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DOMINATION IN SOME CLASSES OF DITREES

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Abstract. Domination and other related concepts in undirected graphs are well studied. Although domination and related topics are extensively studied, the respective analogies on digraphs have not received much attention. Such studies in the directed graphs have applications in game theory and other areas.

A directed graph D is a pair (V, E), where V is a non empty set and E is a set of ordered pairs of elements taken from set V. V is called vertices and E set called directed edges.

Let $D = (V, E)$ be a digraph if $(x, y) \in E$ then arc is directed from x to y and is denoted by $x \to y$. The vertex x is called a predecessor of y and y is called a successor of x. A set $S \subseteq V$ of a digraph D is said to be a dominating set of D if $\forall v \notin S$, v is a successor of some vertex $s \in S$.

In this paper we study domination theory on few well known classes of directed trees. Directed trees are extensively used in path algorithm, scheduling problems, data processing networks, data compression, causal structures like family tree, Bayesian network, moral graphs, influence diagram etc. The concept of dominating function plays a significant role in these models.

1. Introduction

There are vast applications of digraphs in the field of sciences. Some familiar digraph problems are transitive closure, strong connectivity, topological sort, program evaluation and review technique (PERT), critical path method (CPM), shortest path, page rank etc. There is an extensive literature on digraphs. Many of these papers contain not only interesting theoretical results but also important algorithms and applications [**1, 2, 3, 4, 5, 6, 7, 8, 9, 10**]. The underlying graph is a special case of a directed graph which has been extensively studied. The subclasses of digraph are directed trees commonly known ditrees have extensive application in the field of computer science.

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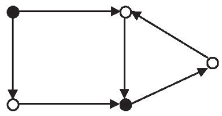

FIGURE 1. Domination set in digraph.

Domination and other related concepts in undirected graphs are well studied. Although domination and related topics are extensively studied, the respective analogs on digraphs have not received much attention. Such studies in directed graphs have its applications in game theory and other areas.

A directed graph (or a digraph) consists of a nonempty finite set $V(D)$ of elements called vertices and a finite set $A(D)$ of ordered pair of distinct vertices called arcs. $V(D)$ is the vertex set and $A(D)$ is called arc set of D.

Let $D = (V, A)$ be a digraph. If $(x, y) \in A$ then the arc is directed from x to y and is denoted by $x \to y$. The vertex x is called a predecessor of y (initial vertex) and y is called a successor of x (terminal vertex). A set $S \subseteq V$ of D is said to be a dominating set of D if $\forall v \notin S$, v is a successor of some vertex $s \in S$.

The directed dominating set of D is minimal directed dominating set if no proper subset of S is a directed dominating set. The minimum cardinality among all the minimal directed dominating set is called domination number of D denoted by $\gamma(D)$ (see Figure 1).

DEFINITION 1.1. If v is a vertex of a digraph D, the number of edges for which v is the initial vertex is called the out-going degree or the **out-degree** of v and the number of edges for which v is the terminal vertex is called the incoming degree or **in-degree** of v. The out degree of v is denoted by $d^+(v)$ and in-degree of v is denoted by $d^-(v)$.

DEFINITION 1.2. The **underlying graph** of a digraph is an undirected graph obtained by replacing each arc of the digraph by a corresponding undirected edge (see Figure 2).

DEFINITION 1.3. A **directed tree** is a directed graph whose underlying undirected graph is a tree.

DEFINITION 1.4. A directed tree T is called a **rooted tree** if T contains a unique vertex called the root, whose in-degree is equal to 0 and the in-degree of all other vertices of T are equal to 1.

DEFINITION 1.5. In a rooted tree a vertex whose out-degree is 0 is called a **leaf** and a vertex which is not a leaf is called an **internal vertex**.

DEFINITION 1.6. A rooted directed tree T is called an m-**ary** tree if every internal vertex of T is of out degree $\leqslant m$.

DEFINITION 1.7. A rooted directed tree T is called a **complete** m-**ary** tree if every internal vertex of T is of out degree m.

