditions for a Γ-semiring to be regular are furnished in the following theorem.

Theorem 4.1. *Following statements are equivalent in S.*
(1) S is regular.
(2) For any bi-ideal B of S, $B\Gamma S\Gamma B = B$.
(3) For any quasi-ideal Q of S, $Q\Gamma S\Gamma Q = Q$.

Proof. (1) ⇒ (2). Suppose that S is regular. Let B be a bi-ideal of S. Hence $B\Gamma S\Gamma B \subseteq B$. For any $b \in B$, $b \in b\Gamma S\Gamma b$ as S is regular. Therefore $b\Gamma S\Gamma b \subseteq B\Gamma S\Gamma B$, since $b \in B$ .Thus $b \in B\Gamma S\Gamma B$. Therefore $B \subseteq B\Gamma S\Gamma B$. Hence $B = B\Gamma S\Gamma B$ as $B\Gamma S\Gamma B \subseteq B$.

(2) ⇒ (3). Every quasi-ideal of S being a bi-ideal of S, the implication follows.

(3) ⇒ (1). Let R be a right ideal and L be a left ideal of S. Then $R \cap L$ is a quasi-ideal of S (see property 11 in [**5**]). Hence by (3) $R \cap L = (R \cap L)\,\Gamma S\Gamma\,(R \cap L)$. Therefore $R \cap L \subseteq (R\Gamma S)\,\Gamma L \subseteq R\Gamma L$. But always $R\Gamma L \subseteq R \cap L$. Therefore $R \cap L = R\Gamma L$. Thus S is a regular Γ-semiring. □

COROLLARY 4.1. *Let S be a regular Γ-semiring. A sub-Γ-semiring B is a bi-ideal of S if and only if $B\Gamma S\Gamma B = B$.*

PROOF. If a sub-Γ semiring B is a bi-ideal of S, then $B\Gamma S\Gamma B = B$ by Theorem 4.1. Conversely, if $B\Gamma S\Gamma B = B$,then B is a bi-ideal of S, by definition. □

COROLLARY 4.2. *Let S be a regular Γ-semiring. A sub-Γ-semiring B is a bi-ideal of S if and only if B is a quasi-ideal of S.*

PROOF. If a sub-Γ-semiring B is a bi-ideal of S, then $B\Gamma S\Gamma B = B$ by Theorem 4.1. But as $B\Gamma S$ is a right ideal, $S\Gamma B$ a left ideal of S and S being a regular Γ-semiring we have $(B\Gamma S) \cap (\mathrm{S}\Gamma B) = (B\Gamma S)\,\Gamma\,(\mathrm{S}\Gamma B)$. Therefore $(B\Gamma S) \cap (\mathrm{S}\Gamma B) = B\Gamma\,(S\Gamma S)\,\Gamma B \subseteq B\Gamma S\Gamma B$, since $S\Gamma S \subseteq S$.Therefore $(B\Gamma S) \cap (\mathrm{S}\Gamma B) \subseteq B$ as $B\Gamma S\Gamma B = B$. Hence B is a quasi-ideal of S. Since every quasi-ideal is a bi-ideal, converse follows. □

Let A and B be bi-ideals of S with $A \subseteq B \subseteq S$, then surely A is a bi-ideal of B. But if A is a bi-ideal of B and B is a bi-ideal of S with $A \subseteq B \subseteq S$, then A need not be a bi-ideal of S. But under the condition of regularity we have

THEOREM 4.2. *Let B be a bi-ideal of S. If B itself is a regular sub-Γ-semiring, then any bi-ideal of B is a bi-ideal of S.*

PROOF. Let A be a bi-ideal of B. Then $A\Gamma B\Gamma A \subseteq A$. For any $b \in A$, $b \in b\Gamma B\Gamma b$ as B is regular. $b\Gamma B\Gamma b = b\Gamma(B\Gamma b) \subseteq A\Gamma B$, since $b \in B$ and $b \in A$. Again $b\Gamma B\Gamma b = (b\Gamma B)\Gamma b \subseteq B\Gamma A$ as $b \in B$ and $b \in A$. Then $A\Gamma S\Gamma A \subseteq (\ A\Gamma B)\,\Gamma S\,(B\Gamma A) = A\Gamma(B\Gamma S\Gamma B)\Gamma A \subseteq A\Gamma B\Gamma A$, since B is a bi-ideal of S. Therefore $A\Gamma S\Gamma A \subseteq A\Gamma B\Gamma A \subseteq A$, since A is a bi-ideal of B. Therefore $A\Gamma S\Gamma A \subseteq A$. This shows that A is a bi-ideal of S. □

A necessary and sufficient condition for a Γ-semiring S to be regular is proved in the following theorem .

