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DECOMPOSITION OF GRAPH INTO DIAMETRAL PATHS

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ABSTRACT. The diametral path of a graph is the shortest path between two vertices which has length equal to diameter of that graph. Diametral path decomposition is introduced as a collection of edge-disjoint diametral paths of a graph so that every edge of the graph appears in exactly one diametral path. Diametral path decomposition number d_e is the cardinality of such a collection. The diametral path decomposition index D_e is the number of such decompositions. In this paper, results on d_e and D_e in some classes of graphs are presented. Also graphs which admit diametral path decomposition are charecterized.

1. INTRODUCTION

The fundamental concept of path decomposition in graphs as introduced by Harary [**6, 8**] continues to be of interest to researchers due to its wide range of applications in real life. The study on decomposition in the context of diametral paths helps us to understand, analyse and design networks effectively. Research in this area helps us analyse problems in transportation, distribution, designing, communication, team formation and event management. Extensive research has been dedicated to the study of various types of decompositions and related parameters in [**1, 2, 3, 4, 9**] in context of paths, cycles and common vertices between the paths. In this paper, a study on decomposition involving diametral paths in simple, connected, undirected and unweighted graphs is undertaken. In section 2, preliminaries relevant to this study are discussed. In section 3, diametral path decomposition is introduced. Also diametral path decomposition number and index

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for some classes of graphs are discussed. In section 4, bounds on number of vertices, edges in graphs which admit diametral path decomposition are proposed.

2. PRELIMINARIES

The definitions and results are in accordance with [**5, 7**]. The length of a path is the number of edges on the path. The distance between two vertices in a graph is the length of shortest path between them. The eccentricity of a vertex is the maximum of distances from it to all the other vertices of that graph. While diameter is the maximum of the eccentricities of all vertices of that graph, the radius is minimum of these. Peripheral vertices are vertices of maximum eccentricity and central vertices are of minimum eccentricity. The diametral path of a graph is the shortest path between two vertices which has length equal to diameter of that graph.
Given below are a few standard results in certain classes of graphs.

(i) Complete graph K_n: diam(K_n) = 1 where n $\geqslant$ 2.
(ii) Wheel W_n: diam(W_n)= 2 where n $\geqslant$ 5.
(iii) Star $K_{1,n}$: diam($K_{1,n}$) = 2 where n $\geqslant$ 2.
(iv) Complete bipartite graph $K_{r,s}$: diam($K_{r,s}$) = 2 where r $\geqslant$ 2 or s $\geqslant$ 2.
(v) Path P_n: diam(P_n) = n-1 and rad(P_n) = $\lfloor n/2 \rfloor$
(vi) Cycle C_n: diam(C_n) = $\lfloor n/2 \rfloor$ where n $\geqslant$ 3.

3. RESULTS ON d_e AND D_e

Definition 3.1. Diametral path decomposition is a collection of edge-disjoint diametral paths of a graph so that every edge of the graph appears in exactly one diametral path.

Definition 3.2. Diametral path decomposition number d_e is the cardinality of such a collection.

Definition 3.3. Diametral path decomposition index D_e is the number of such decompositions.

Proposition 3.1. *Every path $P_n (n \geqslant 2)$ admits a diametral path decomposition. Also $d_e(P_n) = 1 = D_e(P_n)$.*

Proof. Since every path is the diametral path itself, there exists a decomposition which has only one element which is the path itself. Also there is only one such decomposition. Hence $d_e(P_n) = 1 = D_e(P_n)$. □

Theorem 3.1. *(i) Cycle C_3 admits a diametral path decomposition. Also $d_e(C_3) = 3$ and $D_e(C_3) = 1$.*
(ii) If n is even, cycle C_n admits a diametral path decomposition. Also $d_e(C_n) = 2$ and $D_e(C_n) = \frac{n}{2}$.

Proof. (i) Since $diam(C_3) = 1$, each edge is a diametral path.
We get a decomposition by taking all the edges. Since there are 3 edges in C_3, we can conclude that $d_e(C_3) = 3$.

Also $D_e(C_3) = 1$, as there is only one such decomposition which has all the edges.

(ii) Since $diam(C_n) = d = \lfloor \frac{n}{2} \rfloor, n = m$ and n is even, we get $d = \frac{n}{2}$ or $n = 2d$ or $m = 2d$. Since $m = 2d, m$ edges can be distributed in 2 diametral paths. There exists a decomposition by taking these 2 diametral paths. Hence $d_e(C_n) = 2$. Also each decomposition has two diametral paths and they are between a specific pair of vertices. Since there are $\frac{n}{2}$ pairs of vertices of that kind, there are $\frac{n}{2}$ decompositions. Hence $D_e(C_n) = \frac{n}{2}$.