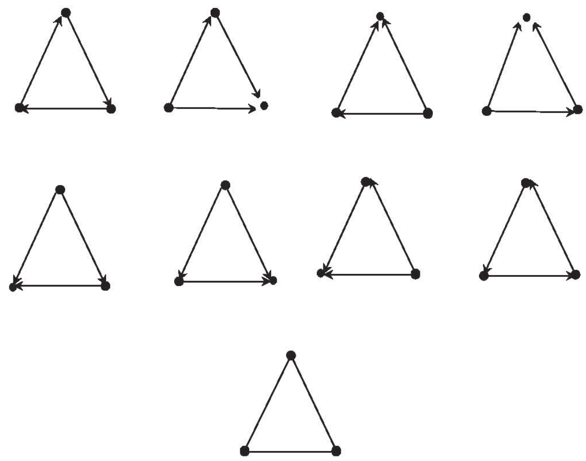

FIGURE 2. Digraphs and the corresponding Underlying graph.

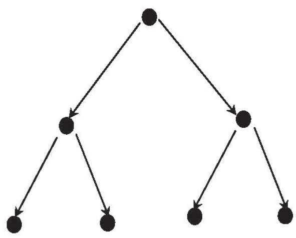

FIGURE 3. Complete full binary directed tree.

DEFINITION 1.8. A complete m-ary directed tree with $m = 2$ is called a **complete binary tree**.

DEFINITION 1.9. A vertex v (other than the root r) of a rooted tree is said to be at the hth **level** or has level number h if the directed path from r to v is of length h. if v_1 and v_2 are two vertices such that v_1 has a lower level number than v_2 and there is a path from v_1 to v_2, then we say that v_1 is an **ancestor** of v_2, or that v_2 is a **descendant** of v_1. In particular, if v_1 and v_2 are such that v_1 has a lower level number than v_2 and there is an directed edge from v_1 to v_2, then v_1 is called the **parent** of v_2, or v_2 is called the **child** of v_1. Two vertices with a common parent are referred to as **siblings**.

DEFINITION 1.10. If T is a rooted directed tree and h is the largest level number achieved by a leaf of T, then T is said to have height h. A rooted directed tree of height h is said to be **balanced** if the level number of every leaf is h or $h - 1$.

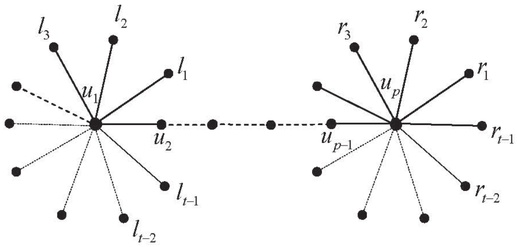

FIGURE 4. Underlying undirected graph of directed thorn rod $P_{p,t}$.

DEFINITION 1.11. Let T be a complete binary directed tree of height h. Then T is called a **full binary tree** if all the leaves in T are at level h (see Figure 3).

DEFINITION 1.12. A digraph D with vertex set $\{v_1, v_2, v_3, \ldots, v_n\}$ is a **tournament** if exactly one of the arcs v_iv_j or v_jv_i is in D for every $i \neq j \in \{1, 2, 3, \ldots, n\}$.

DEFINITION 1.13. **Arborescence** is a directed graph with a vertex u called the root and any other vertex v, there is exactly one directed path from u to v.

In this paper we consider the directed thorn rod, directed thorn star and directed thorn ring [**1, 2**].

DEFINITION 1.14. A **directed thorn rod** $P_{p,t}$ is a digraph whose underlying undirected graph includes a linear chain (termed as a rod) of p vertices and degree t terminal vertices at each of the two rod ends (see Figure 4).

DEFINITION 1.15. A **directed thorn star** is a digraph whose underlying undirected graph is obtained from a k arm star by attaching $t-1$ terminal vertices to each of the star arms and is denoted by $S_{k,t}$ (see Figure 5).

DEFINITION 1.16. The **directed thorny ring** C_n^+ whose underlying undirected graph consists of $2n$ vertices where n vertices on the cycle are of degree three and remaining n vertices are pendant vertices (see Figure 6).