THEOREM 4.3. *S is a regular if and only if $B \cap I \cap L \subseteq B\Gamma I\Gamma L$ for any bi-ideal B, left ideal L and two sided ideal I of S.*

PROOF. Let S be a regular Γ-semiring, B be a bi-ideal , I be a two sided ideal and L be a left ideal of S. Let $a \in B \cap I \cap L$. $a \in S$ and S is regular imply $a \in a\Gamma S\Gamma a$. Therefore $a\Gamma S\Gamma a \subseteq (a\Gamma S\Gamma a)\,\Gamma S\Gamma(a\Gamma S\Gamma a)$. Then

$$a \in (a\Gamma S\Gamma a)\,\Gamma\,(S\Gamma a\Gamma S)\,\Gamma a.$$

Hence $a \in (B\Gamma S\Gamma B)\,\Gamma\,(S\Gamma I\Gamma S)\,\Gamma L$. Therefore $a \in B\Gamma I\Gamma L$, since B is a bi-ideal, I is an ideal and $a \in L$. Therefore $B \cap I \cap L \subseteq B\Gamma I\Gamma L$.

Conversely, let R be a right ideal and L be a left ideal of S. By assumption $R \cap S \cap L \subseteq R\Gamma S\Gamma L$. Hence $R \cap L \subseteq R\Gamma L$. But $R\Gamma L \subseteq R \cap L$ always. Thus we have $R \cap L = R\Gamma L$.

This shows that S is a regular Γ-semiring. □

REMARK 4.1. If Q_1 and Q_2 are quasi-ideals of S, then $Q_1\Gamma Q_2$ need not be a quasi-ideal of S. For this consider the following example. If

$$T = \left\{ \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \mid a\ ,\ b \in R^{+} \right\}$$

then T is a semigroup with respect to usual matrix multiplication. If

$$S = \left\{ \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \mid a\ ,\ b \in R^{+} \right\} \cup \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and Γ= S, then S is a Γ-semiring with usual matrix multiplication and $+$ is defined by $A + B = 0$ if $A,\ B \in\ S$ and $A + 0 = 0 + A = A$, for all $A \in S$.

If

$$Q_1 = \left\{ \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \mid a\ ,\ b \in R^{+}, 0 < a < b \right\} \cup \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$Q_2 = \left\{ \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \mid a\ ,\ b \in R^{+}, a > 0, b > 5 \right\} \cup \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then Q_1 is a right ideal and Q_2 is a left ideals of S. And hence Q_1 and Q_2 are quasi-ideals of S. But $Q_1\ \Gamma Q_2$ is a not a quasi-ideal of S.

$Q_1\Gamma Q_2$ need not be a quasi-ideal of S when both Q_1 and Q_2 are quasi-ideal of S. But in a regular Γ-semiring we have $Q_1\Gamma\ Q_2$ is a quasi-ideal of S.

Theorem 4.4. *Let S be a regular Γ-semiring. If Q_1 is a sub Γ-semiring and Q_2 is a bi-ideal of S, then $Q_1\Gamma Q_2$ and $Q_2\Gamma Q_1$ are quasi-ideals of S.*

Proof. Let Q_1 be a sub-Γ-semiring and Q_2 be a bi-ideal of S. In a regular Γ-semiring quasi-ideals and bi-ideals coincide (see Corollary 4.2). Hence by Theorem 3.2, $Q_1\Gamma Q_2$ and $Q_2\Gamma Q_1$ are bi-ideals of S. Therefore $Q_1\Gamma Q_2$ and $Q_2\Gamma Q_1$ are quasi-ideals of S by Corollary 4.2 . □

Every quasi-ideal of S being a sub-Γ-semiring and a bi-ideal of S, by applying Thoerem 4.4, we get

Corollary 4.3. *Let S be a regular Γ-semiring. If Q_1 and Q_2 are quasi-ideals of S, then $Q_1\Gamma Q_2$ and $Q_2\Gamma Q_1$ are quasi-ideals of S.*

We know that a sub-Γ-semiring of a Γ-semiring S need not be a bi-ideal of S (see example 3.1).