□

EXAMPLE 3.1. Consider cycle C_6 in Figure 1. It can be noted that $diam(C_6) = 3$. It admits a diametral path decomposition with $d_e(C_6) = 2$. The decomposition can be taken as $\{(A, B, C, D), (A, F, E, D)\}$. Also $D_e(C_6) = \frac{6}{2} = 3$ The decompositions are $\{(A, B, C, D), (A, F, E, D)\}, \{(B, C, D, E), (B, A, F, E)\}$ and $\{(C, D, E, F), C, B, A, F)\}$.

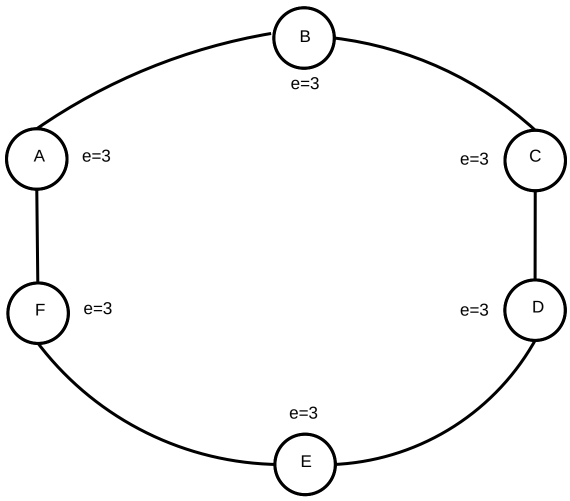

FIGURE 1. Cycle C_6

THEOREM 3.2. *If n is odd, then wheel $W_n (n \geqslant 5)$ admits a diametral path decomposition. Also $d_e(W_n) = n - 1$.*

PROOF. Since $diam(W_n) = 2$ and n is odd, there are $n - 1$(even) peripheral vertices and one central vertex. Hence there are $n-1$ edges with peripheral vertices as end vertices and $n-1$ edges with central vertex as one end vertex and peripheral vertex as the other end vertex. There exists a decomposition by taking $\frac{(n-1)}{2}$ diametral paths through the peripheral vertices and $\frac{(n-1)}{2}$ diametral paths through the central vertex. Hence $d_e(W_n) = \frac{(n-1)}{2} + \frac{(n-1)}{2} = n - 1$. □

EXAMPLE 3.2. Consider wheel W_5 in Figure 2. It can be noted that $diam(W_5) = 2$. It admits a diametral path decomposition with $d_e(W_5) = 5 - 1 = 4$. The decomposition can be taken as $\{(A, B, C), (A, D, C), (A, E, C), (B, E, D)\}$.

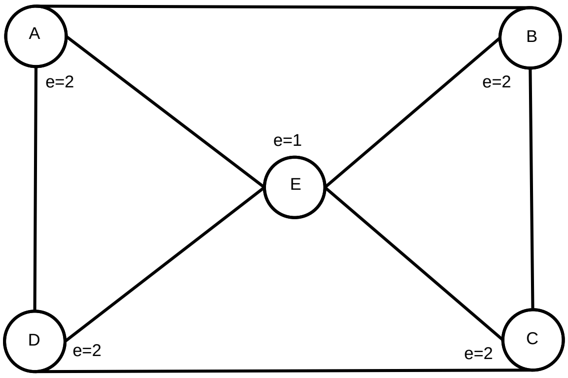

FIGURE 2. Wheel W_5

THEOREM 3.3. *If n is even, then star $K_{1,n}(n \geqslant 2)$ admits a diametral path decomposition. Also $d_e(K_{1,n}) = \frac{n}{2}$ and $D_e(K_{1,n}) = (n-1)(n-3)(n-5)\ldots 1$.*

PROOF. Since $diam(K_{1,n}) = 2$, there are n peripheral vertices and one central vertex. Also there are n edges with central vertex as one end vertex and peripheral vertex as the other end vertex. Since n is even and every two edges form a diametral path, there exists a decomposition by taking $\frac{n}{2}$ diametral paths through the central vertex. Hence $d_e(K_{1,n}) = \frac{n}{2}$.