The **directed thorny ring** $C_n^\vee$ whose underlying undirected graph consists of $n(t-1)$ vertices of which n vertices are in the cycle (each of degree t) and the remaining $n(t-2)$ are pendant vertices (see Figure 7).

2. Preliminary results of digraphs

The following results are found in [**11**].

THEOREM 2.1. *In every digraph D with n vertices, $\sum_{i=1}^{n} d^+(v_i) = \sum_{i=1}^{n} d^-(v_i) = m$, the out degree of v is denoted by $d^+(v)$ and in-degree of v is denoted by $d^-(v)$ and m is the number of edges in D.*

THEOREM 2.2. *For a digraph D, $\gamma(D) \leqslant n-1$.*

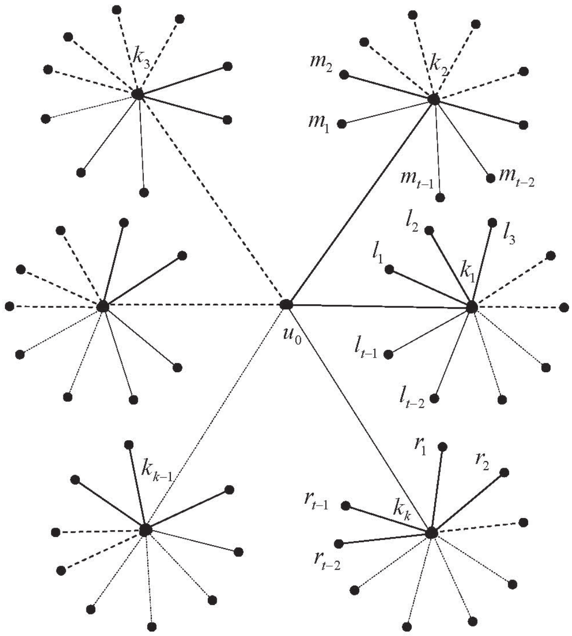

FIGURE 5. Underlying undirected graph of directed thorn star $S_{k,t}$.

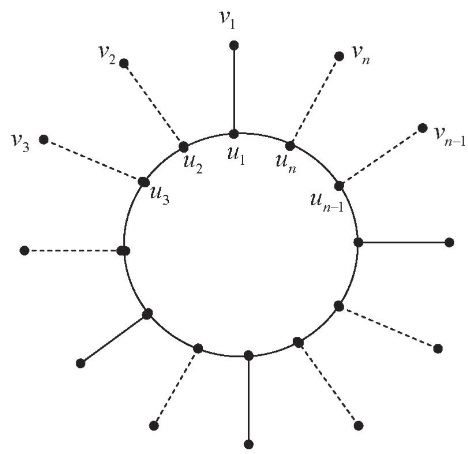

FIGURE 6. Underlying undirected graph of directed thorn ring C_n^+.

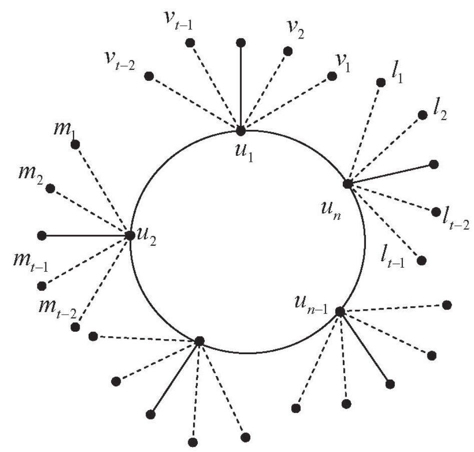

FIGURE 7. Underlying undirected graph of directed thorn ring $C_n^{\vee}$.

