BULLETIN OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4874; ISSN (o) 2303-4955
www.imvibl.org /JOURNALS / BULLETIN
Vol. 6(2016), 135-141

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

NEIGHBOR TOUGHNESS OF GRAPHS

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Abstract. The vulnerability shows the resistance of the network until communication breakdown after the disruption of certain stations or communication links. This study introduces a new vulnerability parameter, neighbor toughness. The neighbor toughness of a graph G is defined as $NT(G) = min\{\frac{|S|}{\omega(G/S)} : \omega(G/S) \geqslant 1\}$, where S is any vertex subversion strategy of G and $\omega(G/S)$ is the number of connected components in the graph G/S. In this paper, the relations between neighbor toughness and other parameters are determined and the neighbor toughness of some specific graphs are obtained.

1. Introduction

Let G be a finite simple graph with vertex set *V(G)* and edge set *E(G)*. The minimum degree in a graph G is denoted $\delta(G)$. A subset S of *V(G)* is called an independent set of G if no two vertices of S are adjacent in G. The independence number of G, $\beta(G)$, is the number of vertices in a maximum independent set of G. The set $N(u) = \{v \in V(G)|v \neq u,\ v \text{ and } u \text{ are adjacent}\}$ is the open neighborhood of u, and $N[u] = u \cup N(u)$ is the closed neighborhood of u. A vertex u in G is said to be subverted if the closed neighborhood of u is removed from G. A set of vertices $S \subseteq V(G)$ is called a vertex subversion strategy of G if each of the vertices in S is subverted from G. By G/S we denote the survival subgraph that remains after each vertex S is subverted from G. A vertex set S is called a cut-strategy of G if the survival subgraphs G/S is disconnected, or is a clique, or is an empty graph.

Graph theory has seen an explosive growth due to interaction with areas like computer science, mathematics, etc. Especially, it has become one of the most powerful mathematical tools in the analysis and study of the architecture of a network. The study of networks has become an important area of multidisciplinary research involving mathematics, informatics, chemistry, social sciences and other theoretical and applied sciences. A network is described as an undirected and

2010 *Mathematics Subject Classification.* 05C40; 68R10; 68M10.
Key words and phrases. Graph theory, Vulnerability, Connectivity, Integrity, Neighbor Integrity, Toughness, Neighbor Toughness.

unweighed graph in which vertices represent the processing and edges represent the communication channel between them [**8, 25**].

It is known that communication systems are often exposed to failures and attacks. The stability of a communication network, composed of processing nodes and communication links, is of prime importance to network designers. As the network begins losing links or nodes, eventually there is a loss in its effectiveness. In the literature, various measures were defined to measure the robustness of network and a variety of graph theoretic parameters have been used to derive formulas to calculate network vulnerability. Graph vulnerability relates to the study of graph when some of its elements (vertices or edges) are removed. The best known measure of reliability of a graph is its connectivity. The vertex (edge) connectivity is defined to be the minimum number of vertices (edges) whose deletion results in a disconnected or trivial graph [**14**]. Then integrity and edge-integrity [**9**], scattering number and edge scattering number [**1, 5, 20**] and toughness and edge-toughness [**11**], etc. have been proposed for measuring the vulnerability of networks. Recently, some average vulnerability parameters such as average lower independence number [**6, 7**], average lower connectivity number [**2**], etc. have been defined. However, most of these parameters do not consider the neighborhoods of the affected vertices. On the other hand, in spy networks, if a spy or a station is captured, then adjacent stations are unreliable. Therefore, neighborhoods should be taken into consideration in spy networks. Gunther and Hartnell [**17, 18, 19**] introduced the idea of modelling a spy network can be modelled by a graph whose vertices represent the agents and whose edges represent lines of communication. Clearly, if a spy is discovered or arrested, the espionage agency can no longer trust any of the spies with whom he or she was in direct communication, and so the betrayed agents become effectively useless to the network as a whole. Such a betrayal is clearly equivalent to the removal of the closed neighborhood of v in the modelling graph, where v is the vertex representing the particular agent who has been subverted. It is clear that to be effective, a spy network must be able to pass messages quickly and easily between its any two agents; it is equally clear, however, that this very need for ease of communication presents great security risks since an agent who knows a lot can also betray a lot [**28**]. Therefore, instead of considering the stability of a communication network in standard sense, some new graph parameters such as vertex-neighbor-connectivity [**16, 17**], vertex-neighbor-integrity [**12, 13**], vertex-neighbor-scattering number [**26**] and vertex-neighbor- rupture degree [**8**] were introduced to measure the stability of communication networks in "*neighbor*" sense. Recent interest in the vulnerability and reliability of networks (communication, computer, transportation) has given rise to a host of other measures, some of which are more global in nature; see, for example, [**3, 4, 10, 15, 21, 22, 23, 24, 25**].

