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A Note on Coefficient Inequalities for (j,i)-Symmetrical Functions with Conic Regions

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Abstract. In this note, the concepts of (j,i)-symmetric points, Janowski functions and the conic regions are combined to define a class of functions in a new interesting domain which represents the conic type regions. Certain interesting coefficient inequalities are deduced.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z : z \in \mathbb{C}\ and\ |z| < 1\}$, and $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of all function which are univalent in $\mathcal{U}$. Given two functions f and g analytic in $\mathcal{U}$, we say that the function f is subordinate to g in $\mathcal{U}$ and write $f(z) \prec g(z)$, if there exists a Schwarz function w, which is analytic in $\mathcal{U}$ with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z))$, $z \in \mathcal{U}$. If g is univalent in $\mathcal{U}$ then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$.
Using the principle of the subordination we define the class $\mathcal{P}$ of functions with positive real part see [**2**].

Definition 1.1. Let $\mathcal{P}$ denote the class of analytic functions of the form $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ defined on $\mathcal{U}$ and satisfying $p(0) = 1$, $\Re\{p(z)\} > 0,\ z \in \mathcal{U}$.

Any function p in $\mathcal{P}$ has the representation $p(z) = \frac{1+w(z)}{1-w(z)}$ where $w(0) = 0$, $|w(z)| < 1$ on $\mathcal{U}$. The class of functions with positive real part $\mathcal{P}$ plays a crucial role in geometric function theory. Its significance can be seen from the fact

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that simple subclasses like class of starlike $\mathcal{S}^*$, class of convex functions $\mathcal{C}$, class of starlike functions with respect to symmetric points have been defined by using the concept of class of functions with positive real part.

DEFINITION 1.2. [**1**] Let $\mathcal{P}[A,B]$, where $-1 \leqslant B < A \leqslant 1$, denote the class of analytic function p defined on $\mathcal{U}$ with the representation $p(z) = \frac{1+Aw(z)}{1+Bw(z)},\ z \in \mathcal{U}$, $w(0) = 0,\ |w(z)| < 1$. $p \in \mathcal{P}[A,B]$ if and only if $p(z) \prec \frac{1+Az}{1+Bz}$.

Geometrically, a function $p(z) \in \mathcal{P}[A,B]$ maps the open unit onto the disk defined by the domain,

$$\Omega[A,B] = \left\{ w : \left| w - \frac{1-AB}{1-B^2} \right| < \frac{A-B}{1-B^2} \right\}.$$

The class $P[A,B]$ is connected the class $\mathcal{P}$ of function with positive real parts by the relation,

$$p(z) \in \mathcal{P} \Leftrightarrow \frac{(A+1)p(z)-(a-1)}{(B+1)p(z)-(B-1)} \in \mathcal{P}[A,B].$$

This class was introduced by Janowski [**1**] and then studied by several authors, Kanas and Wisniowska [**3, 8**] introduced and studied the class $k-UCV$ of k-uniformly convex functions and the corresponding class $k-ST$ of k-starlike functions. These classes were defined subject to the conic region $\Omega_k,\ k \geqslant 0$ [**3, 8**] as

$$\Omega_k = \{u+iv : u > k\sqrt{(u-1)^2+v^2}\}.$$

This domain represents the right half plane for $k=0$, hyperbola for $0<k<1$, a parabola for $k=1$ and ellipse for $k>1$.