THEOREM 2.3. *For a underlying graph G, $\gamma(G) \leqslant \gamma(D)$.*

THEOREM 2.4. *For a digraph D, $\frac{n}{1+\Delta(D)} \leqslant \gamma(D) \leqslant n - \Delta(D)$, $\Delta(D)$ is the maximum out degree of D.*

THEOREM 2.5. *If a digraph D is Hamiltonian, then $\gamma(D) \leqslant \lceil \frac{n}{2} \rceil$.*

THEOREM 2.6. *If a digraph D is strongly connected, then $\gamma(D) \leqslant \lceil \frac{n}{2} \rceil$.*

THEOREM 2.7. *For a digraph D, $1 \leqslant \gamma(D) \leqslant \frac{\delta^-+1}{2\delta^-+1}n$, $\delta^- \geqslant 1$ is the minimum in degree of D.*

THEOREM 2.8. *For a tournament T, $1 \leqslant \gamma(T) \leqslant \lfloor \log_2(n+1) \rfloor$.*

THEOREM 2.9. *For a digraph D, $ir(D) \leqslant \gamma(D) \leqslant i(D) \leqslant \beta(D) \leqslant \Gamma(D) \leqslant IR(D)$.*

THEOREM 2.10. *If a digraph D is transitive, then $\gamma(D) = i(D) = \beta(D) = \Gamma(D) = IR(D)$.*

Recalling the following from [**11**], $\gamma(D)$ is the domination number of D, $i(D)$ is the independent domination number of D, $\beta(D)$ is the maximal independent domination number of D, $\Gamma(D)$ is the upper domination number of D, $ir(D)$ is the irredundance number of D and $IR(D)$ is the upper irredundance number of D.

3. Domination number of few classes of ditrees

THEOREM 3.1. *If D is a complete full binary rooted tree of maximum level l then,* $\gamma(D) = \begin{cases} 1 + \frac{2(2^l-1)}{3}, & \text{for } l = 2k,\ k \in N \\ 1 + \frac{4(2^{l-1}-1)}{3}, & \text{for } l = 2k-1,\ k \in N. \end{cases}$

PROOF. Consider l levels namely $L_0, L_1, L_2, \dots, L_{l-1}, L_l$ of D, each level L_i $(0 \leqslant i \leqslant l)$ has 2^l vertices.

We consider the following cases.

Case (i): For $l = 2k$.

Vertices in L_{l-1} will be labeled '1', therefore vertices of L_l has to be labeled '0'. Vertices in L_{l-3} will be labeled '1', therefore vertices of L_{l-2} has to be labeled '0'. Continuing the process we get the root to be labeled '1'.

Therefore $\gamma(D) = \big(2^{l-1} + 2^{l-3}+, \dots, +2^5 + 2^3 + 2^1\big) + 1$.

Rearranging and finding geometric sum of the series, we get $1 + \big(2^1 + 2^3 + 2^5+, \dots, +2^{l-3} + 2^{l-1}\big)$.

Hence, $\gamma(D) = 1 + \frac{2\left(\left(2^2\right)^{l/2}-1\right)}{2^2-1} = 1 + \frac{2\left(2^l-1\right)}{3}$, for $l = 2k$, given $k \in N$.

Case (ii): For $l = 2k - 1$.

Vertices in L_{l-1} will be labeled '1', therefore vertices of L_l has to be labeled '0'. Vertices in L_{l-3} will be labeled '1', therefore vertices of L_{l-2} has to be labeled '0'. Continuing the process we get the root to be labeled '1'.

Therefore $\gamma(D) = 1 + \big(2^{l-1} + 2^{l-3}+, \dots, +2^6 + 2^4 + 2^2\big)$.

Rearranging and finding the geometric sum of the series, we get $\gamma(D) = 1 + \big(2^2 + 2^4 + 2^6+, \dots, +2^{l-3} + 2^{l-1}\big)$.

Hence, $\gamma(D) = 1 + \frac{4\left(2^{l-1}-1\right)}{3}$, for $l = 2k - 1$, given $k \in N$.

Hence the proof. □

THEOREM 3.2. *If D is a complete full binary rooted tree of maximum level l, then* $\gamma(D) = \left\lceil \frac{n}{1+\Delta(D)} \right\rceil = \left\lceil \frac{n}{1+2} \right\rceil = \left\lceil \frac{n}{3} \right\rceil$.

PROOF. The proof follows since in a complete full binary rooted tree every vertex has out degree 2 and since every vertex except the pendant vertex can dominate two vertices and itself.

