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ON A CLASSIFICATION OF SEMILATTICE VALUED FUNCTIONS

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Abstract. In the framework of fuzzy sets and corresponding techniques, we investigate functions from a nonempty set X into an ordered structure $(S, \leqslant)$ where S is a meet-semilattice.

Each function $\mu : X \to S$ determines a family of subsets of X, which are called cut sets. Vice versa, particular family of subsets of X, indexed by the elements of S uniquely determines a function from X to S.

Further, any function $\mu : X \to S$ determines a semi-closure operator on S, which induces an equivalence relation on the semilattice S. Using the above results, we classify functions in S^X.

1. Introduction

Fuzzy sets are functions, and they can be characterized by a family of subsets of the domain, consisting of cut sets [**3, 4, 5**]. If cuts are considered as non-indexed subsets, then it may happen that different fuzzy sets on the same domain have equal collections of cuts. Conditions under which it happens for lattice valued fuzzy sets are investigated and presented in [**3**]. In case when the codomain is an arbitrary poset these conditions are studied in [**3**] and [**6**]. Clearly, equality of the collections of cuts is an analogue problem in case that the codomain of fuzzy sets is a meet or join-semilattice. Conditions for such equivalence has not been investigated in the literature. In paper [**2**], we have investigated properties of cuts of semilattice valued fuzzy sets. In the mentioned paper we introduced and investigated semi-closure systems and dual semi-closure systems and applied these to investigations

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of collections of cut sets of semilattice valued fuzzy sets. The present article is a continuation of this paper ([**2**]), and we deal with the problem of finding conditions for two semilattice valued fuzzy sets to have equal collections of cuts. Our results are similar to those in papers [**3, 4, 6**], we follow the same ideas, but here we use different techniques, combining lattice and poset theoretic tools. Using properties of the collection of cut sets, we define a binary relation on a semilattice through a particular extension function. In this way, we obtain a partition of the co-domain semilattice. In connection to this partition, following the notions from [**3, 4**], we introduce equivalent semilattice-valued fuzzy sets. We prove that these mappings (fuzzy sets) have equal collections of cuts.

2. Preliminaries

2.1. Order; semilattices. Here we give some basic definitions and basic properties of ordered structures and related notions, which are used throughout the paper. These are a part of the topic presented e.g., in [**1**].

A partially ordered set, a **poset** $(S, \leqslant)$ is a nonempty set equipped with an ordering relation $\leqslant$. For every $p \in S$, the **principal filter generated by** p is denoted by $\uparrow p$:

$$\uparrow p = \{x \in S \mid p \leqslant x\}.$$

Dually, the **principal ideal generated by** p is denoted by $\downarrow p$:

$$\downarrow p = \{x \in S \mid p \leqslant x\}.$$

If S is a poset and $x, y \in S$, then clearly

$$x \leqslant y \quad \text{if and only if} \quad \uparrow y \subseteq \uparrow x.$$

A poset is **bounded** if it has the smallest element, the **bottom**, denoted by 0, and the greatest, the top, denoted by 1.

Let C be a collection of subsets of a nonempty set X, ordered by inclusion. This poset is called a **closure system** on X if it closed under arbitrary (including empty) intersections ([**1**]). A collection of subsets C of X is called a **semi-closure system** on X (see [**2**]) if it is closed under arbitrary non-empty set intersections. A collection of subsets C is called a **dual semi-closure system** if it closed under all intersections of arbitrary nonempty sub-collection ([**2**]).

A **meet-semilattice** is a poset in which for every two-element subset $\{x, y\}$ there is the greatest lower bound (glb, meet, infimum), denoted by $x \wedge y$. A **join-semilattice** is a poset in which for every two-element subset $\{x, y\}$ there is the least upper bound (lub, join, supremum), denoted by $x \vee y$.

A **lattice** is a poset which is a meet-semilattice and a join-semilattice. A meet or a join are binary operations on S, hence a semilattice is also an algebra, denoted by $(S, \wedge)$ or $(S, \vee)$ respectively.

A **complete lattice** is a poset $(S, \leqslant)$ in which every subset possesses the greatest lower bound and the least upper bound. Every complete lattice is bounded.

If $(S, \leqslant)$ and $(T, \leqslant)$ are posets, then a map $f : S \longrightarrow T$ is an **order isomorphism** if it is a bijection compatible with the order in both directions, i.e., if for

$x, y \in S$,

$$x \leqslant y \text{ if and only if } f(x) \leqslant f(y).$$

If S and T are semilattices, then such a bijection has to preserve meets for meet-semilattices (**meet-homomorphism**) or joins for join semi-lattices (**join-homomorphism**) respectively, i.e., for $x, y \in S$,

$$f(x \wedge y) = f(x) \wedge f(y) \text{ or } f(x \vee y) = f(x) \vee f(y).$$

If S and T are lattices, then such a bijection is supposed to preserve both, (finite) meets and joins.