The functions $p_k(z)$ which play the role of extremal functions for these conic regions are given as

$$p_k(z) = \begin{cases} \frac{1+z}{1-z},\ k=0 \\ 1+\frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2,\ k=1. \\ 1+\frac{2}{1-k^2}\sinh^2\left[\left(\frac{2}{\pi}\arccos\ k\right)\arctan h\sqrt{z}\right],\ 0<k<1. \\ 1+\frac{2}{k^2-1}\sin\left[\frac{\pi}{2R(t)}\int_0^{\frac{u(z)}{\sqrt{t}}}\frac{1}{\sqrt{1-x^2}\sqrt{1-(tx)^2}}dx\right]+\frac{1}{k^2-1},\ k>1, \end{cases} \tag{1.2}$$

where $u(z) = \frac{z-\sqrt{t}}{1-\sqrt{tx}},\ t \in (0,1),\ z \in \mathcal{U}$ and z is chosen such that $k = \cosh\left(\frac{\pi R'(t)}{4R(t)}\right)$, $R(t)$ is the Legendre's complete elliptic integral of the first kind and $R'(t)$ is complementary integral $R(t)$; $p_k(z) = 1+\delta_k z + ...$, in [**7**] where

$$\delta_k = \begin{cases} \frac{8(\arccos\ k)^2}{\pi^2(1-k^2)},\ 0 \leqslant k < 1 \\ \frac{8}{\pi^2},\ k=1. \\ \frac{\pi^2}{4(k^2-1)\sqrt{t}(1+t)R^2(t)},\ k>1. \end{cases} \tag{1.3}$$

DEFINITION 1.3. A function p is said to be in the class $k-P[A,B]$, $-1\leqslant B<A\leqslant 1$, $k\geqslant 0$, if and only if,

$$p(z)\prec \frac{(A+1)p_k(z)-(A-1)}{(B+1)p_k(z)-(B-1)},\ ,$$

where $p_k(z)$ is defined by (1.2) and .

DEFINITION 1.4. Let i be a positive integer. A domain $\mathcal{D}$ is said to be i-fold symmetric if a rotation of $\mathcal{D}$ about the origin through an angle $\frac{2\pi}{i}$ carries $\mathcal{D}$ onto itself. A function f is said to be i-fold symmetric in $\mathcal{U}$ if for every z in $\mathcal{U}$

$$f(e^{\frac{2\pi\mathbf{i}}{i}}z)=e^{\frac{2\pi\mathbf{i}}{i}}f(z).$$

The family of all i-fold symmetric functions is denoted by $\mathcal{S}^i$ and for $i=2$ we get class of the odd univalent functions.

The notion of (j,i)-symmetrical functions $(i=2,3,...\ ; j=0,1,2,...,i-1)$ is a generalization of the notion of even, odd, i-symmetrical functions and also generalize the well-known result that each function defined on a symmetrical subset can be uniquely expressed as the sum of an even function and an odd function. The theory of (j,i) symmetrical functions has many interesting applications, for instance in the investigation of the set of fixed points of mappings, for the estimation of the absolute value of some integrals, and for obtaining some results of the type of Cartan uniqueness theorem for holomorphic mappings [**12**].

DEFINITION 1.5. Let $\varepsilon=(e^{\frac{2\pi\mathbf{i}}{i}})$ and $j=0,1,2,..,i-1$ where $i\geqslant 2$ is a natural number. A function $f:\mathcal{U}\mapsto\mathbb{C}$ is called (j,i)-symmetrical if

$$f(\varepsilon z)=\varepsilon^j f(z),\ z\in\mathcal{U}.$$

The family of all (j,i)-symmetrical functions is denoted be $\mathcal{S}^{(j,i)}$. Also $\mathcal{S}^{(0,2)}$, $\mathcal{S}^{(1,2)}$ and $\mathcal{S}^{(1,i)}$ are called even, odd and i-symmetric functions respectively. We have the following decomposition theorem.