Hence the proof. □

Theorems 3.1 and 3.2 leads to the following general result whose proof is similar.

THEOREM 3.3. *If D is a m-ary full rooted tree of maximum level l, then* $\gamma(D) = \begin{cases} 1 + \frac{m\left(m^l-1\right)}{m^2-1}, & \text{for } l = 2k,\ k \in N \\ 1 + \frac{m^2\left(m^{l-1}-1\right)}{m^2-1}, & \text{for } l = 2k-1,\ k \in N. \end{cases}$
and $\gamma(D) = \left\lceil \frac{n}{1+m} \right\rceil$.

Let D_1 be a ditree obtained by reversing all the directions in the rooted full binary tree (see Figure 8). Then we have the following results.

THEOREM 3.4. *If D_1 is a directed tree of maximum level l then,* $\gamma(D_1) = \begin{cases} 1 + \frac{4\left(2^l-1\right)}{3}, & \text{for } l = 2k,\ k \in N \\ \frac{2\left(2^{l+1}-1\right)}{3}, & \text{for } l = 2k-1,\ k \in N. \end{cases}$

PROOF. Consider the l levels namely $L_0, L_1, L_2, \dots, L_{l-1}, L_l$ of D_1, each level $L_i (0 \leqslant i \leqslant l)$, has 2^l vertices.

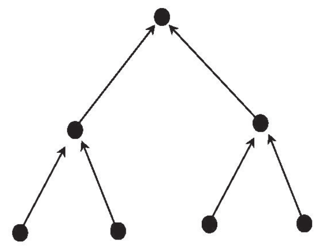

FIGURE 8. Directed tree D_1.

We consider the following cases.

Case (i) For $l = 2k$.

Vertices in L_l will be labeled '1', therefore vertices of L_{l-1} has to labeled '0'. Vertices in L_{l-2} will be labeled '1', therefore vertices of L_{l-3} has to be labeled '0'. Continuing the process we get the root to be labeled '1'.

Therefore $\gamma(D_1) = \left(2^l + 2^{l-2}+, \ldots, +2^6 + 2^4 + 2^2\right) + 1$.

Rearranging and finding the geometric sum of the series, we get $1 + \left(2^2 + 2^4 + 2^6+, \ldots, +2^{l-2} + 2^l\right)$.

Hence, $\gamma(D_1) = 1 + \frac{4\left(\left(2^2\right)^{l/2}-1\right)}{2^2-1} = 1 + \frac{4\left(2^l-1\right)}{3}$, for $l = 2k$, given $k \in N$.

Case (ii) For $l = 2k - 1$.

Vertices in L_l will be labeled '1', therefore vertices of L_{l-1} has to be labeled '0'. Vertices in L_{l-2} will be labeled '1', therefore vertices of L_{l-3} has to be labeled '0'. Continuing the process we get the root to be labeled '0'.

Therefore $\gamma(D_1) = \left(2^l + 2^{l-2}+, \ldots, +2^5 + 2^3 + 2^1\right)$.

Rearranging and finding the geometric sum of the series, we get $\gamma(D_1) = \left(2^1 + 2^3 + 2^5+, \ldots, +2^{l-2} + 2^l\right)$.

Hence, $\gamma(D_1) = \frac{2\left(2^{l+1}-1\right)}{3}$, for $l = 2k - 1$, given $k \in N$.

Hence the proof. □

Let D_2 be a directed graphs whose underlying graph is a complete full m-ary tree. Then Theorem 3.1 and Theorem 3.4 lead to the following results.

COROLLARY 3.1. *If $D = D_2$ is directed tree of maximum level l then,* $\gamma(D_2) \leqslant \begin{cases} 1 + \frac{4\left(2^l-1\right)}{3}, & \text{for } l = 2k,\ k \in N \\ \frac{2\left(2^{l+1}-1\right)}{3}, & \text{for } l = 2k-1,\ k \in N. \end{cases}$ □

COROLLARY 3.2. *If $D = D_2$ is a m-ary directed tree of maximum level l then,* $\gamma(D_2) \leqslant \begin{cases} 1 + \frac{m^2\left(m^l-1\right)}{m^2-1}, & \text{for } l = 2k,\ k \in N \\ \frac{m\left(m^{l+1}-1\right)}{m^2-1}, & \text{for } l = 2k-1,\ k \in N. \end{cases}$ □

For the directed thorn rod $P_{p,t}$ we now have the following result.