Two semilattices are isomorphic as algebras if and only if they are order isomorphic in the above sense.

As usual in the set theory, a **family** of subsets of a set X is a function from an index set I into the power set $P(X)$. The codomain of the family is a corresponding collection of subsets of X.

Let $(S, \leqslant)$ be a meet (join)-semilattice. For arbitrary $M \subseteq S$, we define **the set of upper bounds of** M,

$$M^u := \{p \in S \mid x \leqslant p, \text{ for every } x \in M\}.$$

and **the set of lower bounds of** M,

$$M^l := \{p \in S \mid p \leqslant x, \text{ for every } x \in M\}.$$

2.2. Semilattice-valued functions. Let $(S, \leqslant)$ be a semilattice and $X \neq \emptyset$. A function $\mu : X \to S$ is a **semilattice-valued** or S**-valued** function on S. Following the terminology of the theory of fuzzy sets, we also say that μ is a semilattice-valued (S-valued) **fuzzy set** on X.

By S^X we denote the set of all L-valued functions (fuzzy sets) on X:

$$S^X = \{\mu \mid \mu : X \to S\}.$$

We consider S^X to be an ordered set whose order is induced by the order in S:

$$\mu \leqslant \nu \text{ if and only if for every } x \in X, \mu(x) \leqslant \nu(x).$$

It is clear that under this order, S^X is a semilattice.

Let $\mu \in S^X$ and $p \in S$. Then a cut set (cut) of μ is a subset of μ_p of X defined by

$$\mu_p = \{x \in X \mid \mu(x) \geqslant p\}$$

In other words, a cut set of μ is the inverse image of the principal filter generated by p:

$$\mu_p = \mu^{-1}(\uparrow p).$$

The set of all cut sets of μ is denoted by μ_S:

$$\mu_S = \{\mu_p \mid p \in S\}.$$

In the following, we present some properties of the above introduced functions in connection to cuts; these are presented in [**2**] (see also [**5**]).

PROPOSITION 2.1. *If $\mu : X \to S$ is a function in S^X, then for every $x \in X$,*

$$\mu(x) = \bigvee\{p \in S \mid x \in \mu_p\}.$$

PROPOSITION 2.2. *Let $\{M_i \mid i \in S\}$ be a family of subsets of a nonempty set X, indexed by elements of meet semilattice S in the following way:*

For every $x \in X$, $\bigvee\{p \mid x \in M_p\}$ exists in S and

$$\bigcap\{M_p \mid x \in M_p\} = M_{\bigvee\{p \mid x \in M_p\}}. \tag{2.1}$$

Then, $\{M_i \mid i \in S\}$ is the family of cut sets of S-valued fuzzy set $\mu : X \to S$, defined by

$$\mu(x) = \bigvee\{p \in S \mid x \in M_p\}. \tag{2.2}$$

THEOREM 2.1. *Let F be a semi-closure system over X and its union be X. Then there is a meet semilattice S and an S-valued fuzzy set $\mu : X \to S$ such that $\mu_S = F$.*

Using a dual semi-closure system, one can prove the analogue property for a join semilattice-valued fuzzy set ([**2**]).

3. Main results

Throughout the section, $(S, \leqslant)$ is supposed to be a meet-semilattice, to which we sometimes refer briefly as to a semilattice.

We start with some auxiliary properties of functions in S^X. Some of these were developed for lattice valued functions in [**3**] (see also reference there), and here we adopt them to meet-semilattices.

Let $\mu : X \to S$ be a function from S^X, and $\approx$ a binary relation on S, such that for $p, q \in S$

$$p \approx q \text{ if and only if } \mu_p = \mu_q.$$

It is obvious that $\approx$ is an equivalence relation on S.

Let $\mu : X \to S$ be a function in S^X, then $\mu(X) = \{\mu(x) \mid x \in X\}$. The pair $(\mu(X), \leqslant)$ is a sub-poset of the semilattice $(S, \leqslant)$.

The following is known for posets and lattices ([**3, 4**], but dealing with semilattices, we prove this property here.