A necessary and sufficient condition for a sub-Γ-semiring of a Γ-semiring S to be a bi-ideal of S is given in the following theorem.

Theorem 4.5. *A sub-Γ-semiring of a regular Γ-semiring S is a bi-ideal of S if and only if B can be represented as $B = R\Gamma L$, where R is a right ideal and L is a left ideal of S.*

Proof. Let B be a sub Γ-semiring of a regular Γ-semiring S. Suppose that B is a bi-ideal of S. Hence by Theorem 3.6, there exist a right ideal R and a left ideal L of S such that $R\Gamma L \subseteq B \subseteq R \cap L$. As S is regular, $R\Gamma L = R \cap L$. Therefore $B = R\Gamma L = R \cap L$.Conversely, suppose that $B = R\Gamma L$, for a right ideal R and a left ideal L. Therefore $B\Gamma S\Gamma B = (R\Gamma L)\,\Gamma S\Gamma\,(R\Gamma L) \subseteq R\Gamma S\Gamma L \subseteq R\Gamma L = B$. This shows that B is a bi-ideal of S. □

5. Bi-ideals in Special Γ-semirings

This article deals with various types of Γ-semirings and their characterizations using bi-ideals.

5.1. Intra-regular Γ-semirings.

DEFINITION 5.1. A Γ-semiring S is an intra-regular Γ-semiring if for any $x \in S$, $x \in S\Gamma x\Gamma x\Gamma S$.

Following theorem is a property of a regular and intra-regular Γ-semiring.

THEOREM 5.1. *If a Γ-semiring S is regular and intra-regular, then for any bi-ideal B and a left ideal L of S, $B \cap L \subseteq B\Gamma L\Gamma B$.*

PROOF. Let $a \in B \cap L$. As $a \in S$ and S is regular, $a \in a\Gamma S\Gamma a$. Therefore we have

$$a\Gamma S\Gamma a \subseteq (a\Gamma S\Gamma a)\,\Gamma \mathrm{S}\Gamma\,(a\Gamma S\Gamma a) \subseteq (a\Gamma S\Gamma a)\,\Gamma S\Gamma\,(S\Gamma a\Gamma a\Gamma S)\,\Gamma S\Gamma a.$$

Hence we get

$$a\Gamma S\Gamma a \subseteq (a\Gamma S\Gamma a)\,\Gamma(\mathrm{S}\Gamma S\Gamma a)\Gamma\,(a\Gamma S\Gamma a) \subseteq (B\Gamma S\Gamma B)\,\Gamma(\mathrm{S}\Gamma S\Gamma L)\Gamma\,(B\Gamma S\Gamma B)$$

as $a \in B \cap L$. Thus we get $a\Gamma S\Gamma a \subseteq B\Gamma L\Gamma B$. Therefore $a \in B \cap L$ implies $a \in B\Gamma L\Gamma B$. This shows that $B \cap L \subseteq B\Gamma L\Gamma B$. □

5.2. Duo Γ-semirings.

DEFINITION 5.2. A Γ-semiring S is a right (left) duo Γ-semiring if every right (left) ideal of S is a left (right) ideal of S. A Γ-semiring S is a duo Γ-semiring if every one sided ideal of S is a two sided ideal of S.

We know every bi-ideal need not be one sided or two sided ideal (see in Remark 3.1 Example (1)). But in a regular duo Γ-semiring we have

THEOREM 5.2. *If S is a regular right (left) duo Γ-semiring, then every bi-ideal of S is a left (right) ideal of S.*

PROOF. Let S be a regular right duo Γ-semiring and B be a bi-ideal of S. By Theorem 4.5 , there exist a right ideal R and a left ideal L such that $B = R\Gamma L = R\cap L$. As S is a right duo Γ-semiring, $R\cap L$ is a left ideal of S. Therefore $B = R\cap L$ is a left ideal of S.