In forming a decomposition, we include diametral paths one after the other so that all edges are represented. Beginning with any peripheral vertex, there are $(n-1)$ ways in which another peripheral vertex is chosen to form a diametral path. When we include that diametral path in the collection, there are $(n-2)$ peripheral vertices left. Taking any one of the remaining peripheral vertices, there are $(n-3)$ ways in which another peripheral vertex is chosen to form a diametral path. Proceeding this way till all edges are included, we get a decomposition. Hence the number of decompositions $= D_e(K_{1,n}) = (n-1)(n-3)(n-5)\ldots 1$. □

EXAMPLE 3.3. Consider star $K_{1,4}$ in Figure 3. It admits a diametral path decomposition with $d_e(K_{1,4}) = \frac{4}{2} = 2$ and has a decomposition $\{(A, E, C), (B, E, D)\}$. Also $D_e(K_{1,4}) = (4-1)(4-3)\ldots 1 = 3$. The decompositions are

$$\{(A, E, B), (C, E, D)\}, \{(A, E, C), (B, E, D)\}$$

and

$$\{(A, E, D), (B, E, C)\}.$$

PROPOSITION 3.2. *Complete graph $K_n(n \geqslant 2)$ admits a diametral path decomposition. Also $d_e(K_n) = n_{C_2}$ and $D_e(K_n) = 1$.*

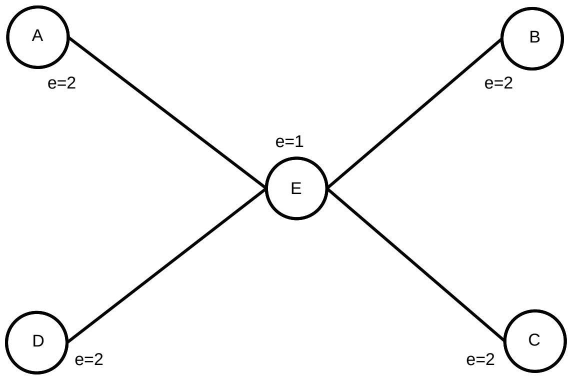

FIGURE 3. Star $K_{1,4}$

PROOF. Since $diam(K_n) = 1$, every edge is a diametral path. Hence there exists a decomposition with all the edges. Also $d_e(K_n) = n_{C_2}$, as there are n_{C_2} edges in K_n. As there is only one such decomposition which has all the edges, $D_e(K_n) = 1$. □

EXAMPLE 3.4. Consider complete graph K_4 in Figure 4. It admits a diametral path decomposition with $d_e(K_4) = 4_{C_2} = 6$. The decomposition is

$$\{(A, B), (B, C), (C, D), (D, A), (A, C), (B, D)\}.$$

Also $D_e(K_4) = 1$, as there is only one such decomposition.

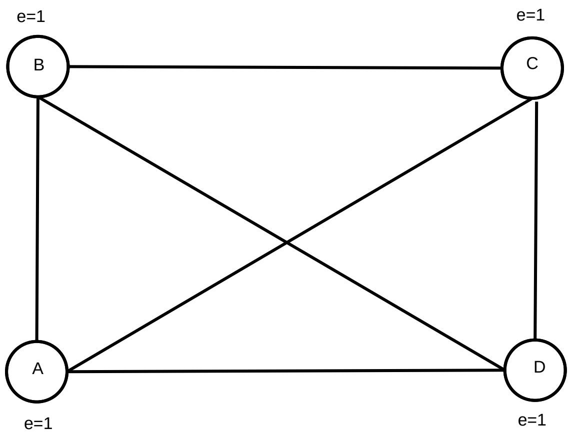

FIGURE 4. Complete Graph K_4

THEOREM 3.4. *If r or s is even, then complete bipartite graph $K_{r,s}(r \geqslant 2$ and $s \geqslant 2)$ admits a diametral path decomposition. Also $d_e(K_{r,s}) = \frac{rs}{2}$.*

PROOF. When r or s is even, rs the number of edges is even. Since $diam(K_{r,s}) = 2$, there exists a decomposition by taking $\frac{rs}{2}$ diametral paths so each edge is uniquely placed in one of them. Hence $d_e(K_{r,s}) = \frac{rs}{2}$. □

EXAMPLE 3.5. Consider complete bipartite graph $K_{2,3}$ in Figure 5. It has a diametral path decomposition with $d_e(K_{2,3}) = \frac{rs}{2} = \frac{6}{2} = 3$. The decomposition is $\{(A, C, B), (A, D, B), (A, E, B)\}$.