PROPOSITION 3.1. *If $\mu \in S^X$ and $p, q \in S$, then*

$$p \approx q \text{ if and only if } {\uparrow}p \cap \mu(X) = {\uparrow}q \cap \mu(X).$$

Proof. The relation $p \approx q$ holds if and only if $\mu_p = \mu_q$ if and only if for every $x \in X, x \in \mu_p$ if and only if $x \in \mu_q$ if and only if $\mu(x) \geqslant p \leftrightarrow \mu(x) \geqslant q$ if and only if $\mu(x) \in {\uparrow}p$ is equivalent with $\mu(x) \in {\uparrow}q$ if and only if $\{x \in X \mid \mu(x) \in {\uparrow}p\} = \{x \in X \mid \mu(x) \in {\uparrow}q\}$ if and only if ${\uparrow}p \cap \mu(X) = {\uparrow}q \cap \mu(X)$. $\square$

PROPOSITION 3.2. *Let $\mu, \nu \in S^X$. If the collections μ_S and ν_S of cuts of these functions coincide, then $\mu(X) \cong \nu(X)$ under an ordered isomorphism.*

Proof. $\mu_S = \nu_S$ is equivalent to

$$(\forall p \in S)(\exists q \in S)\mu_p = \nu_q \text{ and } (\forall p \in S)(\exists q \in S)\nu_p = \mu_q.$$

The mapping $f : \mu(X) \to \nu(X)$ is defined by $f(\mu(x)) = \nu(x)$. We show that f is well defined. To do this, we prove firstly that $\mu(x) = \mu(y)$ if and only if $\nu(x) = \nu(y)$. Suppose that $\nu(x) \neq \nu(y)$. Let $p = \nu(x)$ and $q = \nu(y)$. Then, either $\nu(x) \nleqslant \nu(y)$ or $\nu(y) \nleqslant \nu(x)$ by contraposition of antisymmetry of the relation $\leqslant$ on S. If $\nu(x) \nleqslant \nu(y)$, then we have $p \nleqslant \nu(y)$ which means that $y \notin \nu_p$. By the assumption, $\nu_p = \mu_t$, for some $t \in S$. Hence, $y \notin \mu_t$ and $\mu(y) \ngeqslant t$. On the other hand, $x \in \nu_p$ means $x \notin \mu_t$, by the assumption $\nu_p = \mu_t$. Further, $\mu(x) \geqslant t$. We have $\mu(x) \neq \mu(y)$. The converse is proved analogously. Hence, f is also injective. Furthermore, if $p \in \nu(X)$, then $p = \nu(x)$, for some x in X. Hence, $f(\mu(x)) = p$ and the mapping is "onto". We show that $\mu(x) \leqslant \mu(y)$ if and only if $\nu(x) \leqslant \nu(y)$. Suppose that $\mu(x) \leqslant \mu(y)$. Let $\nu(x) = p$. Then, $x \in \nu_p$ and there is $q \in S$, such that $\nu_p = \mu_q$. Now, $x \in \mu_q$ and $q \leqslant \mu(x) \leqslant \mu(y)$, hence $y \in \mu_q$. Therefore, $y \in \nu_p$ and $\nu(x) = p \leqslant \nu(y)$. Analogously, we prove the converse, i.e. that from $\nu(x) \leqslant \nu(y)$ it follows that $\mu(x) \leqslant \mu(y)$. □

The converse of Proposition 3.2 is not satisfied as presented in the following example.

EXAMPLE 3.1. Let $X = \{x, y, z\}$ and meet semilattice S that is given by its diagram in Figure 1.

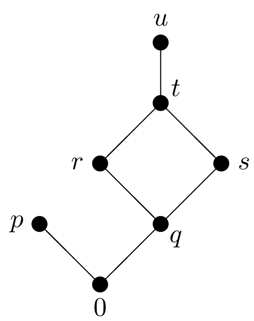
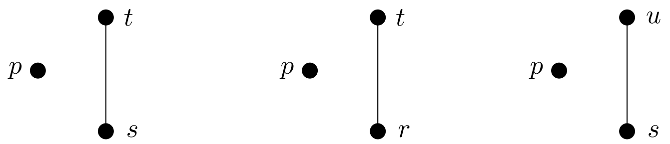


Figure 1: Semilattice S

Let $\mu, \nu, \pi : X \to S$ be three functions from X to S, defined as follows:

$$\mu = \begin{pmatrix} x & y & z \\ p & t & s \end{pmatrix}; \nu = \begin{pmatrix} x & y & z \\ p & t & r \end{pmatrix}; \pi = \begin{pmatrix} x & y & z \\ p & u & s \end{pmatrix}$$

Then all three functions have order isomorphic posets of images, i.e., $\mu(X) \cong \nu(X) \cong \pi(X)$. (See Figure 2).