THEOREM 3.5. *Let $D = P_{p,t}$, $p > 2$, $p, t \in N$. Then,* $\gamma(D) \leqslant (t-1) + \left\lceil \frac{p}{2} \right\rceil$.

PROOF. For $t = 1$, $P_{p,1} \cong P_p$. Hence $\gamma(D) \leqslant \lceil \frac{p}{2} \rceil$.

For $t = 2$, $P_{p,2} \cong P_{p+2}$. Hence $\gamma(D) \leqslant \lceil \frac{p+2}{2} \rceil$.

Let $t > 2$. By definition, $P_{p,t} = G_1 \cup G_2$ where G_1 is a star and G_2 is a path.

Consider G_1. The maximum value of domination number of the G_1 is achieved only when the direction of all the pendant vertices is towards the end vertex of the linear chain.

$$\gamma(G_1) \leqslant (t-1) \tag{3.1}$$

Consider G_2. The linear chain is P_p. Therefore

$$\gamma(G_2) \leqslant \left\lceil \frac{p}{2} \right\rceil \tag{3.2}$$

From Eqns. (3.1) and (3.2), we get $\gamma(D) \leqslant (t-1) + \lceil \frac{p}{2} \rceil$.

Hence the proof. □

For the directed thorn star $S_{k,t}$ we now have the following result.

THEOREM 3.6. *Let $D = S_{k,t}$, $t > 1$, $t, k \in N$. Then, $\gamma(D) \leqslant k(t-1) + 1$.*

PROOF. For $k = 1$, $S_{k,t} \cong K_{1,t}$. Hence $\gamma(D) \leqslant (t-1)$.

For $k = 2$, $S_{k,t} \cong P_{3,t}$. Hence by Theorem 3.5, the proof follows.

Let $k > 2$. Label $S_{k,t}$ with k arms as $k_1, k_2, k_3, \ldots, k_k$ with corresponding siblings as $l_1, l_2, l_3, \ldots, l_{t-1}$, $m_1, m_2, m_3, \ldots, m_{t-1}, \ldots$, $r_1, r_2, r_3, \ldots, r_{t-1}$ also the central vertex where the k arms are incident is labeled as u_0 (see Fig. 5). The maximum value of $\gamma(D)$ is achieved only when the direction of all the pendant vertices is towards the end vertex of the k arms also the central vertex is to be labeled 1.

Hence the proof. □

For the directed thorn ring C_n^+ we now have the following result.

THEOREM 3.7. *Let $D = C_n^+$,$n > 2$, $n \in N$. Then, $\gamma(D) \leqslant n$.*

PROOF. C_n^+ consists of $2n$ vertices out of which 'n' vertices are on the cycle which are of degree three and the remaining 'n' vertices are pendant vertices. The maximum value of $\gamma(D)$ is achieved if either the pendant vertices direction is towards the vertices of the cycle or vice versa.

Hence the proof. □

For the directed thorn ring $C_n^\vee$ we now have the following result.

COROLLARY 3.3. *Let $D = C_n^\vee$, $n > 2$, $n \in N$. Then, $\gamma(D) \leqslant n(t-2)$.*

By Theorem 2.3 we have $\gamma(G) \leqslant \gamma(D)$, where G is the underlying graph of the digraph D. Hence by finding the domination number of G, we can obtain the lower bound of D.

For the undirected graph G which is a thorn rod, thorn star or thorn ring. We now prove the following result.

THEOREM 3.8. *Let $G = P_{p,t}$, $p > 2$, $p, t \in N$. Then, $\gamma(G) = 2 + \lceil \frac{p-4}{3} \rceil$.*

PROOF. For $t = 1$, $P_{p,1} \cong P_p$. Hence $\gamma(G) = \lceil \frac{p}{3} \rceil$.