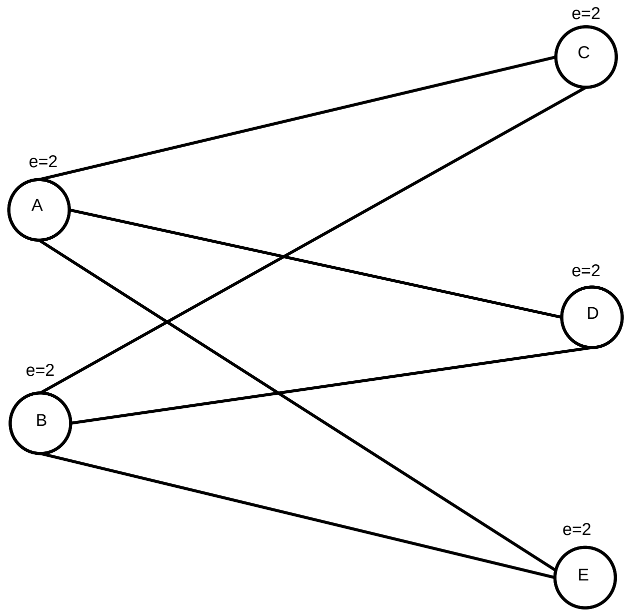

FIGURE 5. Complete bipartite graph $K_{2,3}$

4. BOUNDS

THEOREM 4.1. *If a graph G admits a diametral path decomposition with $d_e(G) < n_{C_2}$, then G is K_n free.*

PROOF. Suppose G has K_n. n_{C_2} edges of K_n are in G. Since $d_e(G) < n_{C_2}$, there should be a diametral path with two or more edges of K_n. Since there is an edge between every pair of vertices in K_n, a diametral path cannot have two or more edges of K_n. This is impossible. Hence we can conclude that G cannot have K_n. Hence G is K_n free. □

THEOREM 4.2. *If a tree with r pendant vertices admits a diametral path decomposition, then*

(i) Every pendant vertex is a peripheral vertex.
(ii) The number of pendant vertices is even.
(iii) $d_e = \frac{r}{2}$.

PROOF. Consider a tree with r pendant vertices which admits diametral path decomposition.

(i) Since every edge is uniquely placed in one of the diametral paths, the pendant edge appears in some diametral path. Since it is a pendant edge, it can appear only at the end of the diametral path. Hence the pendant vertex is a peripheral vertex.

(ii) Since pendant vertices are end vertices of the diametral path and they cannot appear in two diametral paths, the number of pendant vertices is even.

(iii) Since there are r pendant vertices and every two pendant vertices have a diametral path between them, the number of diametral paths in the decomposition $= d_e = \frac{r}{2}$.

□

THEOREM 4.3. *If a graph $G(n, m)$ with $diam(G) = d$ admits a diametral path decomposition with $d_e(G) = k$, then*

(i) $d + 1 \leqslant n < k(d + 1)$.

(ii) $m = kd$.

(iii) If $d_e(G) = 1$, then G is a path.

PROOF. Let $diam(G) = d$ and $d_e(G) = k$.

(i) Since $diam(G) = d$, there is atleast one diametral path. Hence $n \geqslant d + 1$, as there are atleast $d + 1$ vertices of the diametral path. Since $diam(G) = d$ and $d_e(G) = k$, there are k number of diametral paths with $d+1$ vertices on each. But if $n = k(d+1)$, the graph would be a disconnected graph as the diametral paths do not have a common vertex or an edge. Hence $n < k(d + 1)$. Hence we can conclude that $d + 1 \leqslant n < k(d + 1)$.

(ii) Since $diam(G) = d$ and $d_e(G) = k$, there are k number of diametral paths with d edges in each. Since repetition of edges is not allowed, $m = kd$.

(iii) Since $d_e(G) = 1$, there is only one diametral path with all the edges of the graph. Since the graph is the diametral path itself, G is a path.

□

5. CONCLUSION

In this paper, a study has been undertaken on how to partition edges in such a way that every edge appears in some diametral path and no edge appears in two diametral paths in a collection of diametral paths named diametral path decomposition. Further to this study, the focus would be to find the number of diametral path decompositions for wheels $W_n (n \geqslant 5)$ where n is odd and complete bipartite graphs $K_{r,s}$ where r or s is even. Also the future work would be to identify applications of this concept.

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