Similarly we can show that in a regular left duo Γ-semiring S every bi-ideal is a right ideal of S. □

As a duo Γ-semiring S is a both left duo Γ-semiring and right duo Γ-semiring, by Theorem 5.2 we have

THEOREM 5.3. *If S is a regular duo Γ-semiring, then every bi-ideal of S is a two sided ideal of S.*

5.3. Simple Γ-semirings.

DEFINITION 5.3. A Γ-semiring S is a left(right) simple Γ-semiring if S has no proper left (right) ideal. A Γ-semiring S is a simple Γ-semiring if S has no proper ideal.

DEFINITION 5.4. A Γ-semiring S with zero is a left (right) 0-simple. Γ-semiring if S has no non zero proper a left (right) ideal and $S\Gamma S \neq \{0\}$. A Γ-semiring S with zero is 0-simple Γ-semiring if S has no non zero proper ideal and $S\Gamma S \neq \{0\}$.

Every bi-ideal of S need not be right (left) ideal of S (see in Remark 3.1 and Example (1)). But in (left\right) simple Γ-semiring we have:

THEOREM 5.4. *If a Γ-semiring S is a left (right) simple Γ-semiring, then every bi-ideal of S is a right ideal (left ideal) of S.*

PROOF. Let S be a left simple Γ-semiring and B be a bi-ideal of S. Then $S\Gamma B$ is a left ideal and $S\Gamma B \subseteq S$. But S is a left simple Γ-semiring and hence $S\Gamma B = S$. Further $B\Gamma S = B\Gamma S\Gamma B \subseteq B$.This implies that B is a right ideal of S. Similaly we can prove for a right simple Γ-semiring. □

Every quasi-ideal is a bi-ideal in S but not conversely (see in Remark 3.1 and Example (2)). In a 0-simple Γ-semiring we have:

THEOREM 5.5. *If a Γ-semiring S with zero is left (right) 0-simple Γ-semiring, then the set of bi-ideals of S coincide with the set of quasi-ideals of S.*

PROOF. Let S be a left 0-simple Γ-semiring . As every quasi-ideal is a bi-ideal of S , we have to only show that any bi-ideal is a quasi-ideal of S.Let B be a bi-ideal of S.Then $S\Gamma B$ is a left ideal of S. $S\Gamma B \subseteq S\Gamma S \neq \{0\}$ as S is left 0-simple. S is left 0-simple and $S\Gamma B$ is a non zero left ideal of S imply $S\Gamma B = S$. Further $(B\Gamma S) \cap (S\Gamma B) \subseteq B\Gamma S = B\Gamma S\Gamma B \subseteq B$. This shows that B is a quasi-ideal of S. In the same way we can prove for a right 0-simple Γ-semiring. □

5.4. Bi-simple Γ-semirings.

DEFINITION 5.5. A Γ-semiring S is a bi-simple Γ-semiring if S has no bi-ideal other than S itself.

Next theorem gives a characterization of a bi-simple Γ-semiring.

THEOREM 5.6. *If S is a Γ-semiring, then S is a bi-simple Γ-semiring if and only if $a\Gamma S\Gamma a = a$, for all $a \in S$.*

PROOF. Suppose that S is a bi-simple Γ-semiring. For any $a \in S$, $a\Gamma S\Gamma a$ is a sub Γ-semiring of S. By Theorem 3.1(9) $S\Gamma a$ is a right ideal of S. Therefore

$$(a\Gamma S\Gamma a)\,\Gamma S\Gamma\,(a\Gamma S\Gamma a) = (a\Gamma S)\,\Gamma\,(a\Gamma S\Gamma a\Gamma S)\,\Gamma a \subseteq a\Gamma S\Gamma a \quad .$$

Hence $a\Gamma S\Gamma a$ is a bi-ideal of S. $a\Gamma S\Gamma a \subseteq S$ and S is a bi-simple Γ-semiring imply $a\Gamma S\Gamma a = S$, for all $a \in S$.