For $t = 2$, $P_{p,2} \cong P_{p+2}$. Hence $\gamma(G) = \lceil \frac{p+2}{3} \rceil$.

For $t > 2$, label the linear chain as $u_1, u_2, u_3, \ldots, u_p$, left siblings as $l_1, l_2, l_3, \ldots, l_{t-1}$ and right siblings as $r_1, r_2, r_3, \ldots, r_{t-1}$(see Fig. 4). For finding the domination number end vertex of the linear chain has to be labeled '1' i.e. $f(u_1) = f(u_p) = 1$. Therefore the left siblings, right siblings and the vertex adjacent to u_1 and u_p are labeled '0', hence $f(l_i) = 0, \forall i = 1, 2, \ldots, t-1$, $f(r_j) = 0, \forall j = 2, \ldots, t-1$ and $f(u_2) = f(u_{p-1}) = 0$,. The remaining labeling is only for linear chain which is a path P_{p-4}.

Hence the proof. □

THEOREM 3.9. *Let $G = S_{k,t}$, $t > 1$, $t, k \in N$. Then, $\gamma(G) = k$.*

PROOF. For $k = 1$, $S_{k,t} \cong K_{1,t}$. Hence $\gamma(G) = 1$.

For $k > 1$. Label $S_{k,t}$ with k arms as $k_1, k_2, k_3, \ldots, k_k$ with corresponding siblings as $l_1, l_2, l_3, \ldots, l_{t-1}$, $m_1, m_2, m_3, \ldots, m_{t-1}, \ldots$, $r_1, r_2, r_3, \ldots, r_{t-1}$ also the central vertex where the k arms are incident is labeled as u_0 (see Fig. 5). For finding the domination number the vertex of the k arms has to be labeled '1'. i.e. $f(l_i) = f(m_i) = \cdots = f(r_i) = 0 \forall i = 1, 2, \ldots, t-1$ and $f(k_i) = 1$.

Hence the proof. □

THEOREM 3.10. *Let $G = C_n^+$,$n > 2$, $n \in N$. Then, $\gamma(G) = n$.*

PROOF. G consists of 'n' vertices on the cycle and the remaining vertices are pendant vertices. Label the vertices in the cycle as $u_1, u_2, u_3, \ldots, u_n$. For finding the domination number the vertices in the cycle has to be labeled '1'. i.e. $f(u_i) = 1$ and remaining pendant vertices are labeled '0'.

Hence the proof. □

Similar to Theorem 3.10 we now have the following result.

COROLLARY 3.4. *Let $G = C_n^\vee$,$n > 2$, $n \in N$. Then, $\gamma(G) = n$.* □

The results proved in Theorem 3.5 and Theorem 3.8 lead to the following.

COROLLARY 3.5. *Let $D = P_{p,t}$, $p > 2$, $p, t \in N$. Then,*
$2 + \lceil \frac{p-4}{3} \rceil \leqslant \gamma(D) \leqslant (t-1) + \lceil \frac{p}{2} \rceil$. □

The results proved in Theorem 3.6 and Theorem 3.9 lead to the following.

COROLLARY 3.6. *Let $D = S_{k,t}$, $t > 1$, $t, k \in N$. Then, $k \leqslant \gamma(D) \leqslant k(t-1) + 1$.* □

The results proved in Theorem 3.7 and Theorem 3.10 lead to the following.

COROLLARY 3.7. *Let $D = C_n^+$, $n > 2$, $n \in N$. Then, $\gamma(D) = n$.* □

The results proved in Corollarys 3.3 and 3.4 lead to the following.

COROLLARY 3.8. *Let $D = C_n^\vee$, $n > 2$, $n \in N$. Then, $n \leqslant \gamma(D) \leqslant n(t-2)$.* □

4. Conclusion

In this paper we study domination theory on few well known classes of directed trees. Directed trees are extensively used in path algorithm, scheduling problems, data processing networks, data compression, causal structures like family tree, Bayesian network, moral graphs, influence diagram etc. The concept of dominating function plays a significant role in these models.

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