(a) μ_S (a) ν_S (a) π_S

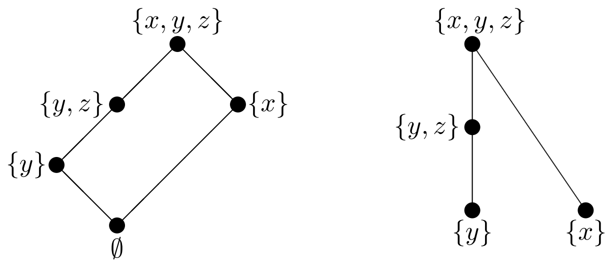
Figure 2: Order isomorphic images of meet fuzzy sets

Cut sets of these functions are given in Table 1.

$z \in S$	μ_s	ν_s	π_s
p	$\{x\}$	$\{x\}$	$\{x\}$
q	$\{y,z\}$	$\{y,z\}$	$\{y,z\}$
r	$\{y\}$	$\{y,z\}$	$\{y\}$
s	$\{y,z\}$	$\{y\}$	$\{y,z\}$
t	$\{y\}$	$\{y\}$	$\{y\}$
u	$\emptyset$	$\emptyset$	$\{y\}$
0	$\{x,y,z\}$	$\{x,y,z\}$	$\{x,y,z\}$

TABLE 1. Collections of cuts of meet fuzzy set μ, ν and π

The following diagram (Figure 3) shows that the collections of cuts of μ and ν coincide but the collection of cuts of μ does not coincide with the corresponding collection for π.



(a) $\mu_s = \nu_s$ π_s

Figure 3: Collections of cuts of meet fuzzy sets

Starting with a function μ from S^X where S is a meet semilattice, we define a special poset ordered by set inclusion, whose elements are certain subsets of the set of all images of μ.
For $\mu \in S^X$, let

$$S_\mu := (\{\uparrow p \cap \mu(X) \mid p \in S\}, \subseteq).$$

In the following, the above collection is considered as a poset ordered by inclusion.

Proposition 3.3. *If $\mu : X \to S$ is a function from S^X, then there is an order isomorphism from the poset S_μ to the poset μ_S of cuts of μ.*

Proof. The function $f : \mu_p \to \uparrow p \cap \mu(X)$ maps the collection μ_S of cuts of μ onto the poset S_μ. By the following sequence of equivalent statements we prove that the mapping is well defined and injective. $\uparrow p \cap \mu(X) = \uparrow q \cap \mu(X)$ if and only if $\{x \mid \mu(x) \geqslant p\} = \{x \mid \mu(x) \geqslant q\}$ if and only if $\mu_p = \mu_q$. Since the fact that f is "onto" is obvious, f is bijective. Now we show that f preserve the order, i.e., that $\mu_p \subseteq \mu_q$ if and only if $f(\mu_p) \subseteq f(\mu_q)$, where $f(\mu_p) = \uparrow p \cap \mu(X)$. Let $\mu_p \subseteq \mu_q$ and let $m \in \uparrow p \cap \mu(X)$. It means that $m \in \uparrow p$ and $m = \mu(x)$ for some $x \in X$. Then $\mu(x) \geqslant p$ for some $x \in X$, i.e., $x \in \mu_p$. It follows that $x \in \mu_q$, by assumption $\mu_p \subseteq \mu_q$. Therefore, $m \in \uparrow q$ and $m = \mu(x)$ for some $x \in X$, i.e., $m \in \uparrow q \cap \mu(X)$. In order to prove the converse, we suppose that $f(\mu_p) \subseteq f(\mu_q)$. Let $x \in \mu_p$, i.e., let $\mu(x) \geqslant p$. We have that $\mu(x) \in \uparrow p \cap \mu(X)$, hence $\mu(x) \in \uparrow q \cap \mu(X)$ by the assumption. Hence, $\mu(x) \geqslant q$ and $x \in \mu_q$, so f is an order isomorphism. □

The following lemma describes a property of complete meet-semilattices which is used in the sequel.