The neighbor connectivity of a graph G is

$$\kappa(G) = min\{|S|\}$$

where S is a subversion strategy of G [**17**].

The neighbor integrity of a graph G is defined to be

$$NI(G) = min\{|S| + c(G/S)\}$$

where S is any vertex subversion strategy of G and $c(G/S)$ is order of the largest connected component of G/S [**12**].

The neighbor scattering number of a graph G is defined as

$$S(G) = max\{\omega(G/S) - |S| : \omega(G/S) \geqslant 1\}$$

where S is any vertex subversion strategy of G and $\omega(G/S)$ is the number of connected components in the graph G/S [**26**].

The neighbor rupture degree of a noncomplete connected graph G is defined to be

$$Nr(G) = max\{\omega(G/S) - |S| - c(G/S) : S \subset V(G), \omega(G/S) \geqslant 1\}$$

where S is any vertex subversion strategy of G and $\omega(G/S)$ is the number of connected components in the graph G/S and $c(G/S)$ is the maximum order of the components of G/S [**8**].

The toughness of a graph G is defined as

$$t(G) = min\{\frac{|S|}{\omega(G-S)} : S \subseteq V(G) \ and \ \omega(G-S) > 1\}$$

where $\omega(G - S)$ denotes the number of components in $G - S$ [**11**].

The paper is organized as follows. In Section 2, we introduce a new vulnerability parameter, the neighbor toughness. Also, we establish relationships between the edge scattering number and some other graph parameters. In Section 3, we compute the neighbor toughness of some special graphs. Conclusions are addressed in Section 4.

2. Neighbor Toughness

We now introduce a new stability measure.

Definition 1. The neighbor toughness of a graph G is defined as

$$NT(G) = min\{\frac{|S|}{\omega(G/S)} : \omega(G/S) \geqslant 1\}$$

where S is any vertex subversion strategy of G and $\omega(G/S)$ is the number of connected components in the graph G/S. In particular, the neighbor toughness of a complete graph K_n is defined to be 0.

Definition 2. A cut-strategy S of G is called an *NT-set* of G if $NT(G) = \frac{|S|}{\omega(G/S)}$.

Similarly to the relation between the neighbor toughness and the neighbor scattering number also differ in showing the vulnerability of networks. This can be shown as follows. For example, consider the graphs G_1 and G_2 in Figure 1.

It can be easily seen that the neighbor scattering number of these graphs are equal.

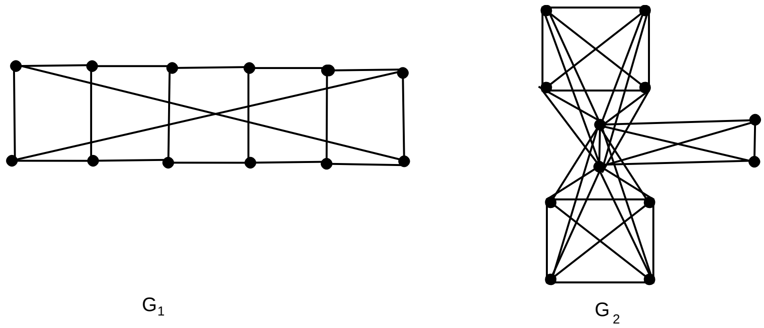

FIGURE 1. The graphs G_1 and G_2.

$$S(G_1) = S(G_2) = 1$$

On the other hand, the neighbor toughness of G_1 and G_2 are different.

$$NT(G_1) = \tfrac{2}{3}$$
$$NT(G_2) = \tfrac{1}{2}$$

Thus, the neighbor toughness is a better parameter then the neighbor scattering number these two graphs. As a new graph parameter, neighbor toughness can be used to measure the vulnerability of spy networks. From the definition of neighbor toughness we know that, in general, the more the neighbor toughness of a graph is, the more stable the graph is.

Next, we give some upper and lower bounds for neighbor toughness via some other well-known graph parameters.

THEOREM 1. *Let G be a connected graph of order n. Then,*

$$NT(G) \leqslant \kappa(G).$$

PROOF. Let S be a cut-strategy of G with $|S| = \kappa(G)$. For any graph G, we have $\omega(G/S) \geqslant 1$. Hence we get

$$\frac{|S|}{\omega(G/S)} \leqslant \kappa(G).$$

Then, by the definition of neighbor toughness, $NT(G) \leqslant \kappa(G)$. □

THEOREM 2. *For any graph G,*

$$NT(G) \leqslant \delta(G).$$

PROOF. For any graph G, we have $\delta(G) \geqslant \kappa(G)$. By Theorem 1 we know $NT(G) \leqslant \kappa(G)$. Thus,

$$NT(G) \leqslant \kappa(G) \leqslant \delta(G).$$

Hence the proof is completed. □

LEMMA 1. [**27**] *For any graph G,*

$$\kappa(G) \leqslant NI(G).$$

The following theorem is easily obtained from Lemma 1 and Theorem 1.