THEOREM 1.1. [**12**] *For every mapping $f:\mathcal{D}\mapsto\mathbb{C}$, and $\mathcal{D}$ is a i-fold symmetric set, there exists exactly a unique sequence of (j,i)- symmetrical functions $f_{j,i}$,*

$$f(z)=\sum_{j=0}^{i-1} f_{j,i}(z). \tag{1.4}$$

From (1.4) we have

$$f_{j,i}(z)=\frac{1}{i}\sum_{v=0}^{i-1}\varepsilon^{-vj}f(\varepsilon^v z)=\frac{1}{i}\sum_{v=0}^{i-1}\varepsilon^{-vj}\left(\sum_{n=1}^{\infty}a_n(\varepsilon^v z)^n\right),$$

then
(1.5)

$$f_{j,i}(z)=\sum_{n=1}^{\infty}\psi_n a_n z^n,\ a_1=1 \qquad \psi_n=\frac{1}{i}\sum_{v=0}^{i-1}\varepsilon^{(n-j)v}=\begin{cases}1, & n=li+j;\\ 0, & n\neq li+j;\end{cases},$$

where

$$(f \in \mathcal{A};\ i = 1, 2, ...; j = 0, 1, 2, ..., i-1).$$

Definition 1.6. A function $f \in \mathcal{A}$ is said to be in the class $k - UC$-uniformly convex if and only if,

$$\Re\left(1 + \frac{(zf'(z))'}{f'(z)}\right) \geqslant k\left|\frac{zf''(z)}{f(z)} - 1\right|,\ z \in \mathcal{U},\ k \geqslant 0.$$

Definition 1.7. A function $f \in \mathcal{A}$ is said to be in the class $k - US$, if and only if,

$$\Re\left(\frac{zf'(z)}{f(z)}\right) \geqslant k\left|\frac{zf'(z)}{f(z)} - 1\right|,\ z \in \mathcal{U},\ k \geqslant 0.$$

Now using the concepts $(j, i)-$symmteric points we define the following

Definition 1.8. A function $f \in \mathcal{A}$ is said to be in the class $k - \mathcal{V}^{(j,i)}[A, B]$, $k \geqslant 0$,
$-1 \leqslant B < A \leqslant 1$, if and only if

$$\Re\left(\frac{(B-1)\frac{zf'(z)}{f_{j,i}(z)} - (A-1)}{(B+1)\frac{zf'(z)}{f_{j,i}(z)} - (A+1)}\right) > k\left|\frac{(B-1)\frac{zf'(z)}{f_{j,i}(z)} - (A-1)}{(B+1)\frac{zf'(z)}{f_{j,i}(z)} - (A+1)} - 1\right|, \tag{1.6}$$

or equivalently,

$$\frac{zf'(z)}{f_{j,i}(z)} \in k - P[A, B].$$

Definition 1.9. A function $f \in \mathcal{A}$ is said to be in the class $k - \mathcal{CV}^{(j,i)}[A, B]$, $k \geqslant 0$,
$-1 \leqslant B < A \leqslant 1$, if and only if

$$\Re\left(\frac{(B-1)\frac{(zf'(z))'}{f'_{j,i}(z)} - (A-1)}{(B+1)\frac{(zf'(z))'}{f'_{j,i}(z)} - (A+1)}\right) > k\left|\frac{(B-1)\frac{(zf'(z))'}{f'_{j,i}(z)} - (A-1)}{(B+1)\frac{(zf'(z))'}{f'_{j,i}(z)} - (A+1)} - 1\right|, \tag{1.7}$$

or equivalently,

$$\frac{(zf'(z))'}{f'_{j,i}(z)} \in k - P[A, B].$$

The following special cases are of interest:

(**i**) $k - \mathcal{V}^{(1,i)}[A, B] = k - ST[A, B, i],\ k - \mathcal{CV}^{(1,i)}[A, B] = k - UCV[A, B, i]$
the classes introduced by Fuad Alsarari and Latha [**4**].

(**ii**) $k - \mathcal{V}^{(1,1)}[A, B] = k - ST[A, B], k - \mathcal{CV}^{(1,1)}[A, B] = k - UCV[A, B]$
the classes introduced by Khalida Inyat Noor and Sarfraz Nawaz Malik [**6**].