Lemma 3.1. *Let $(S, \leqslant)$ be a complete meet-semilattice. If M is a nonempty subset of S and $M^u \neq \emptyset$, then $\bigvee M$ exists.*

Proof. Let $M \subseteq S$. Suppose $M^u = \{p \in S \mid x \leqslant p, \text{ for every } x \in M\} \neq \emptyset$. By the assumption, $\bigwedge M^u$ exists. Now it is easy to prove that $m = \bigwedge M^u$ is a supremum of M. □

Lemma 3.2. *Let $(S, \leqslant)$ be a complete meet-semilattice, X a nonempty set, $\mu \in S^X$ and $p, q \in S$. If p and q have an upper bound, then*

$$(\uparrow(p \vee q)) \cap \mu(X) = (\uparrow p \cap \uparrow q) \cap \mu(X). \tag{3.1}$$

Proof. Suppose that $\{p, q\} \subseteq S$ and $\{p, q\}^u \neq \emptyset$. Then $p \vee q$ exist by Lemma 3.1. For all $p, q \in S$, $p \leqslant p \vee q$ and $q \leqslant p \vee q$. Then $\uparrow(p \vee q) \subseteq \uparrow p$ and $\uparrow(p \vee q) \subseteq \uparrow q$ by properties of filters. This implies $\uparrow(p \vee q) \subseteq \uparrow p \cap \uparrow q$, and we have $\uparrow(p \vee q) \cap \mu(X) \subseteq (\uparrow p \cap \uparrow q) \cap \mu(X)$. Now we show the converse. Let $m \in (\uparrow p \cap \uparrow q) \cap \mu(X)$. In other words, $m \in \uparrow p$, $m \in \uparrow q$ and $m = \mu(x)$ for some $x \in X$. Hence $m \geqslant p$, $m \geqslant q$ and $m = \mu(x) \in \mu(X)$. Therefore, $m \geqslant p \vee q$ and $m = \mu(x)$, for some $x \in X$, i.e., $m \in \uparrow(p \vee q) \cap \mu(X)$. Thus, we have $(\uparrow p \cap \uparrow q) \cap \mu(X) \subseteq (\uparrow(p \vee q)) \cap \mu(X)$. □

Remark 3.1. If p and q do not have an upper bound, then $\uparrow p \cap \uparrow q$ is the empty set.

By the previous lemma, in the following we show that $S_\mu \cup \{\emptyset\}$ is a meet semilattice under inclusion.

Observe that for a mapping $\mu : X \to S$, where $(S, \leqslant)$ is a meet (join)-semilattice, we have

$$(\uparrow p \cap \mu(X)) \cap (\uparrow q \cap \mu(X)) = (\uparrow p \cap \uparrow q) \cap \mu(X). \tag{3.2}$$

By (3.2), in the poset $S_\mu \cup \{\emptyset\}$, in case of non empty set intersection we have:

$$(\uparrow p \cap \mu(X)) \cap (\uparrow q \cap \mu(X)) = (\uparrow(p \vee q)) \cap \mu(X),$$

and in case of empty set intersections:

$$(\uparrow p \cap \mu(X)) \cap (\uparrow q \cap \mu(X)) = \emptyset,$$

for $(\uparrow p \cap \uparrow q) = \emptyset$. So, $S_\mu \cup \{\emptyset\}$ is a meet semilattice.

Example 3.2. For functions μ, ν and π (Example 3.1), we calculate the corresponding subsets of S in Table 2. as follows. Three posets $(S_\mu, \subseteq), (S_\nu, \subseteq)$ and

$z \in S$	$\uparrow z \cap \mu(X)$	$\uparrow z \cap \nu(X)$	$\uparrow z \cap \pi(X)$
p	$\{p\}$	$\{p\}$	$\{p\}$
q	$\{s,t\}$	$\{r,t\}$	$\{s,u\}$
r	$\{t\}$	$\{t,r\}$	$\{u\}$
s	$\{s,t\}$	$\{t\}$	$\{s,u\}$
t	$\{t\}$	$\{t\}$	$\{u\}$
u	$\emptyset$	$\emptyset$	$\{u\}$
0	$\{p,t,s\}$	$\{p,t,r\}$	$\{p,u,s\}$

Table 2. Collections of subsets S

$(S_\pi, \subseteq)$ are presented in Figure 4.

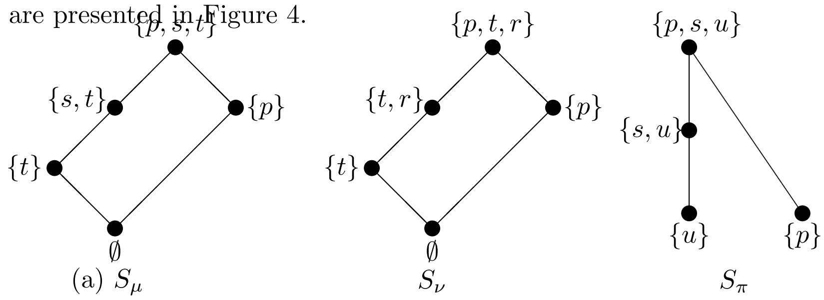


Figure 4: Posets of collections of subsets of S

Next, we introduce our main definition by which we can classify functions in the collection S^X. We start with a meet semilattice using the same definition of the equivalence relation that was introduced in the lattice case [see [**3**]].