THEOREM 3. *For any graph G,*

$$NT(G) \leqslant NI(G).$$

3. Neighbor Toughness of Some Specific Graphs

In this section, we consider the neighbor toughness of some graphs.

THEOREM 4. *Let P_n be a path graph of order $n(\geqslant 3)$. Then,*

$$NT(P_n) = \begin{cases} 1, & \text{if } n = 3,4; \\ \frac{1}{2}, & \text{if } n \geqslant 5. \end{cases}$$

PROOF. The case $n = 3,4$ is trivial, so we assume $n \geqslant 5$. Let S be a cut-strategy of P_n and $|S| = r$. If we remove r vertices from P_n, then we have $\omega(P_n/S) \leqslant r+1$. So,

$$\frac{|S|}{\omega(P_n/S)} \geqslant \frac{r}{r+1}$$
$$NT(P_n) \geqslant \min_r\{\frac{r}{r+1}\}$$

the function $f(r)$ takes its minimum value at $r = 1$, and we get $NT(P_n) \geqslant \frac{1}{2}$.

It can be easily seen that there exist a vertex v in P_n such that $\omega(P_n/S) = 2$. Then, by the definition of neighbor toughness, $NT(P_n) = \frac{1}{2}$.
Hence the proof is completed. □

THEOREM 5. *Let C_n be a cycle graph of order $n(\geqslant 4)$. Then,*

$$NT(C_n) = \begin{cases} 2, & \text{if } n = 6,7; \\ 1, & \text{if } n = 4,5 \text{ or } n \geqslant 8. \end{cases}$$

PROOF. The case $n = 4,5,6,7$ is trivial, so we assume $n \geqslant 8$. Let S be a cut-strategy of C_n. If $|S| = r$ then we have $\omega(C_n/S) \leqslant r$. Thus,

$$\frac{|S|}{\omega(C_n/S)} \geqslant \frac{r}{r}.$$

So, we get $NT(C_n) \geqslant 1$.

It is clear that there is a cut strategy of C_n such that $|S| = 2$ and $\omega(C_n/S) = 2$. From the definition of neighbor toughness we have, $NT(C_n) = 1$. □

THEOREM 6. *Let $K_{m,n}$ be a complete bipartite graph. Then,*

$$NT(K_{m,n}) = \begin{cases} \frac{1}{m-1}, & \text{if } n < m; \\ \frac{1}{n-1}, & \text{if } n \geqslant m. \end{cases}$$

PROOF. We assume $n < m$. Let vertices set of $K_{m,n}$ be $V(G) = V(G_1) \cup (G_2)$ where $V(G_1)$: The set contains m vertices with degree n, $V(G_2)$: The set contains n vertices with degree m. Let S be a cut-strategy of $K_{m,n}$ and $|S| = r$. If we remove r vertices from $K_{m,n}$, then we have $\omega(K_{m,n}/S) \leqslant m - 1$. Thus,

$$\frac{|S|}{\omega(K_{m,n} - S)} \geqslant \frac{1}{m-1}$$

so, we get $NT(K_{m,n}) \geqslant \frac{1}{m-1}$

It can be easily seen that there exist a vertex $v \in V(G_1)$ such that $\omega(K_{m,n}/S) = m - 1$. Then, by the definition of neighbor toughness, $NT(K_{m,n}) = \frac{1}{m-1}$.

Similarly, we obtain $NT(K_{m,n}) = \frac{1}{n-1}$ when $n \geqslant m$. Finally we have

$$NT(K_{m,n}) = \begin{cases} \frac{1}{m-1}, & \text{if } n < m; \\ \frac{1}{n-1}, & \text{if } n \geqslant m. \end{cases}$$

The proof is completed. □

COROLLARY 1. *The neighbor toughness of the star graph $K_{1,n}$ is*

$$NT(K_{1,n}) = \frac{1}{n-1}.$$

4. Conclusion

Reliability and efficiency are important criteria in the design of networks. When we want to design a network, we wish that it is as stable as possible. Any network can be modelled as a connected graph. We investigate a new measure for reliability of a graph called the neighbor toughness which is recently introduced. From the definition of neighbor toughness we know that, in general, the more the neighbor toughness of a graph is, the more stable the graph is.

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Received by editors 12.02.2016; Available online 21.03.2016.

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