(**iii**) $k - \mathcal{V}^{(1,1)}[A, B] = k - \mathcal{V}_b^{\sigma}(A, B)$
the class introduced by Fuad Alsarari and Latha [**11**].

(**iv**) $k - \mathcal{V}^{(1,1)}[1, -1] = k - ST, k - \mathcal{CV}^{(1,1)}[1, -1] = k - UCV$
the well-known classes introduced by Kanas and Wisniowska [**8**].

(v) $0-\mathcal{V}^{(j,i)}[A,B]=\mathcal{S}^{(j,i)}[A,B]$
the class introduced by Fuad Alsarari and S. Latha [**10**].
(vi) $0-\mathcal{V}^{(1,1)}[A,B]=S^{*}[A,B],\ 0-\mathcal{CV}^{(1,1)}[A,B]=C[A,B]$
the well-known classes introduced by Janowski [**1**].

We need the following lemmas to prove our main results

LEMMA 1.1. [**13**] *Let $h(z)=1+\sum_{n=1}^{\infty}c_nz^n$ be subordinate to $H(z)=1+\sum_{n=1}^{\infty}b_nz^n$. If $H(z)$ is univalent in $\mathcal{U}$ and $H(z)$ is convex, then*

$$|c_n|\leqslant|b_1|,\ n\geqslant 1.$$

LEMMA 1.2. [**6**] *Let $h(z)=1+\sum_{n=1}^{\infty}c_nz^n\in P[A,B]$. Then*

$$|c_n|\leqslant|\delta_{AB}|,\ |\delta_{AB}|=\frac{(A-B)|\delta_k|}{2},$$

where δ_k is defined by (1.3).

2. Main results

THEOREM 2.1. *A function $f\in\mathcal{A}$ and of the form* (1.1) *is in the class $k-\mathcal{V}^{(j,i)}[A,B]$, if it satisfies the condition*

$$\sum_{n=2}^{\infty}[2(k+1)(n-\psi_n)+|n(B+1)-\psi_n(A+1)|]|a_n|$$

$$<|(B+1)-(A+1)\psi_1|-2(k+1)|\psi_1-1|, \tag{2.1}$$

where $-1\leqslant B<A\leqslant 1$, $k\geqslant 0$, and ψ_n is defined by (1.5).

PROOF. Assuming that (2.2) holds, then it suffices to show that

$$k\left|\frac{(B-1)\frac{zf'(z)}{f_{j,i}(z)}-(A-1)}{(B+1)\frac{zf'(z)}{f_{j,i}(z)}-(A+1)}-1\right|-\Re\left[\frac{(B-1)\frac{zf'(z)}{f_{j,i}(z)}-(A-1)}{(B+1)\frac{zf'(z)}{f_{j,i}(z)}-(A+1)}-1\right]<1,$$

we get

$$k\left|\frac{(B-1)\frac{zf'(z)}{f_{j,i}(z)}-(A-1)}{(B+1)\frac{zf'(z)}{f_{j,i}(z)}-(A+1)}-1\right|-\Re\left[\frac{(B-1)\frac{zf'(z)}{f_{j,i}(z)}-(A-1)}{(B+1)\frac{zf'(z)}{f_{j,i}(z)}-(A+1)}-1\right].$$

$$\leqslant(k+1)\left|\frac{(B-1)zf'(z)-(A-1)f_{j,i}(z)}{(B+1)zf'(z)-(A+1)f_{j,i}(z)}-1\right|$$

$$=2(k+1)\left|\frac{f_{j,i}(z)-zf'(z)}{(B+1)zf'(z)-(A+1)f_{j,i}(z)}\right|.$$

$$=2(k+1)\left|\frac{(\psi_1-1)z+\sum_{n=2}^{\infty}(\psi_n-n)a_nz^n}{[(B+1)-(A+1)\psi_1]z+\sum_{n=2}^{\infty}[n(B+1)-\psi_n(A+1)]a_nz^n}\right|.$$

$$\leqslant 2(k+1)\frac{|\psi_1 - 1| + \sum_{n=2}^{\infty}|\psi_n - n||a_n|}{|(B+1)-(A+1)\psi_1| - \sum_{n=2}^{\infty}|n(B+1)-\psi_n(A+1)||a_n|}.$$