Our motivation can be seen in Examples 3.1 and 3.2. Observe that the posets of images for all three fuzzy sets are isomorphic, while the posets of cuts are not. In order to make the corresponding distinction, for mapping μ, ν, ..., we investigate the posets S_μ, S_ν, ... and so on.

Definition 3.1. Let S be meet semilattice and let $\sim$ be a relation on S^X, defined as follows:

$$\mu \sim \nu \text{ if and only if the correspondence } f : \mu(x) \to \nu(x), x \in X,$$

is a bijection from $\mu(X)$ onto $\nu(X)$ which has an extension to an isomorphism from S_μ onto S_ν, given by the map:

$$F(\uparrow p \cap \mu(X)) = \uparrow \bigwedge\{\nu(x) \mid \mu(x) \geqslant p\} \cap \nu(X), p \in S. \tag{3.3}$$

REMARK 3.2. If $x \in X$, then $\mu(x)$ and $\nu(x)$ are in the mentioned correspondence. Since this is a bijection, we have that $\mu(x) = \mu(y)$ if and only if $\nu(x) = \nu(y)$.

LEMMA 3.3. *Let $\mu, \nu \in S^X$ and $\mu \sim \nu$, and let for $p \in S, \bigwedge\{\nu(x) \mid \mu(x) \geqslant p\} = q$. Then for any $y \in X$,*

$$\mu(y) \geqslant p \textit{ if and only if } \nu(y) \geqslant q.$$

Proof. Suppose that for $p \in S$, we have $\bigwedge\{\nu(x) \mid \mu(x) \geqslant p\} = q$. If $\mu(y) \geqslant p$, then $\nu(y) \geqslant \bigwedge\{\nu(x) \mid \mu(x) \geqslant p\} = q$.

Now, suppose that $\nu(y) \geqslant q$. Then, by $\mu \sim \nu$, we have that $F(\uparrow p \cap \mu(X)) = \uparrow \bigwedge\{\nu(x) \mid \mu(x) \geqslant p\} \cap \nu(X)$ is an isomorphism from S_μ onto S_ν. By the definition of q, $F(\uparrow p \cap \mu(X)) = \uparrow q \cap \nu(X)$. By $\nu(y) \geqslant q$, we have that $\uparrow \nu(y) \cap \nu(X) \subseteq \uparrow q \cap \nu(X)$. Hence, $\uparrow \mu(y) \cap \mu(X) \subseteq \uparrow p \cap \mu(X)$. Since $\mu(y) \in \uparrow \mu(y) \cap \mu(X)$, we have that $\mu(y) \in \uparrow p \cap \mu(X)$, hence $\mu(y) \geqslant p$. $\square$

PROPOSITION 3.4. *The relation $\sim$ is an equivalence relation on S^X.*

Proof.

- Reflexivity. For every $\mu \in S^X$, we will show that $\mu \sim \mu$, i.e., $F(\uparrow p \cap \mu(X)) = \uparrow \bigwedge\{\mu(x) \mid \mu(x) \geqslant p\} \cap \mu(X), p \in S$. Let $\mu \in S^X$, we have $\mu(x) \in S$. Since f is a bijection, $\mu(x) \to \mu(x)$ is an identity mapping in $\mu(X)$. This mapping determines the identity mapping in extension F as $\uparrow \mu(x) \cap \mu(X) \to \uparrow \mu(x) \cap \mu(X)$ in S_μ. Therefore, $F(\uparrow p \cap \mu(X)) = \uparrow \bigwedge\{\mu(x) \mid \mu(x) \geqslant p\} \cap \mu(X), p \in S$, i.e., $\mu \sim \mu$.
- Symmetry. For every $\mu, \nu \in S^X$, if $\mu \sim \nu$ then $\nu \sim \mu$. Suppose that $\mu \sim \nu$. It means F is an isomorphism and $F(\uparrow p \cap \mu(X)) = \uparrow \bigwedge\{\nu(x) \mid \mu(x) \geqslant p\} \cap \nu(X) = \uparrow q \cap \nu(X), p \in S$. By Lemma 3.3 we have $\mu(x) \geqslant p \leftrightarrow \nu(x) \geqslant q$. Now, we want to show $F^{-1}(\uparrow q \cap \nu(X)) = \uparrow \bigwedge\{\mu(x) \mid \nu(x) \geqslant p\} \cap \mu(X) = \uparrow p \cap \mu(X), q \in S$, that is:
  (i) $\uparrow \bigwedge\{\mu(x) \mid \nu(x) \geqslant q\} \cap \mu(X) \subseteq (\uparrow p \cap \mu(X))$, and
  (ii) $\uparrow p \cap \mu(X) \subseteq \uparrow \bigwedge\{\mu(x) \mid \nu(x) \geqslant q\} \cap \mu(X)$.