Conversely, suppose that $a\Gamma S\Gamma a = S$, for all $a \in S$. Let B be a bi-ideal of S such that $B \subseteq S$. For any $b \in B$, by assumption $b\Gamma S\Gamma b = S$. $S = b\Gamma S\Gamma b \subseteq B\Gamma S\Gamma B \subseteq B$ imply $B = S$. Hence S is a bi-simple Γ semiring. □

5.5. Division Γ-semirings.

DEFINITION 5.6. A Γ-semiring S with zero is a division Γ-semiring if for any non zero $a \in S$ and non zero $\alpha \in \Gamma$ there exist $b \in S$, $\beta \in \Gamma$ such that $a\alpha b\beta x = x$ and $x\beta b\alpha a = x$, for $x \in S$.

In the next theorem we prove a property of a division Γ-semiring.

THEOREM 5.7. *If S is a division Γ-semiring, then S has no proper non zero bi-ideal.*

PROOF. Let S be a division Γ-semiring and B be a non zero bi-ideal of S. Let $0 \neq a \in B$ and $0 \neq \alpha \in \Gamma$. $0 \neq a \in S$, $0 \neq \alpha \in \Gamma$ and S is a division Γ-semiring imply for any non zero $\alpha \in \Gamma$ there exist $b \in S$ and $\beta \in \Gamma$ such that $a\alpha b\beta x = x$ and $x\beta b\alpha a = x$, for $x \in S$. $a\alpha b\beta x = x$ implies $x \in B\Gamma S$ as $a \in B$. As this is true for any $x \in S$, we get $S \subseteq B\Gamma S$. But $B\Gamma S \subseteq S$. Therefore $B\Gamma S = S$. Similarly $x\beta b\alpha a = x$ implies $S\Gamma B = S$. Therefore $S = S\Gamma B = B\Gamma S$. Using these relations we have $S = S\Gamma B = B\Gamma S\Gamma B$. As B is a bi-ideal of S, $B\Gamma S\Gamma B \subseteq B$. Hence $S \subseteq B$. Thus $S = B$ as $B \subseteq S$. This shows that S has no proper non zero bi-ideal. □

References

[1] R. Chinram, A note on quasi-ideals in Γ-semirings. *Int. Math. Forum*, **3**(26)(2008), 1253-1259.

[2] T.K. Dutta and S.K. Sardar, Semi-prime ideals and irreducible ideals of Γ-semiring. *Novi Sad J. Math.*, **30**(1)(2000), 97-108.

[3] R.A. Good and D.R.Hughes, Associated groups for a semigroup. *Bull. Amer. Math. Soc.*, **58**(1952), 624-625.

[4] K.Iseki, Quasi-ideals in a semiring without zero. *Proc. Japan Acad*, **34**(2)(1958), 79-81.

[5] R.D. Jagatap and Y.S. Pawar, Quasi-ideals and Minimal Quasi-ideals in Γ-semirings. *Novi Sad J. Math.*, **39**(2)(2009), 79-87.

[6] S. Lajos , On (m, n) ideals of semigroups. *Abstracts of Second Hungar Math. Congress* Vol. I, (1960), pp. 42-44.

[7] S. Lajos and F. Szasz, On the bi-ideals in Associative ring. *Proc. Japan Acad.*, **46**(6)(1970), 505-507.

[8] N.Nobusawa, On a generalization of the ring theory. *Osaka J. Math.*, **1**(1)(1964), 81-89.

[9] M. M. K. Rao, Γ-semirings 1. *Southeast Asian Bull. Math.*, **19**(1995), 49-54.

[10] M. Shabir, A. Ali and S. Batool, A note on quasi-ideals in semirings. *Southeast Asian Bull. Math.*, **27**(5)(2004), 923-928.

[11] M.F. Smiley, An Introduction to Hestenes Ternary Rings. *American Mathematical Monthly*, **76**(3)(1969), 245-248.

[12] O. Steinfeld, On ideals quotients and prime ideals. *Acta. Math. Acad. Sci. Hungar*, **4**(1953), 289-298.

[13] O. Steinfeld, Uher die quasi-ideals von halbgruppen. *Publ. Math. Debrecen*, **4**(1956), 262-275.

[14] H.S. Vandiver, On Some simple Types of Semirings. *American Mathematical Monthly*, **46**(1939), 22-26.

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