The last expression is bounded above by 1, then

$$\sum_{n=2}^{\infty}[2(k+1)|\psi_n - n| + (1-\gamma)|n(B+1)-\psi_n(A+1)|]|a_n| < \\ |(B+1)-(A+1)\psi_1| - 2(k+1)|\psi_1 - 1|,$$

and this completes the proof. □

THEOREM 2.2. *A function* $f \in \mathcal{A}$ *and of the form* (1.1) *is in the class* $k - \mathcal{CV}^{(j,i)}[A,B]$, *if it satisfies the condition*

$$\sum_{n=2}^{\infty} n\{[2(k+1)(n-\psi_n)+|n(B+1)-\psi_n(A+1)|]\}|a_n|$$

$$< |(B+1)-(A+1)\psi_1| - 2(k+1)|\psi_1 - 1|, \tag{2.2}$$

where $-1 \leqslant B < A \leqslant 1$, $k \geqslant 0$, *and* ψ_n *is defined by* (1.5).

Particular choices in Theorem 2.1 and Theorem 2.2 respectively we get the following corollaries

When $j = 1$, we have the following results, proved by Fuad Alsarari and Latha [**4**].

COROLLARY 2.1. *A function* $f \in \mathcal{A}$ *and of the form* (1.1) *is in the class* $k - ST[A,B,N]$, *if it satisfies the condition*

$$\sum_{n=2}^{\infty}\{2(k+1)(n-\lambda_N(n)) + |n(B+1)-\lambda_N(n)(A+1)|\}|a_n| < |B-A|$$

COROLLARY 2.2. *A function* $f \in \mathcal{A}$ *and of the form* (1.1) *is in the class* $k - UCV[A,B,N]$, *if it satisfies the condition*

$$\sum_{n=2}^{\infty} n\{2(k+1)(n-\lambda_N(n)) + |n(B+1)-\lambda_N(n)(A+1)|\}|a_n| < |B-A|,$$

For $j = i = 1$ we have the following known results, proved by Khalida Inayat Noor and Sarfraz Nawaz Malik [**6**].

COROLLARY 2.3. *A function* $f \in \mathcal{A}$ *and form* (1.1) *in the class* $k - ST[A,B]$, *if it satisfies the condition*

$$\sum_{n=2}^{\infty}\{2(k+1)(n-1) + |n(B+1)-(A-1)|\}|a_n| < |B-A|, \tag{2.3}$$

where $-1 \leqslant B < A \leqslant 1$ *and* $k \geqslant 0$.

COROLLARY 2.4. *A function $f \in \mathcal{A}$ and form* (1.1) *in the class $k-UCV[A,B]$, if it satisfies the condition*

$$\sum_{n=2}^{\infty} n\{2(k+1)(n-1)+|n(B+1)-(A-1)|\}|a_n| < |B-A|, \tag{2.4}$$

where $-1 \leqslant B < A \leqslant 1$ and $k \geqslant 0$.

For $j=i=A=-B=1$, we have following result due to Kanas and Wisniowska [**3**].

COROLLARY 2.5. *A function $f \in \mathcal{A}$ and form* (1.1) *in the class $k-ST$, if it satisfies the condition*

$$\sum_{n=2}^{\infty}\{n+k(n-1)\}|a_n| < 1, \ \ k \geqslant 0.$$

Also for $k=0, j=i=1,\ A=1-2\alpha$ with $0 \leqslant \alpha < 1$, then we have the following known result.