  (i) If $\nu(x) \geqslant q$, by assumption: $\mu(x) \geqslant p$ and $\bigwedge\{\mu(x) \mid \nu(x) \geqslant q\} \geqslant p$. Then we have $\uparrow \bigwedge\{\mu(x) \mid \nu(x) \geqslant q\} \subseteq \uparrow p$ and also $\uparrow \bigwedge\{\mu(x) \mid \nu(x) \geqslant q\} \cap \mu(X) \subseteq \uparrow p \cap \mu(X)$ by properties of filter.

  (ii) Let $\mu(x) \in \uparrow p \cap \mu(X)$. It means that $\mu(x) \geqslant p$, hence, $\nu(x) \geqslant q$ and $\mu(x) \geqslant \bigwedge\{\mu(y) \mid \nu(y) \geqslant q\}$. Therefore $\mu(x) \in \uparrow \bigwedge\{\mu(y) \mid \nu(y) \geqslant q\} \cap \mu(X)$. We have $\uparrow p \cap \mu(X) \subseteq \uparrow \bigwedge\{\mu(x) \mid \nu(x) \geqslant q\} \cap \mu(X)$. From (i) and (ii) we have $\sim$ is a symmetric on S^X.
- Transitivity. For every $\mu, \nu, \rho \in S^X$, we have to prove that from $\mu \sim \nu$ and $\nu \sim \rho$ it follows that $\mu \sim \rho$. Suppose $\mu \sim \nu$, it means that mapping $f : \mu(X) \to \nu(X)$ is a bijection such that $F : S_\mu \to S_\nu$ is an isomorphism

by the definition

$$F(\uparrow p \cap \mu(X)) = \uparrow \bigwedge \{\nu(x) \mid \mu(x) \geqslant p\} \cap \nu(X), p \in S,$$

and suppose $\nu \sim \rho$, it means that a mapping $f : \nu(X) \to \rho(X)$, such that $F : S_\nu \to S_\rho$ is an isomorphism by the definition

$$G(\uparrow q \cap \nu(X)) = \uparrow \bigwedge \{\rho(x) \mid \nu(x) \geqslant q\} \cap \rho(X), q \in S.$$

We show that $\mu \sim \rho$ i.e., that a mapping $f \circ g$ is a bijection such that $F \circ G$ is an isomorphism, i.e., we show that

$$(F \circ G)(\uparrow p \cap \mu(X)) = \uparrow \bigwedge \{\rho(x) \mid \mu(x) \geqslant p\} \cap \rho(X), p \in S.$$

Let $p \in S$ and $\uparrow p \cap \mu(X) \in S_\mu$,

$$(F \circ G)(\uparrow p \cap \mu(X)) = G(F(\uparrow p \cap \mu(X))) = G(\uparrow \bigwedge \{\nu(x) \mid \mu(x) \geqslant p\} \cap \nu(X)).$$

By Lemma 3.3, we have

$$(F \circ G)(\uparrow p \cap \mu(X)) = G(\uparrow q \cap \nu(X))) = \uparrow \bigwedge \{\rho(x) \mid \nu(x) \geqslant q\} \cap \rho(X).$$

Again by Lemma 3.3,

$$(F \circ G)(\uparrow p \cap \mu(X)) = \uparrow \bigwedge \{\rho(x) \mid \mu(x) \geqslant p\} \cap \rho(X), p \in S.$$

It means that $\mu \sim \rho$.

□

We say that semilattice-valued sets μ and ν on X are **equivalent** if $\mu \sim \nu$. The following example shows two meet-fuzzy sets on X that are equivalent.

Example 3.3. For functions μ, ν (Example 3.1), two isomorphic ordered images of two meet-fuzzy sets are given in Figure 2.

All of the subsets $\uparrow z \cap \mu(X)$ and $\uparrow z \cap \nu(X)$, $z \in S$ can be seen in Table 2.

The collections S_μ and S_ν which are isomorphic can be seen in Figure 4.

Therefore $\mu, \nu \in S^X$ have equal families of cuts. Obviously, collections of cuts μ_S and ν_S coincide and we have $S_\mu \cong \mu_S$ and $S_\nu \cong \nu_S$ as lattice isomorphisms with respect to inclusion. Therefore, meet-fuzzy sets $\mu, \nu \in S^X$ are equivalent.

The following example shows two meet fuzzy sets $\mu, \nu \in S^X$ that are not equivalent.

Example 3.4. For functions μ and π (Example 3.1), two ordered isomorphic images of two meet-fuzzy sets are obtained (Figure 2). Furthermore, in Table 2. (Example 3.2) subsets $\uparrow z \cap \mu(X)$ and $\uparrow z \cap \pi(X)$, for every $z \in S$ are presented. Collections S_μ and S_π are not isomorphic (Figure 4). Therefore, meet-fuzzy sets $\mu, \pi \in S^X$ are not equivalent. The collections μ_S and π_S of cuts ordered by inclusion (Figure 3) do not coincide. But it's always true that $S_\mu \cong \mu_S$ and $S_\pi \cong \pi_S$ by Proposition 3.3.

REMARK 3.3. If $F : S_\mu \to S_\nu$ is an ordered isomorphism then we have two collections of cuts μ_S and ν_S that coincide. Furthermore, we have an ordered isomorphism from $\mu(X)$ onto $\nu(X)$. But having an ordered isomorphism $f : \mu(X) \to \nu(X)$, is not sufficient to show that two collections of cuts coincide, as it is illustrated in Example 3.2.

Let $\mu, \nu \in S^X$ and if μ and ν are equivalent meet-fuzzy sets, then by the next theorem they have equal families of cuts.

THEOREM 3.1. *Let $\mu, \nu \in S^X$. Then $\mu \sim \nu$, if and only if fuzzy sets μ and ν have equal families of cuts.*

Proof. Suppose that $\mu \sim \nu$. Then we show that $\mu_S = \nu_S$, i.e., for every $p \in S$, there is $q \in S$ such that $\mu_p = \nu_q$. Now we take $p \in S$, and we have two cases below. Case (1), if $p \in \mu(X)$. Let $p = \mu(x)$ for some $x \in X$. We show that $\mu_p = \nu_q$ for $q = \nu(x)$. Take $y \in \mu_p$ if and only if $\mu(y) \geqslant p = \mu(x)$ if and only if $\nu(y) \geqslant \nu(x) = q$ if and only if $y \in \nu_p$. Case (2), if $p \in S$ is not in $\mu(X)$. Let $x \in \mu_p$. Now we show that $x \in \nu_q$, for $q = \bigwedge\{\nu(z) \mid \mu(z) \geqslant p\}$. We have:

$$\mu(x) \geqslant p \text{ if and only if } {\uparrow}\mu(x) \cap \mu(X) \subseteq {\uparrow}p \cap \mu(X),$$

$$\longleftrightarrow {\uparrow}\nu(x) \cap \nu(X) \subseteq {\uparrow}\bigwedge\{\nu(z) \mid \mu(z) \geqslant p\} \cap \nu(X),$$

$$\longleftrightarrow \bigwedge\{\nu(z) \mid \mu(z) \geqslant p\} \leqslant \nu(x),$$

$$\longleftrightarrow x \in \nu_{\bigwedge\{\nu(z)\mid\mu(z)\geqslant p\}},$$

$$\longleftrightarrow x \in \nu_q.$$

Now we proof the converse, i.e., if $\mu_S = \nu_S$ then $\mu \sim \nu$. By Proposition 3.3 we know $S_\mu \cong \mu_S$, $S_\nu \cong \nu_S$ and $\mu_S = \nu_S$. It implies that $S_\mu \cong S_\nu$. Hence, there is isomorphism from S_μ to S_ν, where ${\uparrow}p \cap \mu(X) \mapsto {\uparrow}p \cap \nu(X)$, i.e., $\mu \sim \nu$. □

4. Conclusion and further work

We have presented an investigation of functions from a set to a semilattice, actually we were dealing with semilattice-valued fuzzy sets. We were concentrated on the situation in which the collections of cuts of these functions coincide. Using the combined techniques of poset and lattice valued fuzzy sets we defined an equivalence relation on S^X, where S is a semilattice and X a nonempty domain. This equivalence relates fuzzy sets with equal collections of cuts.

As a further work, we plan to investigate properties of classes obtained by the mentioned equivalence, in terms of set and order theoretic features of the corresponding collections of cut sets.

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