COROLLARY 2.6. *A function $f \in \mathcal{A}$ and form* (1.1) *in the class $S^*(\alpha)$, if it satisfies the condition*

$$\sum_{n=2}^{\infty}\{(n-\alpha)\}|a_n| < 1-\alpha,$$

where $0 \leqslant \alpha < 1$.

THEOREM 2.3. *Let $f \in k-\mathcal{V}^{(j,i)}[A,B]$ and is of the form* (1.1). *Then for $n \geqslant 2$, $-1 \leqslant B < A \leqslant 1$, $k \geqslant 0$, .*

$$|a_n| \leqslant \prod_{h=1}^{n-1} \frac{|\delta_k(A-B)-2(h-\psi_h)B|}{2(h+1-\psi_{h+1})}, \tag{2.5}$$

where δ_k, ψ_n are defined respectively by (1.3), (1.5).

PROOF. By definition (1.8) we have

$$\frac{zf'(z)}{f_{j,i}(z)} = p(z),$$

where

$$p(z) \in k-P[A,B],$$

then we have we have

$$zf'(z) = f_{j,i}(z)\left(1+\sum_{n=1}^{\infty} c_n z^n\right)$$

$$(1-\psi_1)z + \sum_{n=2}^{\infty}(n-\psi_n)z^n = \sum_{n=1}^{\infty}\psi_n a_n z^n . \sum_{n=1}^{\infty} c_n z^n.$$

Equating coefficients of z^n on both sides, we have

$$(n-\psi_n)a_n = \sum_{h=1}^{n-1} \psi_{n-h} a_{n-h} c_h, \ a_1 = \psi_1 = 1.$$

This implies that

$$|a_n| \leqslant \frac{1}{(n-\psi_n)} \sum_{h=1}^{n-1} \psi_{n-h} a_{n-h} c_h, \ a_1 = \psi_1 = 1.$$

By Lemma 1.2, we get

$$|a_n| \leqslant \frac{|\delta_k|(A-B)}{2(n-\psi_n)} \sum_{h=1}^{n-1} \psi_h |a_h|. \tag{2.6}$$

Now we prove that

$$\frac{|\delta_k|(A-B)}{2(n-\psi_n)} \sum_{h=1}^{n-1} \psi_h |a_h| \leqslant \prod_{h=1}^{n-1} \frac{|\delta_k(A-B) - 2(h-\psi_h)B|}{2(h+1-\psi_{h+1}))}. \tag{2.7}$$

For this, we use the induction method.
For $n = 2$: from (2.6), we have

$$|a_2| \leqslant \frac{|\delta_k|(A-B)}{2(2-\psi_2)}.$$

From (2.8), we have

$$|a_2| \leqslant \frac{|\delta_k|(A-B)}{2(2-\psi_2)}.$$

For $n = 3$: from (2.6), we have

$$|a_3| \leqslant \frac{|\delta_k|(A-B)}{2(3-\psi_3)} \left[1 + \frac{|\delta_k|(A-B)}{2(2-\psi_2)}\psi_2\right].$$

From (2.8), we have

$$\begin{aligned} |a_3| &\leqslant \frac{|\delta_k|(A-B)}{2(2-\psi_2)} \frac{|\delta_k(A-B) - 2(2-\psi_2)B|}{2(3-\psi_3)} \\ &\leqslant \frac{|\delta_k|(A-B)}{2(2-\psi_2)} \frac{|\delta_k|(A-B) + 2(2-\psi_2)}{2(3-\psi_3)} \\ &\leqslant \frac{|\delta_k|(A-B)}{2(3-\psi_3)} \left[1 + \frac{|\delta_k|(A-B)}{2(2-\psi_3)}\right]. \end{aligned}$$

Let the hypothesis be true for $n = m$. From (2.6), we have

$$|a_m| \leqslant \frac{|\delta_k|(A-B)}{2(m-\psi_m)} \sum_{h=1}^{m-1} \psi_h |a_h|, \ a_1 = \psi_1 = 1.$$

From (2.8), we have

$$|a_m| \leqslant \prod_{h=1}^{m-1} \frac{|\delta_k(A-B) - 2(h-\psi_h)B|}{2(h+1-\psi_{h+1})}.$$

$$\leqslant \prod_{h=1}^{m-1} \frac{|\delta_k|(A-B)+2(h-\psi_h)}{2(h+1-\psi_{h+1})}.$$

By the induction hypothesis, we have

$$\frac{|\delta_k|(A-B)}{2(m-\psi_m)} \sum_{h=1}^{m-1} \psi_h |a_h| \leqslant \prod_{h=1}^{m-1} \frac{|\delta_k|(A-B)+2(h-\psi_h)}{2(h+1-\psi_{h+1})}.$$

Multiplying both sides by $\frac{|\delta_k|(A-B)+2(m-\psi_m)}{2(m+1-\psi_{m+1})}$, we have

$$\prod_{h=1}^{m} \frac{|\delta_k|(A-B)+2(h-\psi_h)}{2(h+1-\psi_{h+1})} \geqslant \frac{|\delta_k|(A-B)}{2(m-\psi_m)} \cdot \frac{|\delta_k|(A-B)+2(m-\psi_m)}{2(m+1-\psi_{m+1})} \sum_{h=1}^{m-1} \psi_h |a_h|,$$

$$= \frac{|\delta_k|(A-B)}{2(m+1-\psi_{m+1})} \left[\frac{|\delta_k|(A-B)}{2(m-\psi_m)} \sum_{h=1}^{m-1} \psi_h |a_h| + \sum_{h=1}^{m-1} \psi_h |a_h| \right],$$

$$\geqslant \frac{|\delta_k|(A-B)}{2(m+1-\psi_{m+1})} \left[\psi_m |a_m| + \sum_{h=1}^{m-1} \psi_h |a_h| \right],$$

$$= \frac{|\delta_k|(A-B)}{2(m+1-\psi_{m+1})} \sum_{h=1}^{m} \psi_h |a_h|.$$

$$\frac{|\delta_k|(A-B)}{2(m+1-\psi_{m+1})} \sum_{h=1}^{m} \psi_h |a_h| \leqslant \prod_{h=1}^{m} \frac{|\delta_k|(A-B)+2(h-\psi_h)}{2(h+1-\psi_{h+1})}.$$

Which shows that inequality (2.7) is true for $n = m+1$. Hence the required result. □

THEOREM 2.4. *Let* $f \in k - \mathcal{CV}^{(j,i)}[A,B]$ *and is of the form* (1.1). *Then for* $n \geqslant 2$, $-1 \leqslant B < A \leqslant 1$, $k \geqslant 0$, .

$$|a_n| \leqslant \frac{1}{n} \prod_{h=1}^{n-1} \frac{|\delta_k(A-B) - 2(h-\psi_h)B|}{2(h+1-\psi_{h+1})}, \tag{2.8}$$

where δ_k, ψ_n *are defined respectively by* (1.3), (1.5).

Particular choices in Theorem 2.3 and Theorem 2.4 respectively we get the following corollaries

When $j = 1$, we have the following results, proved by Fuad Alsarari and Latha [**4**].

COROLLARY 2.7. *Let* $f \in k - ST[A,B,N]$ *and is of the form* (1.1). *Then for* $n \geqslant 2$.

$$|a_n| \leqslant \prod_{j=1}^{n-1} \frac{|\delta_k(A-B) - 2(j-\lambda_N(j))B|}{2(j+1-\lambda_N(j+1))},$$

Corollary 2.8. *Let $f(z) \in k-UCV[A,B,N]$ and is of the form* (1.1). *Then for $n \geqslant 2$.*

$$|a_n| \leqslant \frac{1}{n}\prod_{j=1}^{n-1}\frac{|\delta_k(A-B)-2(j-\lambda_N(j))B|}{2(j+1-\lambda_N(j+1))},$$

When $j=i=1$ we have the following results proved Khalida Inayat Noor and Sarfraz Nawaz Malik [**6**]

Corollary 2.9. *Let $f \in k-ST[A,B]$, then*

$$|a_n| \leqslant \prod_{j=0}^{n-2}\frac{|\delta_k(A-B)-2jB|}{2(j+1)}, \ -1 \leqslant B < A \leqslant 1, \ n \geqslant 2.$$

Corollary 2.10. *Let $f \in k-UCV[A,B]$, then*

$$|a_n| \leqslant \frac{1}{n}\prod_{j=0}^{n-2}\frac{|\delta_k(A-B)-2jB|}{2(j+1)}, \ -1 \leqslant B < A \leqslant 1, \ n \geqslant 2.$$

For $A=1$ $B=-1, j=i=1, \gamma=0$ we arrive at Kanas and Wisniowska [**3**]

Corollary 2.11. *Let $f \in k-ST$, then*

$$|a_n| \leqslant \prod_{j=0}^{n-2}\frac{|\delta_k+j|}{(j+1)}, \ n \geqslant 2.$$

Also for $k=0$ $\delta_k=2, j=i=1$ we have the well-known result proved by Janowski [**1**].

Corollary 2.12. *Lat $f \in \mathcal{S}^*[A,B]$, then*

$$|a_n| \leqslant \prod_{j=0}^{n-2}\frac{|(A-B)-jB|}{(j+1)}, \ -1 \leqslant B < A \leqslant 1, \ n \geqslant 2.$$

References

[1] W. Janowski, Some extremal problems for certain families of analytic functions, *Ann. Polon. Math.*, **28** (1973), 297-326.

[2] A. W. Goodman. Univalent Functions, Vols. I-II, *Mariner Publishing Company, Florida, USA*, (1983).

[3] S. Kanas, A. Wisniowska, Conic domains and starlike functions, *Roumaine Math. Pures Appl.* **45**, (2000), 647-657.

[4] F. Alsarari and S. Latha, Conic regions and symmetric points. *Int. J. Pure. App. Math*, **97**(3)(2014), 273-285.

[5] S. Latha, Coefficient inqualities for certain classes of Ruscheweyh type analytic functions, *J. Inequal. Pure. Appl. Math.* **9**(2)(2008), Art. 52, 5 pp.

[6] K. L. Noor and S. N. Malik, On coefficient inequalities of functions associatd with conic domains, *Comput. Math. Appl.*, **62**(2011), 2209-2219.

[7] S. Kanas, Coefficient estimates in subclasses of the Caratheodory class related to conical domains, *Acta. Math. Univ. Comenian*, **74**(2)(2005), 149-161.

[8] S. Kanas and A. Wisniowska, Conic regions and k-uniform convexity, *J. comput. Appl. Math.*, **105**(1999), 327-336.

[9] H. A. Al- Kharsani, A. Sofo, Subordination results on harmonic k-uniformly convex mappings and related classes, *Appl. Math. Comput. Math. Appl.*, **59**(12)(2010), 3718-3726.

[10] Fuad Alsarari and S. Latha, A few results on functions that are Janowski starlike related to (j, k)-symmetric opints, *Octo. Math. Maga*, **21**(2)(2013), 556-563.

[11] F. Alsarari and S.Latha, A note on cofficent inqualities for certain classes of Ruscheweyh type analytic functions in conic regions, accepted for publication in *Mathematical Sciences Letters,*,(2016).

[12] P. Liczberski and J. Polubinski, On (j, k)-symmtrical functions, *Mathematica Bohemica*, **120**(1995), 13-25.

[13] W. Rogosinski, On the coefficients of subordiniate function, *Proc. Lond. Soc.*, **48**(1943), 48-882.

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